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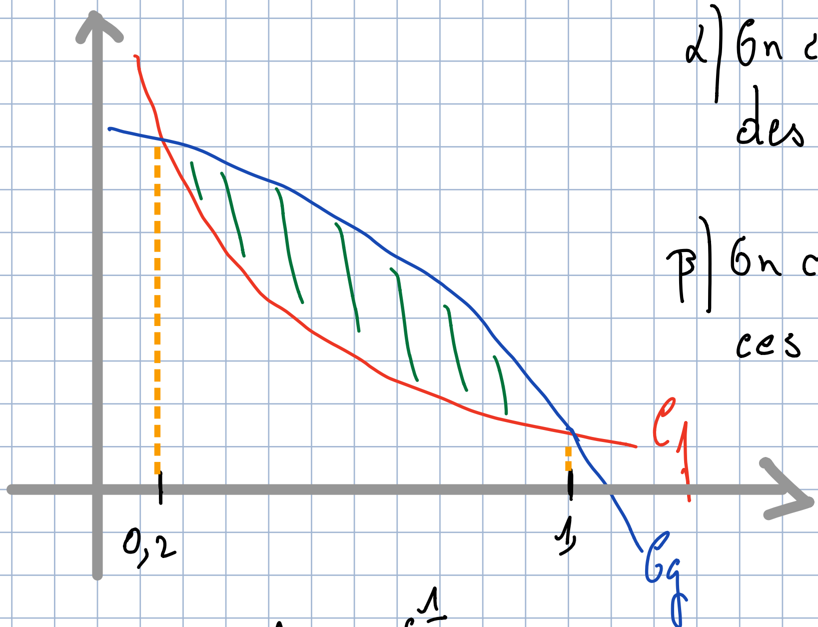
TD 7 : Primitives et Intégrales N°1

Partie 1 :

1) Intégrale comme aire sous la courbe :

On veut calculer l'aire de la région située entre $g(x) = -x^2 + 0,2x + 1$ et $f(x) = \frac{1}{5x}$.



α) On cherche les points d'intersection des deux courbes : ici $x = 0,2$ et $x = 1$

β) On calcule l'intégrale, prise entre ces bornes, de la différence $g(x) - f(x)$

$$A = \int_{0,2 = \frac{2}{10}}^{1} \left((-x^2 + 0,2x + 1) - \frac{1}{5x} \right) dx = \left[-\frac{x^3}{3} + \frac{0,2}{2} x^2 + x - \frac{1}{5} \ln x \right]_{\frac{2}{10}}^{1}$$

$0,1 = \frac{1}{10}$

$$= \left(-\frac{1}{3} + \frac{1}{10} + 1 - 0 \right) - \left(-\frac{1}{3}\left(\frac{2}{10}\right)^3 + \frac{1}{10}\left(\frac{2}{10}\right)^2 + \frac{2}{10} - \frac{1}{5} \ln\left(\frac{2}{10}\right) \right)$$

$$= 0,767 - 0,523 \simeq \boxed{0,24 = A}$$

2. $f(x) = x^2 + 2x - \frac{4}{x^2}$ telle que $F(1) = -1$

$\left(\int u'(x) u^n(x) dx = \frac{1}{n+1} u^{n+1} ; n \neq -1 \in \mathbb{Z} \right)$

avec $u(x) = \frac{1}{x^2} = x^{-2} \Rightarrow \int : -\frac{1}{x}$

↳ $F(x) = \frac{x^3}{3} + x^2 + \frac{4}{x} + C$ et $F(1) = \frac{1}{3} + 1 + 4 + C = -1$

$$\Rightarrow C = -1 - 1 - 4 - \frac{1}{3} = \boxed{-\frac{19}{3} = C}$$

Au final $\boxed{F(x) = \frac{x^3}{3} + x^2 + \frac{4}{x} - \frac{19}{3}}$

3. $g(t) = 3.\cos(3t)$ telle que $G\left(\frac{\pi}{6}\right) = 1$ $\left(\int \cos(at)dt = \frac{1}{a}\sin(at)\right)$

$\hookrightarrow G(t) = \frac{1}{\cancel{3}}.\cancel{3}.\sin(3t) + C$ et $G\left(\frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) + C = 1 \Rightarrow C = 0.$

$\hookrightarrow \frac{\pi}{2}$ et $\sin\left(\frac{\pi}{2}\right) = 1$

Au final $\boxed{G(t) = \sin(3t)}$

Partie 2: calcul des primitives:

1) $f(x) = x^3 - 2x + 1$; $\boxed{F(x) = \frac{x^4}{4} - x^2 + x + C;\ C \in \mathbb{R}}$

