Licence 2 - EEA
Techniques mathématiques (HAE304X)

UNIVERSITÉ DE MONTPELLIER

Contrôle Continu n° 2

Les documents, comme la calculatrice, sont interdits
Toutes les réponses doivent être ***justifiées*** *et les résultats* ***soulignés***
Répondez dans les cases. Si vous manquez de place, continuez au verso

Le 04/11/2025	**NOM, Prénom :**

Exercice 1 (5 pts) Déterminez les primitives suivantes :

a) $\displaystyle\int x \arctan(x)\, \mathrm{d}x$

IPP $u' = x \quad u = \frac{x^2}{2}$
$v = \arctan x \quad v' = \frac{1}{1+x^2}$

$\int = \left[\frac{x^2}{2} \arctan x\right] - \int \frac{1}{2} \frac{x^2}{1+x^2}\, dx$ or $\frac{x^2}{1+x^2} = \frac{x^2 - 1 + 1}{1+x^2} = 1 - \frac{1}{1+x^2}$

$\int = \left[\frac{x^2}{2} \arctan x\right) - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \text{\_\_\_\_\_\_} - \frac{1}{2}\left[x - \arctan x\right] = \frac{1}{2}\left[(x^2+1)\arctan x - x\right] + C$ 1,5

b) $\displaystyle\int \left(\ln(x)\right)^2 \mathrm{d}x$

IPP $u' = 1 \quad u = x$
$v = \ln^2 x \quad v' = 2\frac{1}{x} \ln x$ $\rightarrow \int = \left[x \ln^2 x\right] - \int 2 \frac{x}{x} \ln x\, dx$ 2

$\int \ln x\, dx = ?$ $\rightarrow$ IPP $u' = 1 \quad u = x$
$v = \ln x \quad v' = \frac{1}{x}$ $\rightarrow \int \ln x = x \ln x - x$

$\Rightarrow \int \ln^2 x\, dx = x \ln^2 x - 2 x \ln x + 2x = x\left[\ln^2 x - 2 \ln x + 2\right] + C$

c) $\displaystyle\int \frac{3x+1}{(x+1)^2}\, \mathrm{d}x = \int \frac{\frac{3}{2}(2x+2) - 2}{(x+1)^2}\, dx = \frac{3}{2}\int \frac{2x+2}{(x+1)^2} - \int \frac{2}{(x+1)^2}\, dx$

$= \frac{3}{2} \int \frac{2x+2}{(x+1)^2}\, dx - \int \frac{2\, dx}{(x+1)^2}$

$= \frac{3}{2} \ln(x+1)^2 + 2(x+1)^{-1}$

$= 3 \ln(x+1) + \frac{2}{x+1} + C$ 1,5

Exercice 2 (6 pts) À l'aide de changements de variable, calculer les intégrales :

a) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{1+e^x}}\,\mathrm{d}x$ Soit $X = e^x \to dX = e^x dx$

Bornes : $x = 0 \to X = 1$

$x = 1 \to X = e$

$$\to \int_1^e \frac{X}{\sqrt{1+X}} \frac{1}{X} dX = \int_1^e \frac{1}{\sqrt{1+X}} dX = \left[2(1+X)^{\frac{1}{2}}\right]_1^e$$

$$= 2\left[(1+e)^{\frac{1}{2}} - 2^{\frac{1}{2}}\right]$$

2

b) $\displaystyle\int_{-1}^1 \sqrt{2-x^2}\,\mathrm{d}x = \int_{-1}^{1} \sqrt{2\left(1-\frac{x^2}{2}\right)}\, dx = I$

$X = \frac{x}{\sqrt{2}}$ $dX = \frac{1}{\sqrt{2}} dx$ Bornes : $x = 1 \to X = \frac{1}{\sqrt{2}}$

$x = -1 \to X = -1/\sqrt{2}$

$$\to I = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \sqrt{2}\,(1-X^2)^{\frac{1}{2}} \sqrt{2}\, dX = 2\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-X^2)\, dX$$

$X = \sin t \to dX = \cos t\, dt$ $X = \frac{1}{\sqrt{2}} \to t = \pi/4$

$X = -1/\sqrt{2} \to t = -\pi/4$ 2

$$\to I = 2\int_{-\pi/4}^{\pi/4} (1-\sin^2 t)^{\frac{1}{2}} \cos t\, dt = 2\int_{-\pi/4}^{\pi/4} \cos^2 t\, dt$$

$$= 2\int \frac{1+\cos 2t}{2} dt = \left[x + \frac{\sin 2t}{2}\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{2} + 1$$

c) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,\mathrm{d}x$ *(changement de variable $x = \tan(t)$)*

$dx = (1+\tan^2 t)\, dt$

Bornes : $x = 1 \to t = \frac{\pi}{4}$

$x = 0 \to t = 0$

$$\int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan^2 t)^2} (1+\tan^2 t)\, dt = \int_0^{\frac{\pi}{4}} \frac{dt}{1+\tan^2 t} \quad \text{or } 1+\tan^2 t = \frac{1}{\cos^2 t}$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 t\, dt = \int_0^{\frac{\pi}{4}} \frac{1+\cos 2t}{2} dt = \frac{1}{2}\left[x + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}(1+0)\right] = \frac{\pi}{8} + \frac{1}{4}$$

