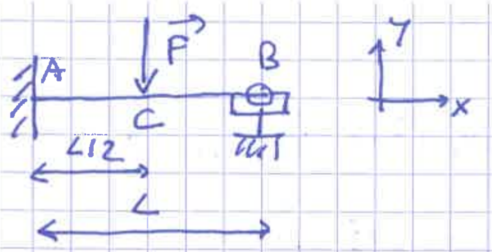
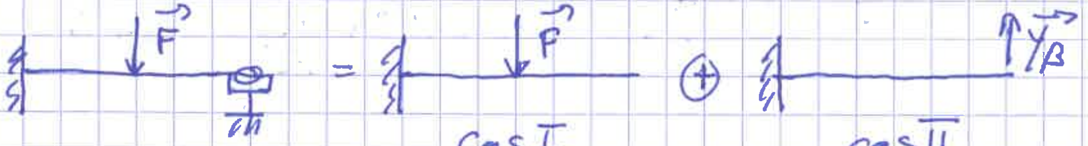
(I)



Y_B ? $\vec{R}_A$? $v(\frac{L}{2})$?

* Principe de superposition



cas I — cas II

$$v_I(L) + v_{II}(L) = 0 \Rightarrow Y_B$$

$$v_I(L) = -\frac{5FL^3}{48EI} \quad v_{II}(L) = \frac{Y_B L^3}{3EI} \quad \Rightarrow Y_B = \frac{5F}{16}$$

Rq : Les flèches ($v_I(L)$ et $v_{II}(L)$) sont obtenues en utilisant les équations d'équilibre local et la loi de comportement (calcul de la déformée)

* Equation d'équilibre locale + Th^me énergétique

$$\begin{cases} \frac{dN(x)}{dx} = 0 \\ \frac{dT(x)}{dx} = 0 \\ \frac{dM(x)}{dx} + T(x) = 0 \end{cases}$$

↳ Pour $0 \leq x < \frac{L}{2}$

$$\begin{cases} N^-(x) = K_N^- \\ T^-(x) = K_T^- \\ M^-(x) = -K_T^- x + K_M^- \end{cases}$$

↳ Pour $\frac{L}{2} < x \leq L$

$$\begin{cases} N^+(x) = K_N^+ \\ T^+(x) = K_T^+ \\ M^+(x) = -K_T^+ + K_M^+ \end{cases}$$

CL : $N^+(L) = 0 \Rightarrow K_N^+ = 0$

$T^+(L) = Y_B \Rightarrow K_T^+ = Y_B$

$M^+(L) = 0 \Rightarrow K_M^+ = Y_B L$

Donc $\boxed{N^+(x) = 0}$; $\boxed{T^+(x) = Y_B}$; $\boxed{M^+(x) = Y_B(L-x)}$

↳ Saut de $\vec{R}(x)$

$$[\![\vec{R}(x)]\!]_{x=\frac{L}{2}} + \vec{F} = 0 \Rightarrow N^+(\tfrac{L}{2}) = N^-(\tfrac{L}{2}) \Rightarrow \boxed{N^-(x) = 0}$$

$$T^+(\tfrac{L}{2}) - T^-(\tfrac{L}{2}) - F = 0$$

$$\Leftrightarrow Y_B - K_T^- - F = 0 \Rightarrow \boxed{T^-(x) = Y_B - F}$$

→ continuité de $\vec{M}(x)$:

$$M^+\left(\frac{L}{2}\right) = M^-\left(\frac{L}{2}\right) \Rightarrow (F - Y_B)\frac{L}{2} + K_M^- = Y_B \frac{L}{2}$$

Donc $$\boxed{M^-(x) = (F - Y_B)x + \left(Y_B - \frac{F}{2}\right)L}$$

→ Th^{me} de Ménabrea : $\dfrac{\partial W^*}{\partial Y_B} = 0$ avec $W^* = \dfrac{1}{2EI}\displaystyle\int_0^L M(x)^2 dx$