2) $f(x) = x + \frac{1}{\sqrt{x}}$; $\frac{1}{\sqrt{x}} = (x)^{-1/2} \xrightarrow{\int} \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} = +2x^{\frac{1}{2}} = +2\sqrt{x}$

$\hookrightarrow \boxed{F(x) = \frac{x^2}{2} + 2\sqrt{x} + C}$

$f(u(x)) = u'(x)\, u^n(x)$

$\hookrightarrow \int f(u(x))dx = \frac{1}{n+1} u^{n+1}$

3) $f(x) = \sin x - 2\cos x$; $\boxed{F(x) = -\cos x - 2\sin(x) + C}$

4) $f(x) = \frac{1}{x^2} - x^2$; $\boxed{F(x) = -\frac{1}{x} - \frac{x^3}{3} + C}$; $C \in \mathbb{R}$

5) $f(x) = 1 - \frac{1}{\cos^2(x)}$; $\boxed{F(x) = x - \tan(x) + C}$

6) $f(x) = \cos\left(\frac{x-\pi}{4}\right)$; $\boxed{F(x) = 4\sin\left(\frac{x-\pi}{4}\right) + C}$ $\left(\int \cos(ax)dx = +\frac{1}{a}\sin(ax)\right)$

7) $f(x) = (x-9)^3$ (du type $u'u^n \xrightarrow{\int} \frac{1}{n+1} u^{n+1}$)

$\hookrightarrow \boxed{F(x) = \frac{1}{4}(x-9)^4 + C}$ $\begin{cases} u = x-9 \\ u' = 1 \\ n = 3 \end{cases}$ $\frac{1}{3+1} u^{3+1} + C$; $C \in \mathbb{R}$

8). $f(x) = \sin(x)\cos^2(x)$ (du type $-u'u^n$ avec $\begin{cases} u(x) = \cos(x) \\ n = 2 \end{cases}$) (3)

↳ $\boxed{F(x) = -\frac{1}{3}\cos^3(x) + C}$

9) $f(x) = x\,(x^2+1)^2$ (du type $\frac{u'u^n}{2}$ avec $u(x) = x^2+1$, $n = 2$)

Si $u(x) = x^2+1$ alors $u'(x) = 2x$ et $f(u(x)) = \frac{u'u^2}{2}$

Donc $F(u(x)) = \frac{1}{2} \cdot \left(\frac{1}{2+1} u^3(x)\right) = \boxed{\frac{1}{6}(x^2+1)^3 + C = F(x)}$; $C \in \mathbb{R}$

10) $f(x) = \frac{1}{x^2}\left(1 + \frac{1}{x}\right)^4$ si $u(x) = 1 + \frac{1}{x}$ alors $u'(x) = -\frac{1}{x^2}$

↳ $f(u(x)) = -u'u^4 \longrightarrow F(u(x)) = -\frac{1}{5}u^5(x)$

$\longrightarrow \boxed{F(x) = -\frac{1}{5}\left(1 + \frac{1}{x}\right)^5 + C \; ;}$ $C \in \mathbb{R}$

11) $f(x) = \frac{1}{\sqrt{x}}(1+\sqrt{x})^2$; $u(x) = \sqrt{x} + 1$; $u'(x) = \frac{1}{2\sqrt{x}}$

$\longrightarrow f(u(x)) = 2u'u^2 \longrightarrow F(u(x)) = \frac{2}{3}u^3$

$\longrightarrow \boxed{F(x) = \frac{2}{3}(1+\sqrt{x})^3 + C \; ; \; C \in \mathbb{R}}$

12) $f(x) = \sin(x) \cdot \cos^{-2}(x) \longrightarrow \boxed{F(x) = (+)\cos^{-1}(x) + C \; ; \; C \in \mathbb{R}}$

car
- $n = -2 \Rightarrow \frac{1}{n+1} = -1$
- $f(u(x)) = -u'(x)u^{-2}(x)$

$\Longrightarrow F(u(x)) = -(-u^{-1}) + C$

(4)

13) $f(x) = x \cdot (x^2+3)^{-3}$; $u(x) = x^2+3$ et $n=-3$

$$F(x) = \frac{1}{2} \cdot \left(-\frac{1}{2}\right)(x^2+3)^{-2} = \boxed{-\frac{1}{4(x^2+3)^2} + C = F(x)}$$

14) $f(x) = x^3 \cdot (x^4+1)^{-3}$; $u(x) = x^4+1$; $n=-3$ $\left(\frac{1}{4} u' u^n ; n=-3\right)$

$\hookrightarrow F(x) = \frac{1}{4} \cdot \left(-\frac{1}{2}\right) \cdot (x^4+1)^{-2} = \boxed{-\frac{1}{8(x^4+1)^2} + C = F(x)}$