2

Exercice 3 (3 pts) Soit la fonction à deux variables $f(x, y) = \cos(x\,y) - x\,\ln(y)$:

a) Calculer ses dérivées partielles premières $\dfrac{\partial f}{\partial x}$ et $\dfrac{\partial f}{\partial y}$

$f(x,y) = \cos(xy) - x \ln y$

$\dfrac{\partial f}{\partial x} = -y\sin(xy) - \ln y$ 0,5

$\dfrac{\partial f}{\partial y} = -x\sin(xy) - \dfrac{x}{y}$ 0,5

b) Calculer ses dérivées partielles secondes $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x\,\partial y}$ et $\dfrac{\partial^2 f}{\partial y^2}$

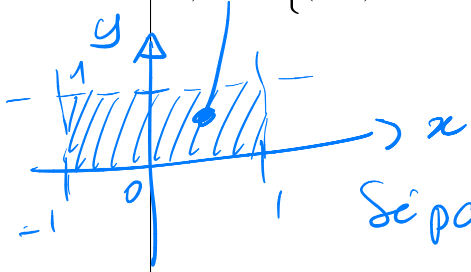
$\dfrac{\partial^2 f}{\partial x^2} = -y^2\cos(xy)$ 0,5

$\dfrac{\partial^2 f}{\partial y^2} = -x^2\cos(xy) + \dfrac{x}{y^2}$ 0,5

$\dfrac{\partial^2 f}{\partial x\partial y} = -\left(\sin(xy) + xy\cos(xy)\right) - \dfrac{1}{y}$ 1

Exercice 4 (6 pts) Représenter les domaines Δ suivants, et calculer les différentes intégrales $\displaystyle\iint_\Delta f(x,y)\,\mathrm{d}x\,\mathrm{d}y$:

a) $\Delta = \Big\{(x,y)\in\mathbb{R}^2,\ x\in[-1,1],\ y\in[0,1]\Big\}$ et $f(x,y) = \dfrac{y^2}{\sqrt{1-x^2}}$



$$\int_{-1}^{1}\int_0^1 \frac{y^2}{\sqrt{1-x^2}}\,dy\,dx$$

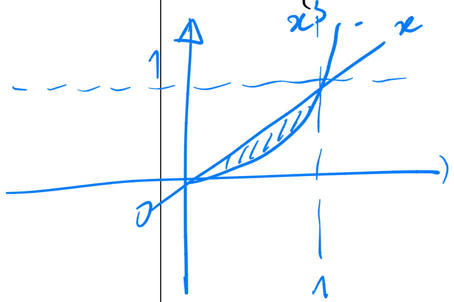
Séparables ! $\rightarrow$ $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx \int_0^1 y^2\,dy$

$$= \Big[\arcsin x\Big]_{-1}^{1}\Big[\frac{y^3}{3}\Big]_0^1$$

$$= \Big(\frac{\pi}{2}+\frac{\pi}{2}\Big)\Big(\frac{1}{3}\Big) = \frac{\pi}{3}$$

1

b) $\Delta = \Big\{(x,y)\in\mathbb{R}^2,\ x\in[0,1],\ x^3\le y\le x\Big\}$ et $f(x,y) = x\,y$



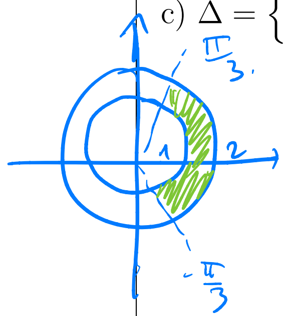
$$\underbrace{\int_0^1}_{x}\underbrace{\int_{x^3}^{x}}_{y} xy\,dy\,dx = \int_0^1 \Big[x\frac{y^2}{2}\Big]_{x^3}^{x}\,dx$$

$$= \int_0^1 \frac{x}{2}\Big[x^2 - x^6\Big]\,dx$$

2

$$= \int_0^1 \frac{1}{2}\big(x^3 - x^7\big)\,dx = \frac{1}{2}\Big[\frac{x^4}{4} - \frac{x^8}{8}\Big]_0^1 = \frac{1}{2}\Big[\frac{1}{4}-\frac{1}{8}\Big] = \frac{1}{16}$$

c) $\Delta = \Big\{(r,\theta)\in\mathbb{R}^+\times[0,2\pi[,\ r\in[1,2],\ \theta\in[-\frac{\pi}{3},\frac{\pi}{3}]\Big\}$ et $f(x,y) = x^2\,y$



en polaires ! $\rightarrow$ $x = r\cos\theta$, $y = r\sin\theta$ $\qquad dx\,dy \rightarrow r\,d\theta\,dr$

$$\underbrace{\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}}_{\theta}\underbrace{\int_1^2}_{r} (r^2\cos^2\theta)(r\sin\theta)\,r\,d\theta\,dr$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2\sin\,d\theta \int_1^2 r^4\,dr$$

2

$$= \Big[\frac{1}{3}\cos^3\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\Big[\frac{r^5}{5}\Big]_1^2$$

$$= \frac{1}{3}\Big(\Big(\frac{1}{2}\Big)^3 - \Big(\frac{1}{2}\Big)^3\Big)\Big(\frac{1}{5}(2^5-1)\Big) = 0$$

(se voit sur la figure !!)