On Paths in Planar Graphs

Daniel P. Sanders*
DEPARTMENT OF MATHEMATICS
THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210-1174
e-mail: dsanders@math.ohio-state.edu

ABSTRACT

This paper generalizes a theorem of Thomassen on paths in planar graphs. As a corollary, it is shown that every 4-connected planar graph has a Hamilton path between any two specified vertices $x, y$ and containing any specified edge other than $xy$. © 1997 John Wiley & Sons, Inc.

1. INTRODUCTION

In 1956, Tutte [5] showed that every 4-connected planar graph has a Hamilton circuit. He proved this by showing that every plane graph has a special kind of path. In this paper, it will be called a Tutte path, and is a generalization of a Hamilton path. It is convenient to consider only 2-connected graphs.

In a 2-connected plane graph $G$, the exterior circuit is the circuit bounding the infinite face and will be denoted $X_G$. For a subgraph $H$ of $G$, the bridges of $H$ in $G$ are defined as follows. A trivial bridge of $H$ in $G$ is an edge in $E(G) \setminus E(H)$ with both ends in $V(H)$. A non-trivial bridge of $H$ in $G$ is a component $K$ of $G \setminus H$ with all vertices of $H$ adjacent to vertices of $K$ added and all edges with one end in $H$ and the other in $K$ added. The vertices of attachment of a bridge $B$ of $H$ in $G$ are $V(B) \cap V(H)$. A bridge is attached to its vertices of attachment. A path (circuit) $P$, subgraph of a plane graph $G$, is a Tutte path (circuit) if and only if each bridge of $P$ has at most three vertices of attachment and each bridge containing an edge of $X_G$ has at most two vertices of attachment.

Lemma 1 (Tutte). Let $G$ be a 2-connected plane graph. Let $x, y$, and $\alpha$ be two vertices and an edge, respectively, of $X_G$. Then $G$ has a Tutte path from $x$ to $y$ containing $\alpha$.

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A Tutte path in a 4-connected graph is also a Hamilton path. Tutte's Theorem follows by choosing $x$ and $y$ to be adjacent. In 1983, Thomassen [4] improved this result by removing the restriction on the location of $y$.

**Lemma 2 (Thomassen).** Let $G$ be a 2-connected plane graph. Let $x$ and $\alpha$ be a vertex and edge, respectively, of $X_G$, and let $y$ be any vertex of $G$ distinct from $x$. Then $G$ has a Tutte path from $x$ to $y$ containing $\alpha$.

Thomassen’s Theorem that every 4-connected plane graph is Hamilton-connected easily follows from this lemma. Freeing the vertex $y$ from being in $X_G$ allows the ends of the path to become arbitrary, giving the necessary Hamilton paths. For a proof of Lemma 2, see [4] and [1].

The main result of this paper generalizes Lemma 2, by removing the restriction on the location of $x$.

**Theorem.** Let $G$ be a 2-connected plane graph. Let $\alpha$ be an edge of $X_G$, and let $x$ and $y$ be arbitrary distinct vertices of $G$. Then $G$ has a Tutte path $P$ from $x$ to $y$ containing $\alpha$.

Since $x$ and $\alpha$ no longer have to share a face, this allows for more powerful results. For example, consider a 4-connected plane graph $G$. While Tutte’s theorem shows that $G$ contains a Hamilton circuit through any edge of $G$, the theorem presented here shows that $G$ has a Hamilton circuit through any two edges of $G$. Further, let $x$ and $y$ be distinct vertices of $G$. Thomassen’s theorem gives a Hamilton path in $G$ from $x$ to $y$. The theorem above guarantees a Hamilton path in $G$ from $x$ to $y$ through any edge of $G$ (except $xy$). These two results are stated below as Corollaries 1 and 2.

**Corollary 1.** Let $G$ be a 4-connected plane graph, $x$, $y$ be vertices of $G$, and $\alpha \neq xy$ be an edge of $G$. Then $G$ has a Hamilton path from $x$ to $y$ through $\alpha$.

**Corollary 2.** Every 4-connected plane graph has a Hamilton circuit through any two of its edges.

The theorem above is as strong as possible in several ways.