15) $f(x) = \cos(x) \cdot (2+\sin(x))^{-\frac{1}{2}}$; $u(x) = 2+\sin(x)$; $n = -\frac{1}{2}$

$\hookrightarrow \boxed{F(x) = 2\sqrt{2+\sin(x)} + C}$; $C \in \mathbb{R}$

Partie 3 : polynômes

1) Divisions Euclidiennes

a) $P(x) = x^3 + x^2 - 16x + 20$ par $(x-2)$.

$$
\begin{array}{rrrr|l}
x^3 & + x^2 & -16x & +20 & x-2 \\
-(x^3 & -2x^2) & & & x^2 + 3x - 10 \\
\hline
 & 3x^2 & -16x & +20 & \\
 & -(3x^2 & -6x) & & \\
\hline
 & & -10x & +20 & \\
 & & -(-10x & +20) & \\
\hline
 & & & 0 &
\end{array}
$$

classé par degré décroissant

Au final : $\dfrac{P(x)}{x-2} = x^2 + 3x - 10$

ou $P(x) = (x-2)\,Q(x)$

Soit $Q(x) = x^2 + 3x - 10$. On cherche les racines x_1 et x_2 telles que $Q(x) = 0$:

$Q(x) = 0 \iff x^2 + 3x - 10 = 0$

$$\Delta = 49 = 7^2 \Longrightarrow \begin{cases} x_1 = \dfrac{-3-7}{2} = -5 \\ x_2 = \dfrac{-3+7}{2} = 2 \end{cases}$$

Alors $Q(x) = (x - x_1)(x - x_2) = (x+5)(x-2)$

et $P(x) = (x-2)\cdot(x^2 + 3x - 10) = \boxed{(x-2)^2 (x+5) = P(x)}$

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b) $P(x) = x^3 + 3x - 2j$ divisé par $x - j$

$j^2 = -1$

$$\begin{array}{r|l} x^3 + 3x - 2j & x - j \\ \cline{2-2} -(x^3 - jx^2) & x^2 + jx + 2 \\ \hline jx^2 + 3x - 2j & \\ -(jx^2 + x) & \\ \hline 2x - 2j & \\ -(2x - 2j) & \\ \hline 0 & \end{array}$$

Au final : $\dfrac{P(x)}{(x-j)} = x^2 + jx + 2$

$P(x) = (x-j)(x^2 + jx + 2)$

$Q(x) = 0 \Rightarrow x^2 + jx + 2 = 0$; $\Delta = -1 - 8 = -9 = -(3)^2 = (3j)^2$

$x_1 = \dfrac{-j + 3j}{2} = j$; $x_2 = \dfrac{-j - 3j}{2} = -2j$

$P(x) = (x-j)(x-j)(x+2j) = \boxed{(x-j)^2(x+2j) = P(x)}$

c) $P(x) = 3x^5 - 6x^4 - 6x^3 + 8x^2 + 4x + 15$ divisé par $Q(x) = x^4 - 3x^3 + x^2 + 4$

$$\begin{array}{r|l} 3x^5 - 6x^4 - 6x^3 + 8x^2 + 4x + 15 & x^4 - 3x^3 + x^2 + 4 \\ \cline{2-2} -(3x^5 - 9x^4 + 3x^3 + 0 + 12x) & 3x + 3 \\ \hline 3x^4 - 9x^3 + 8x^2 - 8x + 15 & \\ -(3x^4 - 9x^3 + 3x^2 + 0 + 12) & \\ \hline 5x^2 - 8x + 3 & \end{array}$$

Au final : $\boxed{\dfrac{P(x)}{Q(x)} = (3x+3) + \dfrac{5x^2-8x+3}{Q(x)}}$

2) Si $P(x) = x^3 + x^2 - 5x + 3$ admet "1" comme racine double alors $P(x) = (x-1)^2(x-a)$; avec a la racine simple.

$P(1) = P'(1) = 0$ et $P''(1) \neq 0$
↳ "1" est racine dble !

