CONTINUOUS ASSESSMENT 2 (90 minutes)

Exam instructions:

1. All course documents available on moodle are authorized. The use of internet is strictly **forbidden**.
2. At the end of the exam, you must provide one program per exercise. Your programs must be **executable** with the command `python3 filename.py` and **display their results**.

I. Period of motion of an anharmonic oscillator

Consider a point mass m moving in a one-dimensional even and convex potential $V(x)$ whose minimum is at $x = 0$.

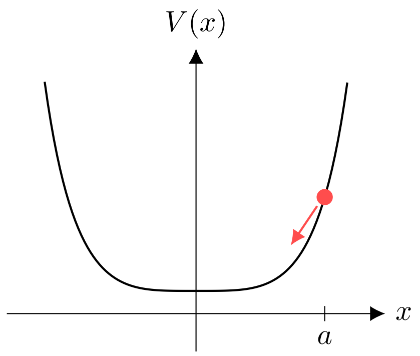

Figure 1: **Sketch of a point mass evolving in a potential $V(x)$.**

The point mass starts from $x = a$ with zero initial velocity. The oscillation period T as a function of the amplitude a is then given by

$$T(a) = \sqrt{8\,m} \int_0^a \frac{1}{\sqrt{V(a) - V(x)}}\,\mathrm{d}x. \tag{1}$$

Question: For $m = 1$ and $V(x) = \cosh(x)$, compute $T(a)$ for 100 equally spaced values of a between $a = 0.1$ and $a = 5$. Use Gaussian quadrature with $N = 50$ nodes, avoiding redundant calculations if possible. Print out your numerical results on the screen, and plot the function $T(a)$.

II. Shape of a soap film

We want to determine the profile of a soap film stretched between two circular rings of radius $R = 2\,\mathrm{cm}$, the two rings being separated by a distance $h = 2.5\,\mathrm{cm}$, see Fig. 2. The soap film forms a surface of revolution called catenoid and parametrized by its radius $\rho(z)$ as a function of the height $z \in [-h/2,\, h/2]$ in cylindrical coordinates. The radius $\rho(z)$ verifies the following ordinary differential equation (ODE) and boundary conditions:

$$\rho(z)\frac{\mathrm{d}^2\rho}{\mathrm{d}z^2}(z) = 1 + \left[\frac{\mathrm{d}\rho}{\mathrm{d}z}(z)\right]^2, \quad \rho(-h/2) = R, \quad \rho(h/2) = R. \tag{2}$$

The objective of this exercise is to solve this problem using a Runge-Kutta 4 method to integrate the above ODE from $z = -h/2$ to $z = h/2$. For that, we define

$$y_0(z) = \rho(z), \quad y_1(z) = \frac{\mathrm{d}\rho}{\mathrm{d}z}(z), \quad \vec{y} = (y_0,\, y_1),$$

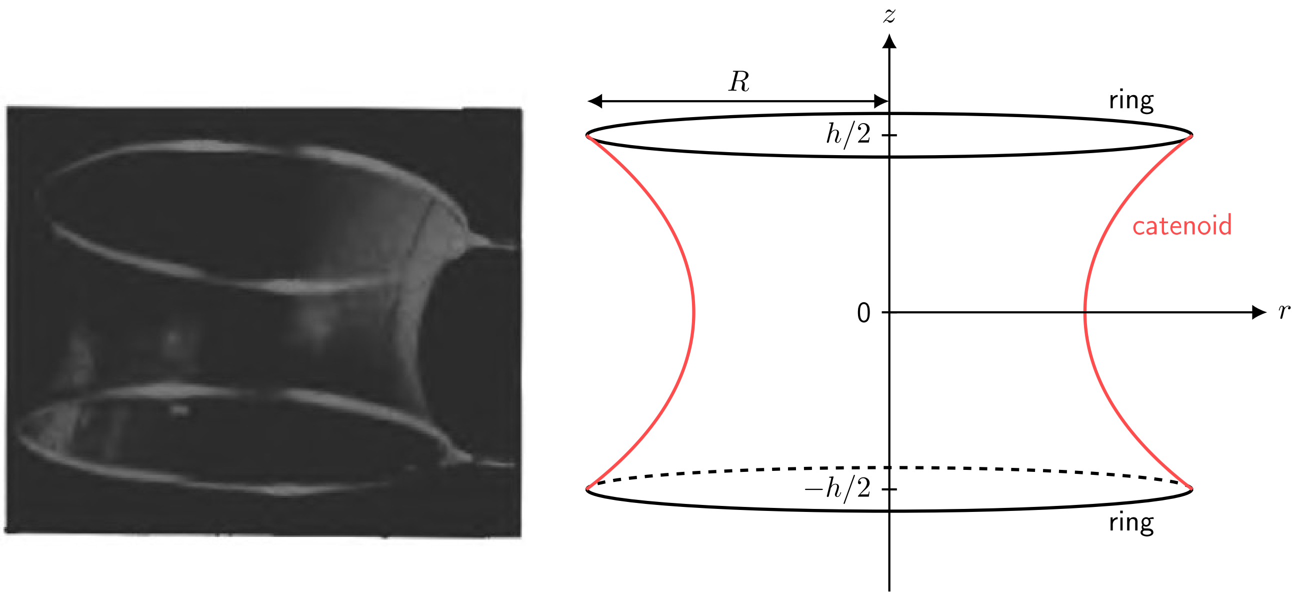

Figure 2: **Shape of a soap film between two circular rings of radius R.** Left: photograph of the soap film from *Capillary and Wetting Phenomena. Drops, Bubbles, Pearls, Waves.* P.-G. de Gennes, F. Brochard-Wyart, D. Quéré (2004). Right: sketch of the soap film with the two supporting rings and its parametrization.

and rewrite the above ODE as

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}z} = f(\vec{y}), \quad f(y_0, y_1) = (y_1,\, (1 + y_1^2)/y_0). \tag{3}$$

The initial condition for y_0 is known: $y_0(-h/2) = R$. However, the initial condition for y_1, namely, $\theta = y_1(-h/2)$, is unknown.

Question 1: All lengths are expressed in centimeters. Define a function `soap_profile(theta)` which takes as input θ, implements the Runge-Kutta 4 method with a step size $\delta = 10^{-2}$ cm, and returns two one-dimensional Numpy arrays `array_z` and `array_rho` containing the values of z and $y_0(z) = \rho(z)$ respectively.

Question 2: The value of θ is such that $y_0(h/2) = R$. Define a function `deviation_end_radius(theta)` which takes as input θ and returns $[y_0(h/2) - R]^2$. You must call the function `soap_profile(theta)`.

Question 3: The value of θ is obtained by finding the location of the minimum of `deviation_end_radius(theta)`. With the commands (on a single line)

```
import scipy.optimize; res = scipy.optimize.minimize(lambda
THETA:deviation_end_radius(THETA[0]), np.array([0.])); theta_opt = res.x[0]
```

one obtains the value θ_{opt} such that `deviation_end_radius(theta)` is minimum. Plot the projection of the shape of the soap film in one plane containing the z-axis, for instance the $(x,\, z)$ plane, considering it is a surface of revolution. The plot should also include the projection of the two supporting rings at $z = \pm h/2$. The graph must have a title, a legend, and labels on axes.