The edge $\alpha$ cannot be removed from $X_G$, even if the vertices $x$ and $y$ are required to be on $X_G$. In the graph of Figure 1, there is no Tutte path between the marked vertices containing the marked edge.

Keeping $\alpha$ on $X_G$, the direction that $\alpha$ is traversed cannot be specified, even if the vertices $x$ and $y$ are again both on $X_G$. This is easily seen from planarity.

In general, there is no theorem to find a path through two edges of $X_G$, even if the vertices $x$ and $y$ are both required to be on the exterior circuit. Consider the circuit $C_4$ with vertices $a, b, c, d$. 

![Figure 1. The edge cannot be removed from the exterior circuit.](image-url)
and edges $ab, ad, bc, cd$, embedded in the plane. There is no Tutte path in $C_4$ from $a$ to $c$ through both $ab$ and either $ad$ or $cd$. Let $D := C_4 + bd$, with $bd$ embedded in the interior face of $C_4$. Then $D$ has no Tutte path from $a$ to $c$ through both $ab$ and $bc$. On the other hand, given a subset $S$ of $E(X_G)$, it may be possible to classify the structure of 2-connected plane graphs that do not have a Tutte path from $x$ to $y$ containing $S$. As a corollary, this would give necessary and sufficient conditions for a 2-connected plane graph without interior component 3-cuts (see [2]) to have a Hamilton circuit.

### 2. A PROOF OF THE THEOREM

One more lemma is required. It is a useful tool for many planar problems, and will be referred to as the Three Edge Lemma. For a proof, see [2] or [3].

**The Three Edge Lemma.** Every 2-connected plane graph $G$ has a Tutte circuit through any three edges of $X_G$.

If $P$ is a path, and $x$ and $y$ are two vertices of $P$, then $xPy$ will represent the subpath of $P$ from $x$ to $y$. If $H$ and $J$ are subgraphs of a graph $G$, then an $H, J$-connector in $G$ is a bridge of $H \cup J$ in $G$ with vertices of attachment in both $H$ and $J$.

**Proof of the Theorem.** The proof is by induction on the number of vertices of $G$. Clearly, the theorem is true for $|V(G)| \leq 4$. Let 2-connected plane graph $G$ be given. Let $e$ be an arbitrary edge of $X_G$, and let $x$ and $y$ be arbitrary vertices of $G$. From Lemma 2, the theorem follows trivially if $x$ or $y$ is a vertex of $X_G$, so assume not. Also, if there is an edge $e \in E(G)$ such that $x$ or $y$ is a vertex of $X_G$, then the theorem follows trivially if Lemma 2 is applied correctly to $G - e$, so assume not. A Tutte path in a graph $H$ with $|V(H)| < |V(G)|$ from $u$ through $\omega$ to $v$ found by induction will be called a $u\omega v$-path in $H$.

Assume first that there are subgraphs $L$ and $R$ of $G$ such that $L \cup R = G$, $V(L) \cap V(R) = \{a, b\} \subset V(X_G)$, $x \in V(L)$, $y \in V(R)$, and $R$ is 2-connected (or a similar structure with $x$ and $y$ swapped). Let $c \notin V(G)$ be given. Let $\beta := ab$, $L' := L + c + ac + bc$, with $c$ embedded where $R$ used to be, and $R' := R + \beta$, with $\beta$ embedded where $L$ used to be. Since $R$ is 2-connected, and $y \notin V(X_{R'})$, $|V(R')| \geq 4$, and thus $|V(L')| < |V(G)|$. Find an $x\alpha e$-path $P_L$ in $L'$ by induction. Without loss of generality, assume $b \in V(P_L)$.

**Case 1.** $a \notin V(P_L)$.

Let $\gamma$ be an edge of $X_R$ containing $a$ and $P_R$ be a $b\gamma y$-path in $R$ by induction. Then set $P := xP_Lb \cup bP_Ry$.

**Case 2.** $a \in V(P_L)$.

Let $P_L$ be an $a\beta y$-path in $R'$ by induction. Then set $P := xP_Lb \cup bP_Ry$.

The path $P$ is as desired in each case.