On a : $(x-1)^2 = x^2 - 2x + 1$

et :

$$
\begin{array}{rl|l}
 & x^3 + x^2 - 5x + 3 & x^2 - 2x + 1 \\
\cline{3-3}
- & (x^3 - 2x^2 + x) & \boxed{x+3} \\
\hline
 & 3x^2 - 6x + 3 & \\
- & (3x^2 - 6x + 3) & \\
\hline
 & 0 &
\end{array}
$$

$\boxed{x+3} \Rightarrow$ "-3" est la racine simple

Au final : $\boxed{P(x) = (x-1)^2 \cdot (x+3)}$

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3) $P(x)$ admet une racine triple ; alors :

$$P(x) = (x - a)^3 (x - b)$$

↳ racine simple car $Deg(P) = 4$

On cherche d'abord "a" : Si "a" est racine triple alors : $P(a) = P'(a) = P''(a) = 0$ et $P'''(a) \neq 0$

$$P(x) = x^4 - 7x^3 - 12x^2 + 176x - 320$$

$$P'(x) = 4x^3 - 21x^2 - 24x + 176$$

$$P''(x) = 12x^2 - 42x - 24 \quad \text{et} \quad P^{(3)}(x) = 24x - 42$$

↳ On peut trouver les racines de ce polynôme

$$P''(x) = 0 \Rightarrow 12x^2 - 42x - 24 = 0$$

. $\Delta = (54)^2$

donc . $x_1 = -\frac{1}{2}$; $x_2 = 4$

"x_2" racine de $P(x)$ et $P'(x)$?

$P(4) = P'(4) = P''(4) = 0$ et $P^{(3)}(4) \neq 0$

"$-\frac{1}{2}$" racine de $P(x)$ et $P'(x)$?

$P(-\frac{1}{2}) \neq 0 \Rightarrow$ "$-\frac{1}{2}$" n'est pas la racine triple.

$\Rightarrow$ "4" est racine triple de $P(x)$.

On cherche ensuite "b" : $(x-4)^3 = x^3 - 12x^2 + 48x - 64$

$$
\begin{array}{l|l}
\Rightarrow x^4 - 7x^3 - 12x^2 + 176x - 320 & x^3 - 12x^2 + 48x - 64 \\
-\left(x^4 - 12x^3 + 48x^2 - 64x \right) & \\
\hline
\quad 5x^3 - 60x^2 + 240x - 320 & x + 5 \\
\quad -\left(5x^3 - 60x^2 + 240 - 320 \right) & \\
\hline
\qquad\qquad\qquad 0 &
\end{array}
$$

"-5" est la racine simple de $P(x)$.

Au final : $\boxed{P(x) = (x-4)^3 (x+5)}$

Partie 4 : fractions rationnelles

1) Éléments de 1ère espèce

exemple :

$$\frac{1}{\underbrace{(x-1)}\underbrace{(x-2)}} = \frac{P(x)}{Q(x)}$$

1) On compare Deg(P) et Deg(Q) : $\overset{0}{Deg(P)} < \overset{2}{Deg(Q)}$

2) les racines sont simples ou multiples

3) On décompose :

$$\underbrace{\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}}_{1^{ère}\ racine} + \underbrace{\frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \dots \frac{B_n}{(x-b)^n}}_{2^{nde}\ racine}$$

$$\boxed{\frac{1}{(x-1)(x-2)} = \frac{A_1}{(x-1)} + \frac{B_1}{(x-2)}}$$; trouver A_1 et B_1 !

1ère méthode : identification

$$\frac{1}{(x-1)(x-2)} = \frac{A_1(x-2) + B_1(x-1)}{(x-1)(x-2)} = \frac{x(A_1+B_1) - 2A_1 - B_1}{(x-1)(x-2)}$$

- Degré 2 : $0 = 0$
- degré 1 : $0 = A_1 + B_1$
  $\hookrightarrow A_1 = -B_1$
- Degré 0 : $1 = -2A_1 - B_1$ $\Rightarrow$ $\boxed{B_1 = +1 \ ; \ A_1 = -1}$

$$\Rightarrow \boxed{\frac{1}{(x-1)(x-2)} = \frac{-1}{(x-1)} + \frac{1}{(x-2)}}$$

2^{eme} méthode : limites.

$$H(x) = \frac{1}{(x-1)(x-2)} = \frac{A_1}{(x-1)} + \frac{B_1}{(x-2)}$$

On cherche A_1 : $(x-1)H(x)\Big|_{x\to 1} = \frac{1}{(x-2)}\Big|_{x\to 1} = \left(A_1 + B_1 \cdot \frac{(x-1)}{(x-2)}\right)\Big|_{x\to 1}$

$$\Rightarrow -1 = A_1 + 0 \Rightarrow \boxed{A_1 = -1}$$

On cherche B_1 : $(x-2)H(x)\Big|_{x\to 2} = \frac{1}{(x-1)}\Big|_{x\to 2} = \left(A_1 \frac{(x-2)}{(x-1)} + B_1\right)\Big|_{x\to 2}$

$$\Longrightarrow +1 = 0 + B_1 \Rightarrow \boxed{B_1 = 1}$$

$$\Longrightarrow \boxed{\frac{1}{(x-1)(x-2)} = \frac{-1}{(x-1)} + \frac{1}{(x-2)}}$$