Pour $0 \leq x \leq \frac{L}{2}$

$$M^-(x) = (F - Y_B)x + \left(Y_B - \frac{F}{2}\right)L$$

$$M^-(x)^2 = (F - Y_B)^2 x^2 + 2(F - Y_B)\left(Y_B - \frac{F}{2}\right)Lx + \left(Y_B - \frac{F}{2}\right)^2 L^2$$

$$W_-^* = \frac{1}{2EI}\int_0^{L/2} M^-(x)^2 dx = \frac{1}{2EI}\left[(F - Y_B)^2 \frac{x^3}{3}\right]_0^{L/2}$$

$$+ \left[2(F - Y_B)\left(Y_B - \frac{F}{2}\right)L\frac{x^2}{2}\right]_0^{L/2} + \left[\left(Y_B - \frac{F}{2}\right)^2 L^2 x\right]_0^{L/2}$$

$$= \frac{1}{2EI}\left((F - Y_B)^2 \frac{L^3}{24} + 2(F - Y_B)\left(Y_B - \frac{F}{2}\right)\frac{L^3}{8} + \left(Y_B - \frac{F}{2}\right)^2 \frac{L^3}{2}\right)$$

$$\Rightarrow \frac{\partial W_-^*}{\partial Y_B} = \frac{1}{2EI}\left(-\frac{5FL^3}{24} + \frac{7}{12}Y_B L^3\right)$$

Pour $\frac{L}{2} \leq x \leq L$

$$M^+(x) = Y_B(L - x) \qquad M^+(x)^2 = Y_B^2 (L - x)^2$$

$$W_+^* = \frac{1}{2EI}\int_{L/2}^{L} Y_B^2 (L - x)^2 dx = \frac{1}{2EI}\left[-Y_B^2 \frac{(L - x)^3}{3}\right]_{L/2}^{L} = \frac{1}{2EI} Y_B^2 \frac{L^3}{24}$$

$$\frac{\partial W_+^*}{\partial Y_B} = \frac{1}{2EI} Y_B \frac{L^3}{12}$$

On a donc $$\frac{1}{2EI}\left(-\frac{5FL^3}{24} + \frac{7}{12}Y_B L^3 + Y_B \frac{L^3}{12}\right) = 0$$

$$\Rightarrow \boxed{Y_B = \frac{5F}{16}}$$

* Equations d'équilibre local + calcul de la déformée

On part de $EI\,v''(x) = M(x)$ ~~avec M(x) = M(x) =~~

↳ Pour $0 \leq x < \frac{L}{2}$:

- $EI v_-''(x) = Y_B (L-x) + F\left(x - \frac{L}{2}\right)$
- $EI v_-'(x) = -Y_B \frac{(L-x)^2}{2} + F \frac{(x-\frac{L}{2})^2}{2} + C_-'$

$v'(0) = 0 \Rightarrow C_-' = Y_B \frac{L^2}{2} - \frac{FL^2}{8}$

Donc $EI v'(x) = -Y_B \frac{(L-x)^2}{2} + F \frac{(x-L/2)^2}{2} + \left(-\frac{FL^2}{8} + Y_B \frac{L^2}{2}\right)$

- $EI v_-(x) = Y_B \frac{(L-x)^3}{6} + F \frac{(x-L/2)^3}{6} + \left(Y_B \frac{L^2}{2} - \frac{FL^2}{8}\right) x + C_-$

$v(0) = 0 \Rightarrow C_- = \frac{FL^3}{48} - Y_B \frac{L^3}{6}$

Donc $\boxed{EI v_-(x) = Y_B \frac{(L-x)^3}{6} + F \frac{(x-L/2)^3}{6} + \left(Y_B \frac{L^2}{2} - \frac{FL^2}{8}\right) x + \frac{FL^3}{48} - Y_B \frac{L^3}{6}}$

↳ Pour $\frac{L}{2} \leq x < L$

- $EI v_+''(x) = Y_B (L-x)$
- $EI v_+'(x) = -Y_B \frac{(L-x)^2}{2} + C_+'$
- $EI v_+(x) = Y_B \frac{(L-x)^3}{6} + C_+' x + C_+$

$v(L) = 0 \Rightarrow C_+ = -C_+' L$

Donc $EI v_+(x) = Y_B \frac{(L-x)^3}{6} + C_+'(x-L)$

- Continuité de la déformée

↳ $v_+'\left(\frac{L}{2}\right) = v_-'\left(\frac{L}{2}\right) \Rightarrow C_+' = Y_B \frac{L^2}{2} - \frac{FL^2}{8}$

