Structural Properties of Plane Graphs Without Adjacent Triangles and an Application to 3-Colorings

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ABSTRACT

If in a plane graph with minimum degree \( \geq 3 \) no two triangles have an edge in common, then: (1) there are two adjacent vertices with degree sum at most 9, and (2) there is a face of size between 4 and 9 or a 10-face incident with ten 3-vertices. It follows that every planar graph without cycles between 4 and 9 is 3-colorable. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

In 1959, Grötsch [5] proved that every planar graph without 3-cycles is 3-colorable. In 1976, Steinberg (see [7, p. 229]) conjectured that every planar graph without 4- and 5-cycles is 3-colorable. (Both 4-cycles and 5-cycles must be excluded as shown by \( K_4 \) and a graph due to Havel [4, Fig. 2] ) In 1990, Erdős (see [7, p. 229]) suggested the following relaxation of Steinberg’s conjecture: Is there an integer \( k \geq 5 \) such that every planar graph without \( i \)-cycles, \( 4 \leq i \leq k \), is 3-colorable? Abbott and Zhou [1] proved that \( k = 11 \) is suitable. In [2], I strengthened this to \( k = 10 \) and now one more step is done:

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Theorem 1. If $G$ is a planar graph without $i$-cycles, $4 \leq i \leq 9$, then $G$ is 3-colorable.

This result was announced in Chapter 2 of a forthcoming monograph by Jensen and Toft [6]. The proof is based on a structural property of plane graphs without adjacent triangles. Denote the minimum degree of a graph by $\delta(G)$, the degree of a vertex $v$ by $d(v)$, and the size of a face $f$ by $s(f)$.

Theorem 2. Let $G$ be a plane graph without two triangles sharing an edge. Then the following statements are valid in which all numerical parameters are best possible:

(a) $\delta(G) \leq 4$;
(b) if $\delta(G) \geq 3$, then there are adjacent vertices $x, y$ such that $d(x) + d(y) \leq 9$;
(c) if $\delta(G) \geq 3$, then there is either an $i$-face where $4 \leq i \leq 9$ or 10-face incident with ten 3-vertices and adjacent to five triangles.

Proof of Theorem 1. Let $G$ be a counterexample minimum on the number of vertices. It has $\delta(G) \geq 3$ and by (c) of Theorem 2 has a 10-face, $f$, with all incident vertices having degree 3. The graph obtained from $G$ by removing all the vertices incident with $f$ is 3-colorable. Take such a coloring and extend it to $G$: since for every vertex on the boundary of $f$ two colors are admissible, and since $s(f)$ is even, this can be done. This contradiction completes the proof.

Proof of Theorem 2. To show the parameters in (a), (b), and (c) are the best possible, take: (a) the line graph of the cube; (b) the graph on Fig. 1 in [3]; (c) the dodecahedron and saw all its corners off.

Clearly, (b) implies (a). Assume $G$ is a connected plane graph with $\delta(G) \geq 3$ and without two triangles having an edge in common. If $V$ and $F$ denote the set of vertices and faces of $G$, respectively, then it follows from Euler's formula that

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (s(f) - 4) = -8 = \sum_{x \in V \cup F} z(x).$$

Let $z(x)$ above be a charge assigned to $x \in V \cup F$. To prove (b) and (c), we introduce rules $R_b$ and $R_c$, respectively, of modifying $z$ to a new charge $z^*$ such that

$$\sum_{x \in V \cup F} z^*(x) = \sum_{x \in V \cup F} z(x) = -8. \quad (*)$$

In both cases, as we shall verify, $z^*(x) \geq 0$ for every $x \in V \cup F$, which contradicts $(*)$. Now the proof divides.

Proof of (b). Assume the contrary, and construct $z^*$ according to the following rules:

(Rb1) For every edge $e = (x, y)$ where $d(x) \leq 4$, transfer from $y$ to $x$ the following charge:
- 1/2 if $d(x) = 3$, and $e$ is incident with a triangle;
- 1/3 if $d(x) = 3$, but $e$ is not incident with a triangle;
- 1/6 if $d(x) = 4$, and $e$ is not incident with a triangle.

(Rb2) Every vertex transfers 1/3 to every incident triangle.