a) $f(x) = \dfrac{1}{(x-1)(x+2)}$; • A_1 ? $(x-1)f(x)\Big|_{x\to 1} = \frac{1}{3} = A_1 + 0$

$$\Rightarrow \boxed{A_1 = \frac{1}{3}}$$

• B_1 ? $(x+2)f(x)\Big|_{x\to -2} = \frac{1}{x-1}\Big|_{x\to -2} = \left(A_1 \frac{(x+2)}{(x-1)} + B_1\right)\Big|_{x\to -2} \Rightarrow \boxed{B_1 = -\frac{1}{3}}$

$$\boxed{f(x) = \frac{1}{(x-1)(x+2)} = \frac{1}{3(x-1)} - \frac{1}{3(x+2)}} \Rightarrow \boxed{F(x) = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2|}$$

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b) $f(x) = \dfrac{3x-1}{(x-3)(x+1)} = \dfrac{P}{Q}$; $\text{Deg } P < \text{Deg } Q \Longrightarrow$ Décomp. éléments simples

$\hookrightarrow f(x) = \dfrac{A_1}{(x-3)} + \dfrac{B_1}{(x+1)}$

$$(x-3)\,f(x)\Big|_{\substack{x \to 3\\ \to +0}} = \frac{3x-1}{x+1}\Big|_{\substack{x\to 3\\ \to +\infty}} = \left(A_1 + B_2 \frac{(x-3)}{(x+1)}\right)\Big|_{\substack{x\to 3\\ \to +\infty}}$$

$\Longleftrightarrow 2 = A_1 + 0 \Longrightarrow \boxed{A_1 = 2}$

$\Longleftrightarrow 3 = A_1 + B_1 \Longrightarrow \boxed{B_1 = 1}$

donne que $f(x) = \dfrac{2}{(x-3)} + \dfrac{1}{(x+1)}$ et la primitive

$$\boxed{F(x) = 2\ln|x-3| + \ln|x+1| + C ;\ C \in \mathbb{R}}$$

c) $f(x) = \dfrac{x^5 + 5x^2 + 3x + 4}{(x-1)(x+2)} = \dfrac{P(x)}{Q(x)}$; ⚠ $\text{Deg}(P) > \text{Deg}(Q)$

$\Longrightarrow$ Division Euclidienne

(13)

$$\begin{array}{r|l} x^5 + \qquad\qquad 5x^2 + 3x + 4 & x^2 + x - 2 \\ -(x^5 + x^4 - 2x^3 \qquad\qquad) & \hline x^3 - x^2 + 3x \\ \hline -x^4 + 2x^3 + 5x^2 + 3x + 4 & \\ -(-x^4 - x^3 + 2x^2 \qquad\qquad) & \\ \hline 3x^3 + 3x^2 + 3x + 4 & \\ -(3x^3 + 3x^2 - 6x \qquad) & \\ \hline 9x + 4 & \end{array}$$

$$f(x) = (x^3 - x^2 + 3x) + \frac{9x+4}{x^2+x-2}$$

⟺ A décomposer en éléments simples

$$\frac{9x+4}{x^2+x-2} = \frac{9x+4}{(x-1)(x+2)} = \frac{A_1}{(x-1)} + \frac{B_1}{(x+2)}$$

Avec la méthode de votre choix, on trouve

$$A_1 = \frac{13}{3} \text{ et } B_1 = \frac{14}{3}$$

$$\Rightarrow \quad f(x) = x^3 - x^2 + 3x + \frac{13}{3(x-1)} + \frac{14}{3(x+2)}$$

Don la primitive est : $\boxed{F(x) = \frac{x^4}{4} - \frac{x^3}{3} + \frac{3x^2}{2} + \cdots \cdots + \frac{13}{3}\ln|x-1| + \frac{14}{3}\ln|x+2| + C,\ C \in \mathbb{R}}$

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d) $f(x) = \dfrac{x^2+x+1}{(x-1)(x+1)(x+3)} = \dfrac{P(x)}{Q(x)}$; $Deg(P) = 2 < Deg(Q) = 3$ ↳ ok.

