Problem 1: Decay of a scalar particle (⋆)

Consider the following Lagrangian, involving two real scalar fields $\phi$ and $\chi$:

$$
L = \frac{1}{2} \partial_\mu \phi \partial^{\mu} \phi + \frac{1}{2} \partial_\mu \chi \partial^{\mu} \chi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} M^2 \chi^2 - \frac{1}{2} \mu \chi \phi^2.
$$

Suppose that $M > 2m$, so that the decay $\chi \to \phi \phi$ is kinematically possible. Calculate the lifetime of $\chi$ to leading order in the coupling $\mu$.

Hints: The Feynman rule associated to the $\chi \phi^2$ interaction term, which gives rise to a vertex attaching to two $\phi$ propagators and one $\chi$ propagator, is

$$
\begin{array}{c}
\end{array} = \mu.
$$

From this it is easy to deduce the tree-level matrix element $\mathcal{M}$, which you can insert into the formula for the differential decay width

$$
d\Gamma = \frac{1}{2M} \prod_f \frac{d\vec{p}'}{2\pi} |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left( p_i - \sum_f p'_f \right).
$$

Problem 2: The Clifford algebra

1. (⋆) Given a set of four matrices $\gamma^\mu$ which satisfy the Clifford algebra

$$
\{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu},
$$

show that the matrices $\gamma^{\mu\nu} \equiv \frac{i}{2} \{ \gamma^\mu, \gamma^\nu \}$ satisfy the Lorentz algebra:

$$
[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] = i \left( g^{\lambda\rho} \gamma^{\kappa\sigma} - g^{\kappa\rho} \gamma^{\lambda\sigma} - g^{\lambda\sigma} \gamma^{\kappa\rho} + g^{\kappa\sigma} \gamma^{\lambda\rho} \right).
$$

2. Show that the Clifford algebra is satisfied by both the Weyl representation of $\gamma$ matrices

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.
$$

and the Dirac-Pauli representation

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}
$$

and find the unitary transformation that takes one into the other.
3. Defining $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$, calculate

$$\{\gamma^5, \gamma^\mu\} \quad \text{and} \quad [\gamma^5, \gamma^{\mu\nu}].$$

**Problem 3: The Dirac field (⋆)**

1. Show that

$$(1 + \frac{i}{2} \omega_{\rho\sigma} \gamma^{\rho\sigma}) \gamma^\mu \left(1 - \frac{i}{2} \omega_{\rho\sigma} \gamma^{\rho\sigma}\right) = \left(1 - \frac{i}{2} \omega_{\rho\sigma} M^{\rho\sigma}\right)^\mu \gamma^\nu + O(||\omega||^2),$$

where the $M^{\rho\sigma}$ generate the vector representation of $\mathfrak{so}(1,3)$,

$$(M^{\kappa\lambda})_{\mu\nu} = i \left(\delta^\kappa_\mu \delta^\lambda_\nu - \delta^\kappa_\nu \delta^\lambda_\mu\right).$$

Use this result to conclude that the Dirac Lagrangian

$$L = \overline{\psi} \left(i \gamma^\mu \partial_\mu - m\right) \psi$$

is invariant under proper orthochronous Lorentz transformations.

2. Find the Euler-Lagrange equations obtained from the Dirac Lagrangian.