Clearly, $z^*(f) \geq 0$ where $f \in F$, for if $s(f) = 3$ then $z^*(f) = z(f) + 3 \cdot 1/3 = 3 - 4 + 1 = 0$ and if $s(f) \geq 4$, $z^*(f) = z(f) = s(f) - 4 \geq 0$.

Assume $v \in V$; if $d(v) \leq 4$, then it is easily seen that in all five cases $z^*(v) = 0$. If $d(v) = 5$, then $v$ is not adjacent to 3- or 4-vertices and $z^*(v) \geq 5 - 4 - 2 \cdot 1/3 \geq 0$. (Here,
and in the sequel, we make use of the fact that the number, \( t(v) \), of triangles incident with \( v \) is at most \( [d(v)/2] \).

If \( d(v) = 6 \) then \( v \) is not adjacent to 3-vertices. Therefore it transfers at most \( 6 \cdot 1/6 = 1 \) to adjacent vertices and at most \( 3 \cdot 1/3 = 1 \) to incident triangles, i.e., \( z^*(v) \geq 0 \).

Assume \( d(v) \geq 7 \). Then \( v \) transfers \( t(v)/3 \) to incident triangles, and at most \( t(v)/2 \) along the edges incident with triangles (because in a triangle \( uvw \), either \( d(u) \geq 4 \) or \( d(w) \geq 4 \)), and at most \( (d(v) - 2t(v))/3 \) along the edges not incident with triangles. It follows, \( z^*(v) \geq d(v) - 4 - t(v)/2 - t(v)/3 - (d(v) - 2t(v))/3 = 2d(v)/3 - 4 - t(v)/6 = (4d(v) - 49/2 - d(v))/6 \geq 0 \).

Due to the above remark, this completes the proof of (b).

**Proof of (c).** Assume the contrary and construct \( z^* \) as follows:

**(Rc1)** Every nontriangular face \( f \) transfers to each incident vertex the following charge:

\[
\begin{align*}
2/3 & \quad \text{if } d(v) = 3, \text{ and } v \text{ is incident with a triangle;} \\
1/3 & \quad \text{if } d(v) = 3, \text{ but } v \text{ is not incident with a triangle, or if } d(v) = 4, \text{ and } v \text{ is incident with either two triangles or one triangle not adjacent to } f.
\end{align*}
\]

**(Rc2)** Every vertex transfers 1/3 to every incident triangle.

Let \( f \in F \). If \( s(f) = 3 \), then \( z^*(f) = 3 - 4 = 3 - 1/3 = 0 \). If \( s(f) \geq 12 \), then \( z^*(f) \geq s(f) - 4 - s(f) \cdot 2/3 = (s(f) - 12)/3 \geq 0 \). If \( r(f) = 11 \), then at least one vertex incident with \( f \) is not a 3-vertex incident with a triangle by parity. This vertex receives at most 1/3 from \( f \) which implies \( z^*(f) \geq 11 = 4 - 10 \cdot 2/3 - 1/3 = 0 \). If \( r(f) = 10 \), then \( f \) cannot be incident with ten vertices getting 2/3 each by assumption. On the other hand, if \( f \) is incident with more than one vertex receiving at most 1/3, then \( z^*(f) \leq 10 - 4 \cdot 2/3 - 2 \cdot 1/3 = 0 \). Thus it remains to consider the case when \( f \) is incident with precisely nine vertices of degree 3, each of which being incident with a triangle. But then the last vertex, \( v \), has \( d(v) \geq 3 \) and fails to receive 1/3 by Rc1; therefore \( z^*(f) = 10 - 4 - 9 \cdot 2/3 = 0 \).

Now assume \( v \in V \) and recall that \( t(v) \leq [d(v)/2] \). If \( d(v) \leq 4 \), we have five cases, and in each of them \( z^*(v) = 0 \). But if \( d(v) \geq 5 \), then \( z^*(v) \geq d(v) - 4 - [d(v)/2]/3 \geq d(v) - 25/6 - d(v)/6 = 5(d(v) - 5)/6 \geq 0 \).

This completes the proof of Theorem 2.

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**References**


**NOTE ADDED IN PROOF**

I have been informed by the referees that Theorem 1 was also proved in "A Note on the Three Color Problem" by D.P. Sanders and Y. Zhao (submitted to Graphs and Combinatorics).

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