↳ (x^2-1) identité remarquable

Décomposition éléments simples: 3 racines simples "−1"; "+1"; "−3".

$$f(x) = \frac{x^2+x+1}{(x-1)(x+1)(x+3)} = \frac{A_1}{(x-1)} + \frac{B_1}{(x+1)} + \frac{C_1}{(x+3)}$$

$$(x-1)f(x) \Big|_{\to 1} = \frac{x^2+x+1}{(x+1)(x+3)}\Big|_{\to 1} = \left(A_1 + B_1\frac{(x-1)}{x+1} + C_1\frac{(x-1)}{(x+3)}\right)\Big|_{\to 1}$$

$\to +\infty$; $\to +\infty$; $\to +\infty$

. $\lim\limits_{x\to 1} (x-1)f(x) = \boxed{\frac{3}{8} = A_1}$. $\lim\limits_{x\to +\infty} (x-1)f(x) = 1 = A_1 + B_1 + C_1$

$$(x+3)f(x)\Big|_{\to -3} = \frac{x^2+x+1}{(x-1)(x+1)}\Big|_{\to -3} = \left(A_1\frac{(x+3)}{(x-1)} + B_1\frac{(x+3)}{(x+1)} + C_1\right)\Big|_{\to 3}$$

. $\lim\limits_{x\to -3} (x+3)f(x) = \boxed{\frac{+7}{8} = C_1}$ or $B_1 = 1 - A_1 - C_1$

$= \dfrac{8-3-7}{8} = \boxed{\dfrac{-1}{4} = B_1}$

Autre méthode pour B_1:

$$(x+1)f(x) = \frac{x^2+x+1}{(x-1)(x+3)}\Big|_{\to -1} = \left(A\frac{(x+1)}{(x-1)} + B_1 + C_1\frac{(x+1)}{(x+3)}\right)\Big|_{\to -1}$$

$\Rightarrow \boxed{\dfrac{-1}{4} = B_1}$

Au final : $\boxed{f(x) = \frac{3}{8(x-1)} - \frac{1}{4(x+1)} + \frac{7}{8(x+3)}}$

et $\boxed{F(x) = \frac{3}{8}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{7}{8}\ln|x+3| + C \ ; \ C \in \mathbb{R}}$

e) $f(x) = \dfrac{x}{(x+1)^2(x-1)}$; $Deg(P) = 1 < Deg(Q) = 3$

"1" racine simple ; "-1" racine double

$$f(x) = \frac{x}{(x+1)^2(x-1)} = \frac{A_1}{(x+1)} + \frac{A_2}{(x+1)^2} + \frac{B_1}{(x-1)}$$

On cherche en premier la racine simple puis l'ordre le plus élevé de la racine double. Ensuite on trouve une relation entre A_1 ; A_2 et B_1 pour déduire A_1.

- $(x-1)f(x)\Big|_{\to 1} = \dfrac{x}{(x+1)^2}\Big|_{\to 1} = \left(\left(\dfrac{A_1}{(x+1)} + \dfrac{A_2}{(x+1)^2}\right)\cdot(x-1) + B_1\right)\Big|_{\to 1}$

$$\lim_{x \to 1} (x-1)f(x) = \boxed{\frac{1}{4} = B_1}$$

- $(x+1)^2 f(x) = \dfrac{x}{(x-1)} = A_1(x+1) + A_2 + B_1\dfrac{(x+1)^2}{(x-1)}$

$$\lim_{x \to -1} (x+1)^2 f(x) = \boxed{\frac{1}{2} = A_2}$$

- $\lim\limits_{x \to +\infty} (x+1)f(x) = 0 = \left(A_1 + \dfrac{A_2}{(x+1)} + B_1\dfrac{(x+1)}{(x-1)}\right)\Big|_{\to +\infty} \Rightarrow \boxed{A_1 = -B_1 = -\frac{1}{4}}$

Au final : $\boxed{f(x) = \frac{-1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}}$

Donne que : $\boxed{F(x) = \frac{+1}{4} \ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\frac{1}{(x+1)} + C \,;\, C \in \mathbb{R}}$

avec $\int \frac{1}{2}(x+1)^{-2}\,dx = -\frac{1}{2(x+1)} + C.$

5) Elements simple de 2nde espèce :

- On doit toujours verifier que $Deg(P) < Deg(Q)$
- 2nde espèce : quand la racine $\in \mathbb{C}$ alors que l'on travaille dans $\mathbb{R}$.

ex : $f(x) = \frac{4x+2}{(x^2+1)} = \frac{A.x+B}{(x^2+1)}$ → il faudra trouver A et B.

↳ racines : $x_1 = +j$ et $x_2 = -j \in \mathbb{C}$

a) $f(x) = \frac{x^2+1}{(x-2)(x^2+x+1)} = \frac{A_1}{x-2} + \frac{Bx+C}{(x^2+x+1)}$

↳ $\Delta = -3 < 0$

↳ 2nd espèce !