If there are no such $L$ and $R$ as above, then $x$ and $y$ are in the same component of $G \setminus X_G$. From elementary graph theory, there is a “path” of blocks of $G \setminus X_G$, the first having $x$ as a vertex and the last having $y$ as a vertex. Let $B_1, B_2, \ldots, B_k$ be the unique such “path” of blocks of $G \setminus X_G$ with $x \in V(B_1), y \in V(B_k)$, and $k$ minimal. Let $b_0 := x, b_i := B_i \cap B_{i+1}$ for $i := 1, \ldots, k - 1, b_k := y$, and $H := \bigcup_{i=1}^k B_i$.

The $(X_G \cup H)$-bridges will now be grouped. Let $s$ be a vertex of $X_G$ which shares a face with a vertex of $H$. For each $(X_G \cup H)$-bridge $X$, let $Q_X$ be the minimal path in $X_G$ including all the vertices of attachment of $X$ in $X_G$ such that $s$ is not an interior vertex of $Q_X$. Further, let $p_X(q_X)$ be the most counterclockwise (clockwise) vertex of $Q_X$. Note that for two $(X_G \cup H)$-bridges
X, Y, either $Q_X \subset Q_Y, Q_Y \subset Q_X, \text{ or } E(Q_X) \cap E(Q_Y) = \emptyset$. Let an $(X_G \cup H)$-bridge $X$ be \textit{maximal} if there is no $(X_G \cup H)$-bridge $Y$ distinct from $X$ such that $Q_X \subset Q_Y$, and $Q_X \neq Q_Y$.

Let the \textit{group} of a maximal $(X_G \cup H)$-bridge $X$ be the union of $X$ and all $(X_G \cup H)$-bridges $Y$ such that $Q_Y \subset Q_X$, and $Q_Y \neq Q_X$. Let the $(X_G \cup H)$-\textit{bridge groups} be the groups of its maximal bridges.

For each $X_G, H$-connector group $K$, let $v_K$ be the unique vertex of attachment of $K$ in $H$ and $i(K)$ be the least (greatest) integer such that $v_K \in V(B_{i(K)}) \in V(B_{j(K)})$. Let $K_\alpha$ be the $X_G, H$-connector group such that $\alpha \in E(Q_{K_\alpha})$, or if there is none, an $X_G, H$-connector group with $p_{K_\alpha}$ nearest to $\alpha$ counterclockwise from it.

Since $G$ is 2-connected, exchanging clockwise and counterclockwise in the definitions above if necessary, there is an $X_G, H$-connector group $K$ with $p_K \neq p_{K_\alpha}$. Let $L_\alpha$ be an $X_G, H$-connector group with $q_{L_\alpha}$ nearest counterclockwise to $Q_{K_\alpha}$ such that $p_{L_\alpha} \neq p_{K_\alpha}$. Let $K_1, \ldots, K_m$ be all the $X_G, H$-connector groups with $p_{K_i} = p_{K_\alpha}$. Notice $K_\alpha = K_i$ for some $i$, and thus $m \geq 1$. Let $L_1, \ldots, L_n$ be all the $X_G, H$-connector groups with $p_{L_j} \neq p_{K_\alpha}, q_{L_j} = q_{L_m}$. Notice $p_{L_m} \neq p_{K_\alpha}$, and thus $L_\alpha = L_j$ for some $j$, and $n \geq 1$.

Let $a_1$ and $a_2$ be vertices not in $G$, and let $\gamma := a_1 a_2$, $\delta_i := a_1 v_{K_i}$, and $\epsilon_j := a_2 v_{L_j}$. Let $f := \min\{\min\{j_K|1 \leq i \leq m\}, \min\{j_{L_j}|1 \leq j \leq n\}\}$. Let $l := \max\{\max\{i_K|1 \leq i \leq m\}, \max\{i_{L_j}|1 \leq j \leq n\}\}$. If $q_{L_m} \neq p_{K_\alpha}$, let $\beta := \gamma$ and $J := (\bigcup_{i=l}^{f} B_i) + a_1 + a_2 + a_1 + \delta_i + \cdots + \delta_m + \epsilon_1 + \cdots + \epsilon_n$, with the extra vertices and edges embedded in a planar way in the infinite face. If $q_{L_m} = p_{K_\alpha}$, note $n = 1$ and let $a_1 = a_2$, $\beta := \epsilon_1$, and $J := (\bigcup_{i=l}^{f} B_i) + a_1 + \delta_i + \cdots + \delta_m + \epsilon_1$, with the extra vertex and edges embedded in a planar way in the infinite face.

