Théorie de la décision

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1 Risk premium: Arrow-Pratt approximation

The DM has sure wealth w and any additive risk $\widetilde{x}$. His/her random wealth is thus written $\widetilde{w} = w + \widetilde{x}$. The risk premium is defined by:

$$u\left(E\widetilde{w} - \Pi\right) = Eu\left(\widetilde{w}\right) \tag{1}$$

or, equivalently,

$$u\left(w + E\left(\widetilde{x}\right) - \Pi\right) = E\left(u\left(w + \widetilde{x}\right)\right) \tag{2}$$

Using Taylor approximations, as in Eeckhoudt et al. (2005), we obtain:

$$u\left(E\widetilde{w} - \Pi\right) = u\left(E\widetilde{w}\right) + \left[-\Pi\right] u'\left(E\widetilde{w}\right) \tag{3}$$

and

$$u\left(\widetilde{w}\right) = u\left(E\widetilde{w}\right) + \left[\widetilde{w} - E\widetilde{w}\right] u'\left(E\widetilde{w}\right) + \frac{1}{2}\left[\widetilde{w} - E\widetilde{w}\right]^2 u''\left(E\widetilde{w}\right) \tag{4}$$

From (4) we get:

$$\begin{aligned} Eu\left(\widetilde{w}\right) &= u\left(E\widetilde{w}\right) + u'\left(E\widetilde{w}\right) \underbrace{E\left[\widetilde{w} - E\widetilde{w}\right]}_{=0} + \frac{1}{2}u''\left(E\widetilde{w}\right) E\left[\widetilde{w} - E\widetilde{w}\right]^2 \\ &= u\left(E\widetilde{w}\right) + \frac{1}{2}u''\left(E\widetilde{w}\right)\sigma^2_{\widetilde{w}} \end{aligned} \tag{5}$$

where $\sigma^2_{\widetilde{w}} = E\left[\widetilde{w} - E\widetilde{w}\right]^2$ is the variance of $\widetilde{w}$. Observe that $\sigma^2_{\widetilde{w}} = \sigma^2_{\widetilde{x}}$. Finally, combining (1), (3) and (5) we get:

$$\left[-\Pi\right] u'\left(E\widetilde{w}\right) = +\frac{1}{2}u''\left(E\widetilde{w}\right)\sigma^2_{\widetilde{w}} \tag{6}$$

Rearranging terms yields we obtain the Arrow-Pratt approximation of the risk premium:

$$\Pi = \frac{1}{2}\frac{-u''\left(E\widetilde{w}\right)}{u'\left(E\widetilde{w}\right)}\sigma^2_{\widetilde{w}} \tag{7}$$

or, equivalently,

$$\Pi = \frac{1}{2}\frac{-u''\left(w + E\widetilde{x}\right)}{u'\left(w + E\widetilde{x}\right)}\sigma^2_{\widetilde{x}} \tag{8}$$

Example 1 *Consider sure wealth* $w = 10$ *and a risk* $\widetilde{x} = \left(2, \frac{1}{2}; 0, \frac{1}{2}\right)$. *Thus the random wealth is written* $\widetilde{w} = w + \widetilde{x} = \left(12, \frac{1}{2}; 0, \frac{1}{2}\right)$. *Note that* $E\widetilde{w} = 11$, $E\widetilde{x} = 1$ *and* $\sigma^2_{\widetilde{w}} = \sigma^2_{\widetilde{x}} = 1$. *The risk premium is thus* $\Pi = \frac{1}{2}\frac{-u''(11)}{u'(11)}$. *The DM is ready to pay* Π *to get* $E\widetilde{x}$ *rather than* $\widetilde{x}$.

Example 2 *Consider sure wealth* $w = 11$ *and a zero-mean risk* $\widetilde{z} = \left(1, \frac{1}{2}; -1, \frac{1}{2}\right)$. *Observe that* $w = 10 + E\widetilde{x}$ *and* $\widetilde{z} = \widetilde{x} - E\widetilde{x}$. *Observe also that the random wealth is exactly the same with* $\widetilde{w} = w + \widetilde{z} = \left(12, \frac{1}{2}; 0, \frac{1}{2}\right)$, *and that* $\sigma^2_{\widetilde{w}} = \sigma^2_{\widetilde{x}} = \sigma^2_{\widetilde{z}} = 1$. *The risk premium is thus the same:* $\Pi = \frac{1}{2}\frac{-u''(11)}{u'(11)}$. *The DM is ready to pay* Π *to get nothing rather than* $\widetilde{z}$.

These two examples may help to recognize that to dislike any zero-mean risk at any wealth level ($\Pi < 0$) is equivalent to always prefer the expected value of a random wealth rather than the random wealth itself.