$\bullet\ (x-2)f(x) = \frac{x^2+1}{(x^2+x+1)} = A_1 + \frac{Bx+C}{(x^2+x+1)} \cdot (x-2)$

$\hookrightarrow \lim\limits_{x\to 2}(x-2)f(x) = \boxed{\frac{5}{7} = A_1}$

$\hookrightarrow \lim\limits_{x\to +\infty}(x-2)f(x) = 1 = A_1 + B \Rightarrow \boxed{B = \frac{2}{7}}$

Or $f(0) = -\frac{1}{2} = -\frac{A_1}{2} + C \Rightarrow C = -\frac{1}{2} + \frac{5}{14} = \frac{-2}{14} = \boxed{C = \frac{-1}{7}}$

Au final : $\boxed{f(x) = \frac{5}{7(x-2)} + \frac{2x-1}{7(x^2+x+1)}}$

b) $f(x) = \frac{1}{\underbrace{x^2}_{\text{racine double}}\underbrace{(x^2+1)}_{\Delta<0}}$; $Deg(P) < Deg(Q) \Rightarrow$ OK !

$= \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{Bx+C}{(x^2+1)}$ or $f(x)$ est paire $\Rightarrow f(x) = f(-x)$

$= \frac{-A_1}{x} + \frac{A_2}{x^2} + \frac{-Bx+C}{(x^2+1)} = f(-x)$

$\Rightarrow A_1 = -A_1 \Leftrightarrow A_1 = 0$ et $B = -B \Rightarrow B = 0$

Alors $f(x) = \frac{1}{x^2(x^2+1)} = \frac{A_2}{x^2} + \frac{C}{x^2+1}$

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- $x^2 f(x) = \frac{1}{x^2+1} = A_2 + \frac{Cx^2}{x^2+1}$

$\rightarrow \lim_{x \to 0} x^2 f(x) = 1 = A_2$ $\longrightarrow \lim_{x \to +\infty} x^2 f(x) = 0 = A_2 + C$

$\Rightarrow C = -A_2 = -1$

Au final : $\boxed{f(x) = \frac{1}{x^2} - \frac{1}{1+x^2}} \Rightarrow \boxed{F(x) = -\frac{1}{x} - \arctan(x) + C}$

$C \in \mathbb{R}$

c) $f(x) = \frac{1}{x^3(x^2+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{Bx+C}{x^2+1}$

$= +\frac{A_1}{x} - \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{Bx-C}{x^2+1} = \underbrace{-f(-x)}_{f \text{ impaire.}}$

$\Rightarrow$ - $A_2 = -A_2 \Rightarrow A_2 = 0$

- $C = -C \Rightarrow C = 0$

$f(x) = \frac{1}{x^3(x^2+1)} = \frac{A_1}{x} + \frac{A_3}{x^3} + \frac{Bx}{x^2+1}$

- $x^3 f(x) = \frac{1}{(x^2+1)} = A_1 x^2 + B\frac{x^4}{x^2+1} + A_3$

- $\lim_{x \to 0} x^3 f(x) = \boxed{A_3 = 1}$

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$\cdot\ x f(x) = \dfrac{1}{x^2(x^2+1)} = A_1 + \dfrac{A_3}{x^2} + B\dfrac{x^2}{x^2+1}$

$\cdot \lim\limits_{x \to +\infty} x f(x) = 0 = A_1 + B \Rightarrow \boxed{A_1 = -B}$

$\cdot\ f(1) = \dfrac{1}{2} = A_1 + A_3 + \dfrac{B}{2} \Leftrightarrow -\dfrac{1}{2} = -B + \dfrac{B}{2} = -\dfrac{B}{2}$

$\Rightarrow \boxed{B = 1}$ et $\boxed{A_1 = -1}$

Au final : $\boxed{f(x) = -\dfrac{1}{x} + \dfrac{1}{x^3} + \dfrac{x}{x^2+1}}$

$\Rightarrow \boxed{F(x) = -\ln|x| - \dfrac{1}{2x^2} + \dfrac{1}{2}\ln|x^2+1| + C\ ;\ C \in \mathbb{R}}.$

3) Primitives des fractions rationnelles.

16) $f(x) = \dfrac{x^2-2x}{(x-1)^2}$; $\mathrm{Deg}(P) = \mathrm{Deg}(Q) \Rightarrow$ division Euclidienne.

$$\begin{array}{r|l} x^2 - 2x & x^2 - 2x + 1 \\ \cline{2-2} -(x^2 - 2x + 1) & 1 \\ \cline{1-1} -1 & \end{array}$$

