FINAL ASSESSMENT (90 minutes)

Exam Instructions:

1. All course documents available on moodle are authorized. The use of internet is strictly **forbidden**.
2. At the end of the exam, you must provide one program per exercise. Your programs must be **executable** with the command `python3 filename.py` and **display their results**.

I. Photon statistics

In a cavity at thermal equilibrium at temperature T, the average number of photons in a state of angular frequency ω is given by the Bose-Einstein distribution and reads

$$\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}, \tag{1}$$

with $\beta = 1/(k_{\mathrm{B}}T)$. We want to compute this average using a Monte Carlo method. We denote by n the instantaneous number of photons of angular frequency ω in the cavity. The energy of the system is then $E_n = n\hbar\omega$, and the probability to have n photons in the cavity follows the Maxwell-Boltzmann distribution $p_n = e^{-\beta E_n}/Z$, with Z a normalization constant. We propose the following Markov chain for the evolution of n, starting from $n = 0$.

1. At each step, decide at random to perform a trial move to increase or decrease n by 1.
2. If the trial move leads to a negative value of n, reject the move.
3. Otherwise, accept or reject the move following detailed balance according to the Metropolis rule.

We work with natural units such that $\hbar = k_{\mathrm{B}} = 1$.

Question 1: Define a function `occupation_theory(w,T)` which takes as input the angular frequency ω and the temperature T and returns the theoretical value of $\langle n \rangle$ given by Eq. (1).

Question 2: Define a function `occupation(w,T)` which takes as input the angular frequency ω and the temperature T and returns the value of $\langle n \rangle$ computed from $N = 100\,000$ iterations of the Markov chain described above.

Question 3: Compute the occupation number $\langle n \rangle$ for 20 equally spaced values of ω between 0.2 and 3 (in natural units) and for three different temperatures $T = 0.5,\ 1$, and 2 (in natural units) with the two functions defined above.

Question 4: Plot all results on a single graph with the angular frequency on the x-axis and the average occupation number on the y-axis. You must use one color for each temperature (for instance blue, green, and red), the exact results must be represented with full lines, and the results from Monte Carlo simulations with symbols only. The graph must have a title, a legend, and labels on the axes.