Notice $J$ is 2-connected, and since only at most two vertices were added while $V(X_G)$, containing at least three vertices, was deleted, $|V(J)| < |V(G)|$. Thus induction gives $P_j$, a $b_{f-1} \beta \eta$-path in $J$. For $i < f$ or $i > l$, let $\xi_i$ be an edge of $X_B$, and $P_i$ be a $b_{i-1} \beta \xi_i$-path in $B_i$.

There is exactly one integer $i$ such that $\delta_i \in E(P_j)$; let $K_\delta := K_i$. There are two cases on how to define $P_{K_\delta}$.

\textbf{Case 1.} $K_\delta$ is a trivial $X_G, H$-connector.

Let $P_{K_\delta} := K_\delta$.

\textbf{Case 2.} $K_\delta$ is a non-trivial $X_G, H$-connector group.

Let $\eta := v_{K_\delta} q_{K_\delta}$ and $M := K_\delta \cup Q_{K_\delta} + \eta$, with $\eta$ embedded in the infinite face such that $V(Q_{K_\delta}) \subset V(X_M)$. Let $\theta$ be an edge of $X_M$ containing $p_{K_\delta}$. There are two cases on how to define a circuit $C$.

\textbf{Case 2a.} $\alpha \notin E(Q_{K_\delta})$.

Let $C$ be a Tutte path in $M$ from $v_{K_\delta}$ to $q_{K_\delta}$ through $\theta$ by Lemma 1.

\textbf{Case 2b.} $\alpha \in E(Q_{K_\delta})$.

Let $C$ be a Tutte circuit through $\alpha, \eta, \theta$ in $M$ by the Three Edge Lemma.

For each of Cases 2a and 2b, let $P_{K_\delta} := C - \eta$.

There is exactly one integer $j$ such that $\epsilon_j \in E(P_j)$; let $L_\epsilon := L_j$. There are two cases on how to define $P_{L_\epsilon}$.

\textbf{Case 1.} $L_\epsilon$ is a trivial $X_G, H$-connector.

Let $P_{L_\epsilon} := L_\epsilon$.

\textbf{Case 2.} $L_\epsilon$ is a non-trivial $X_G, H$-connector group.

\textbf{Case 2a.} $q_{L_\epsilon} \neq p_{K_\alpha}$.

Let $\iota := v_{L_\epsilon} p_{L_\epsilon}$, and $N_\iota := L_\epsilon \cup Q_{L_\epsilon} + \iota$, with $\iota$ embedded in the infinite face such that $V(Q_{L_\epsilon}) \subset V(X_{N_\iota})$. Let $\lambda$ be an edge of $X_{N_\iota}$ containing $q_{L_\epsilon}$. Let $D$ be the Tutte path in $N_\iota$ from $v_{L_\epsilon}$ to $p_{L_\epsilon}$ through $\lambda$ by Lemma 1. Let $P_{L_\epsilon} := D - \iota$. 


Case 2b. \( q_{L_k} = p_{K_n} \).

Let \( \kappa := \epsilon_{L_k} q_{L_k} \) and \( N_b := L_k \cup Q_{L_k} + \kappa \), with \( \kappa \) embedded in the infinite face such that \( V(Q_{L_k}) \subset V(X_{N_b}) \). Let \( P_b \) be a \( \epsilon_{L_k} q_{L_k} \)-path in \( N_b \) by induction. Let \( P_{L_k} := v_{L_k} P_b q_{L_k} \).

Let \( P_X \) be the path in \( X_G \) from \( P_{L_k} \) counterclockwise to \( q_{K_n} \). Finally, let \( T := P_j \cup (\bigcup_{i=1}^{j-1} P_i) \cup (\bigcup_{i=1}^{j+1} P_i) \cup P_{K_n} \cup P_{L_k} \cup P_X = a_1 - a_2 \).

