

INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS (2)

Objectives:

- ▶ Integrate a Poisson-like partial differential equation numerically;
- ▶ Implement the Jacobi method;
- ▶ Implement the Gauss-Seidel method;
- ▶ Implement the overrelaxation method;
- ▶ Set up Dirichlet boundary conditions.

No list manipulation is allowed in this tutorial!

I. Introduction

In this tutorial, we determine the electrostatic potential inside a square domain, 1 meter on each side, bounded by four conductors at fixed potential, see Fig. 1. We assume that the space between the conductors is empty.

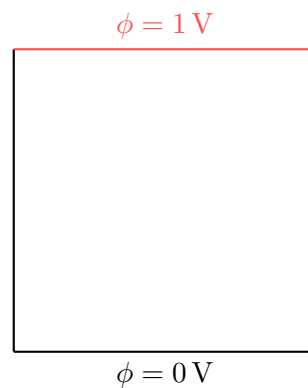


Figure 1: **Electrostatic problem in vacuum to solve.** An empty space is delimited by 4 conductors. Three of them (in black) are at zero potential, the last one (in pink) is at a potential of 1 volt.

The boundary-value problem to solve is thus (with distances expressed in meters, and the potential expressed in volts):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi(0, y) = 0, \quad \phi(1, y) = 0, \quad \phi(x, 0) = 0, \quad \phi(x, 1) = 1. \quad (1)$$

The objective is to compare the speed of resolution and the accuracy of different methods: the Jacobi method, the Gauss-Seidel method, and the overrelaxation method. For all methods, we discretize space as follows: $x_j = j\delta$, $y_k = k\delta$, with δ the discretization step size, $j, k \in \llbracket 0, M \rrbracket$, and $M\delta = 1$.

II. The Jacobi method

Question 1: We recall that all methods described in the lecture notes to integrate Eq. (1) are based on the fact that the solution to the diffusion-like partial differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \quad (2)$$

converges to the solution to Eq. (1) when $t \rightarrow +\infty$.

- Within the FTCS scheme, derive the recurrence relation between ϕ_{jk}^{n+1} and the $\{\phi_{j',k'}^n\}_{j',k' \leq M}$'s for $j, k \in \llbracket 1, M-1 \rrbracket$ and for a discretization timestep h .
- The above scheme is stable as long as $h \leq \delta^2/4$. By taking $h = \delta^2/4$, show that the recurrence relation for $j, k \in \llbracket 1, M-1 \rrbracket$ simplifies to:

$$\phi_{jk}^{n+1} = \phi_{jk}^n + \frac{1}{4} (\phi_{j+1,k}^n + \phi_{j-1,k}^n + \phi_{j,k+1}^n + \phi_{j,k-1}^n). \quad (3)$$

- What are the values of $\phi_{j,0}^n$, $\phi_{j,M}^n$, $\phi_{0,k}^n$ and $\phi_{M,k}^n$ for all $j, k \in \llbracket 0, M \rrbracket$?

Question 2: We first propose to solve Eq. (1) using the Jacobi method. The latter consists in directly applying the recurrence relation given by Eq. (3).

- When should you stop to iterate this recurrence relation? Propose a quantitative stopping criterion based on how much the solution changes between successive iterations..
- Iterate the recurrence relation given by Eq. (3) and plot a heat map of the solution. You can take $\delta = 0.01$ (in meters).
- How many steps are required for the method to converge? How long does it take (in seconds)?

III. The Gauss-Seidel and the overrelaxation methods

Question 3: We now propose to solve Eq. (1) using the Gauss-Seidel method.

- How is the recurrence relation given by Eq. (3) modified?
- Implement the Gauss-Seidel method and plot a heat map of the solution. Use the same value of δ as used for the Jacobi method.
- How many steps are required for the method to converge? How long does it take (in seconds)?

Question 4: We finally propose to solve Eq. (1) using the overrelaxation method.

- How is the recurrence relation given by Eq. (3) modified?
- Implement the overrelaxation method and plot a heat map of the solution. Use the same value of δ as used for the two previous methods. You can take $\omega = 1.8$.
- How many steps are required for the method to converge? How long does it take (in seconds)?

IV. A quantitative comparison between the different methods

Question 5: The exact solution to Eq. (1) is known and is given by:

$$\phi(x, y) = \frac{4}{\pi} \sum_{m=0}^{+\infty} \frac{\sin[(2m+1)\pi x] \sinh[(2m+1)\pi y]}{(2m+1) \sinh[(2m+1)\pi]}. \quad (4)$$

We define the relative error between the numerical solution and the exact solution as

$$e = \frac{\sum_{j,k} |\phi_{jk} - \phi(x_j, y_k)|}{\sum_{j,k} |\phi(x_j, y_k)|}. \quad (5)$$

- For the three methods implemented above, compute e .
- Which solution is a good compromise between computation time and accuracy?

Question 6 (bonus): Explore how the above conclusion is sensitive to the choice of ω in the overrelaxation method.