$\Rightarrow f(x) = 1 - \dfrac{1}{(x-1)^2} = 1 - (x-1)^{-2}$

NB: $x^2 - 2x = x^2 - 2x + 1 - 1 = (x-1)^2 - 1$

$\Longrightarrow \dfrac{x^2-2x}{(x-1)^2} = \dfrac{(x-1)^2 - 1}{(x-1)^2}$

$\Rightarrow \boxed{F(x) = x + \dfrac{1}{(x-1)} + C\ ;\ C \in \mathbb{R}}$

17) $f(x) = \dfrac{2x^3 + 13x^2 + 24x + 2}{(x+3)^2}$; $\mathrm{Deg}\,P > \mathrm{Deg}\,Q \Rightarrow$ division Eucl.

$$
\begin{array}{l|l}
2x^3+13x^2+24x+2 & x^2+6x+9 \\
\hline
-(2x^3+12x^2+18x) & 2x+1 \\
\quad x^2+6x+2 & \\
-(x^2+6x+9) & \\
\quad -7 &
\end{array}
$$

$$\Rightarrow \int f(x) = (2x+1) - \frac{7}{(x+3)^2}$$

$$\Rightarrow \boxed{F(x) = x^2 + x + \frac{7}{x+3} + C;\ C \in \mathbb{R}}$$

18) $$\int f(x) = \frac{\overset{u'}{2x}}{\underset{u}{x^2+1}} + \frac{3x}{4} \Rightarrow \boxed{\int f(x) = \ln|x^2+1| + \frac{3x^2}{8} + C;\ C \in \mathbb{R}}$$

19) $$\int f(x) = \frac{1}{x(x+1)} = \frac{A_1}{x} + \frac{B_1}{x+1} = \frac{1}{x} - \frac{1}{x+1}$$

$$\Rightarrow \boxed{F(x) = \ln\left|\frac{x}{x+1}\right| + C;\ C \in \mathbb{R}}$$

20) $$\int f(x) = \frac{1}{x^2(x+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x+1} + \frac{B_2}{(x+1)^2}$$

ni paire ni impaire

$$\cdot\ x^2 \int f(x)\Big|_{x\to 0} = \frac{1}{(x+1)^2}\Big|_{x\to 0} = \left(A_1 x + A_2 + B_1 \frac{x^2}{x+1} + B_2 \frac{x^2}{(x+1)^2}\right)\Big|_{x\to 0}$$

$$\Rightarrow \boxed{1 = A_2}, \quad \frac{1}{4} = A_1 + A_2 + \frac{B_1}{2} + \frac{B_2}{4}$$

$$\cdot\ x \int f(x)\Big|_{x\to \infty} = \frac{1}{x(x+1)^2}\Big|_{x\to \infty} = \left(A_1 + \frac{A_2}{x} + B_1 \frac{x}{x+1} + B_2 \frac{x}{(x+1)^2}\right)\Big|_{x\to \infty}$$

$$\Rightarrow \boxed{0 = A_1 + B_1}$$

$\bullet\ (x+1)^2 f(x) = \dfrac{1}{x^2} = A_1 \dfrac{(x+1)^2}{x} + A_2 \dfrac{(x+1)^2}{x^2} + B_1 (x+1) + B_2$

$$\lim_{x\to -1} (x+1)^2 f(x) = \boxed{1 = B_2}$$

$\bullet\ (x+1)\, f(x) = \dfrac{1}{x^2(x+1)} = A_1 \dfrac{(x+1)}{x} + A_2 \dfrac{(x+1)}{x^2} + B_1 + \dfrac{B}{(x+1)}$

$$\lim_{x\to \infty} (x+1) f(x) = A_1 + 0 + B_1 = 0 \Rightarrow \text{déjà obtenue}$$

il manque tjs une équation.

$\bullet\ f(1) = \dfrac{1}{4} = A_1 + A_2 + \dfrac{B_1}{2} + \dfrac{B_2}{4}$

$\iff 1 = 4A_1 + 4A_2 + 2B_1 + B_2$

$\iff 1 = -4B_1 + 2B_1 + 4 + 1 \Rightarrow -4 = -2B_1 \iff \boxed{B_1 = 2}$

et $\boxed{A_1 = -2}$

Au final : $f(x) = \dfrac{-2}{x} + \dfrac{1}{x^2} + \dfrac{2}{(x+1)} + \dfrac{1}{(x+1)^2}$

$\hookrightarrow \boxed{F(x) = 2 \ln\left|\dfrac{x+1}{x}\right| - \dfrac{1}{x} - \dfrac{1}{(x+1)} + C ;\ C \in \mathbb{R}}$

2)) $f(x) = \dfrac{1}{(x+2)^5}$; $u = x+2 \Rightarrow u' = 1 \Rightarrow f(u(x)) = u'(x)\, u^{-5}$

$\Rightarrow F(u(x)) = -\dfrac{1}{4} u^{-4} \Rightarrow \boxed{F(x) = \dfrac{-1}{4(x+2)^4} + C}$