Now let \( T \) be modified to become a Tutte path \( P \) in \( G \). For a bridge \( A \) of \( T \) with \( V(A) \cap V(X_G) \neq \emptyset \), let \( Q_A, p_A, q_A \) be defined as with \( X_G, H \)-connectors. Also, let the bridges be grouped as before.

If there is a non-trivial bridge group \( A \) of \( T \) with all its vertices of attachment in \( V(X_G) \cap V(T) \), then let \( \mu := p_{A \cup A} \) and \( M := A \cup Q_A + \mu \), with \( \mu \) embedded in the infinite face such that \( V(Q_A) \subset V(X_M) \). If \( \alpha \in E(Q_A) \), then let \( \beta := \alpha \), else let \( \beta \) be any edge of \( X_M \) distinct from \( \mu \). Let \( C \) be a Tutte path in \( M \) from \( p_A \) to \( q_A \) through \( \beta \) by Lemma 1. Modify \( T \) by replacing \( Q_A \) by \( C \) in \( T \).

If there is a bridge group \( A_m \) of \( T \) remaining with \( \alpha \in E(Q_{A_m}) \), then let \( \nu := v_{A_m}, p_{A_m}, \xi := v_{A_m} q_{A_m}, N := A_m \cup Q_{A_m} + \nu + \xi \), with \( \xi \) embedded in the infinite face such that \( V(Q_{A_m}) \subset V(X_N) \). (Notice that \( A_m = K_n \).) Let \( C \) be the Tutte circuit through \( \alpha, \nu, \xi \) in \( N \) from the Three Edge Lemma. Modify \( T \) by replacing \( Q_{A_m} \) by \( C - v_{A_m} \) in \( T \).

Note that every bridge group of \( T \) has at most two vertices of attachment not in \( V(X_G) \). In each case, a portion of \( T \) which is a subgraph of \( X_G \) will be replaced by a path through the corresponding bridge group.

Let \( R_2 \) be a bridge group of \( T \) with two vertices of attachment not in \( V(X_G) \). Let \( u_1 \) (\( u_2 \)) be the most counterclockwise (clockwise) vertices of attachment of \( R_2 \) in \( X_G \). Let \( U \) be the path in \( X_G \) from \( u_1 \) clockwise to \( u_2 \). Notice \( U \) is a subgraph of \( T \). Let \( v_1 \) and \( v_2 \) be the vertices of attachment of \( R_2 \) not in \( X_G \) such that \( u_1 \) and \( v_1 \) are in the boundary of some face \( F \) of \( G \). Let \( \pi_1 := u_1 v_1, \rho := v_1 v_2, \) and \( H_2 := R_2 \cup U + \pi_1 + \pi_2 + \rho \), with \( \pi_1, \pi_2, \rho \) embedded in the infinite face such that \( V(U) \subset V(X_H) \). Note \( H_2 \) is 2-connected and plane; thus there is a Tutte circuit \( C_2 \) through \( \pi_1, \pi_2, \rho \) by the Three Edge Lemma. Let \( P_2 \) be the path from \( u_1 \) to \( u_2 \) in \( C_2 \) not through \( v_1 \). Now \( T \) is modified by replacing \( U \) with \( P_2 \). Repeat this process for all such \( R_2 \).

Let \( R_1 \) be a bridge group of \( T \) with one vertex of attachment not in \( V(X_G) \). Let \( u_1 \) (\( u_2 \)) be the most counterclockwise (clockwise) vertices of attachment of \( R_1 \) in \( X_G \). Let \( U \) be the path in \( X_G \) from \( u_1 \) clockwise to \( u_2 \). Notice \( U \) is a subgraph of \( T \). Let \( t \) be the vertex of attachment of \( R_1 \) not in \( X_G \). Let \( \sigma_1 := u_1 t, \) and \( H_1 := R_1 \cup U + \sigma_1 + \sigma_2 \), with \( \sigma_1, \sigma_2 \) embedded in the infinite face such that \( V(U) \subset V(X_H) \). Note \( H_1 \) is 2-connected and plane; thus there is a Tutte path \( P_1 \) from \( u_1 \) to \( t \) through \( \sigma_2 \) by Lemma 1. Now \( T \) is modified by replacing \( U \) with \( u_1 P_1 u_2 \). Repeat this process for all such \( R_1 \).

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**References**


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