↳ $v_+\left(\frac{L}{2}\right) = v_-\left(\frac{L}{2}\right) \Rightarrow C_+' = \frac{FL^2}{12} - Y_B \frac{L^2}{6}$

$$\Rightarrow \boxed{Y_B = \frac{5F}{16}}$$

* Réaction au point A :

↳ PFS : $\begin{cases} X_A = 0 \\ Y_A + Y_B - F = 0 \\ M_A - \frac{FL}{2} + Y_B L = 0 \end{cases}$

↳ CL en effort : $N(0) = -X_A \quad M(0) = -M_A$

$T(0) = -Y_A$

* Déplacement du point C

Castigliano : $\frac{\partial W^*}{\partial F} = v_C$ avec $Y_B = \frac{5F}{16}$

$\hookrightarrow W^*_- = \frac{1}{2EI}\left[\left(F-\frac{5F}{16}\right)^2\frac{L^3}{24} + 2\left(F-\frac{5F}{16}\right)\left(\frac{5F}{16}-\frac{F}{2}\right)\frac{L^3}{8} + \left(\frac{5F}{16}-\frac{F}{2}\right)^2\frac{L^3}{2}\right]$

$= \frac{L^3}{2EI}\left[\left(\frac{11F}{16}\right)^2 \times \frac{1}{24} + 2\,\frac{11F}{16}\,\frac{3F}{16}\,\frac{1}{8} + \left(\frac{3F}{16}\right)^2\frac{1}{2}\right]$

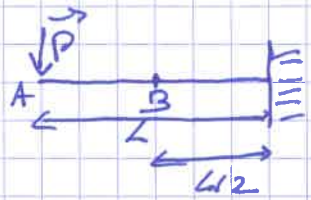
$= \frac{L^3F^2}{512EI}\left(\frac{121}{24} + \frac{33}{4} + \frac{9}{2}\right) = \frac{427L^3F^2}{512}$

$\hookrightarrow W^*_+ = \frac{1}{2EI}\left(\frac{5F}{16}\right)^2\frac{L^3}{24} = \frac{25F^2L^3}{512\times 24}$

$\Rightarrow W^* = \frac{1}{2EI}\,\frac{24\times 427L^3F^2 + 25F^2L^3}{512\times 24} = \frac{10273}{12288}F^2L^3 \times \frac{1}{2EI}$

$\Rightarrow \frac{\partial W^*}{\partial F} = \frac{10273}{12288}\,\frac{FL^3}{EI}$

(II)



1] $M_f(x) = -Px$

$W^* = \int_0^L \frac{P^2x^2}{2EI}dx = \frac{P^2L^3}{2EI} \Rightarrow v_A = \frac{\partial W}{\partial P} = \frac{PL^3}{3EI}$ vers le bas

2] Déplacement de B : charge fictive Q en B

$M_f(x) = \begin{cases} -Px & 0 \leq x \leq L/2 \\ -Px - Q\left(x-\frac{L}{2}\right) & \frac{L}{2} \leq x \leq L \end{cases}$

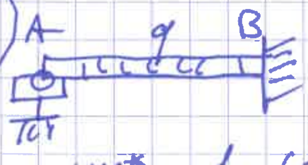
$W^* = \frac{L^3}{48EI}\left(8P^2 + Q^2 + 5PQ\right) \quad v_B = \left.\frac{\partial W^*}{\partial Q}\right|_{Q=0} = \frac{5PL^3}{48EI}$

2] Rotation de la section A charge fictive M_A en A

$M_f(x) = -Px - M_A \Rightarrow W^* = \frac{1}{2EI}\left(\frac{P^2L^3}{3} + PM_A L^2 + M_A^2 L\right)$

$\Rightarrow \theta_A = \left.\frac{\partial W^*}{\partial M_A}\right|_{M_A=0} = \frac{PL^2}{2EI}$

(III)



$M_f(x) = R_A x - q\frac{x^2}{2}$

$W^* = \frac{1}{2EI}\left(R_A^2\frac{L^3}{3} - qR_A\frac{L^4}{4} + \frac{qL^5}{5}\right)$ en $\frac{\partial W^*}{\partial R_A} = 0 \Rightarrow R_A = \frac{3qL}{8}$