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Notes, Comments, and Letters to the Editor

Risk Aversion in the Theory of Expected Utility with Rank Dependent Probabilities

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Expected utility with rank dependent probability theory is a model of decision-making under risk where the preference relations on the set of probability distributions is represented by the mathematical expectation of a utility function with respect to a transformation of the probability distributions on the set of outcomes. This paper defines, based on Gâteaux differentiability, measures of risk aversion for such preferences which characterize the relation "more risk averse" and applies these measures to the analysis of unconditional and conditional portfolio choice problems. *Journal of Economic Literature* Classification Numbers: 026,521. © 1987 Academic Press, Inc.

1. INTRODUCTION

Expected utility with rank-dependent probabilities theory, henceforth EURDP, is a model of decision making under risk according to which the preference relation on the set of probability distributions is represented by the mathematical expectation of a utility function with respect to a transformation of the probability distributions on the set of outcomes. The transformed measure is non-additive and depends on the rank of the outcome in the induced preference ordering on the set of outcomes. Quiggin [9] was the first to axiomatize and EURDP model. Yaari [12] developed independently an axiomatization of an EURDP model with linear utility. Segal [11] and Chew [3] extended these results. The interest in EURDP theory, however, is not confined to its mathematical foundations, rather it is a result of its ability to explicate experimental evidence such as the Allais paradox, the common ratio effect (Segal [11], Quiggin [10]) and the

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"preference reversal" phenomenon (Karni and Safra [6]). It is interesting, therefore, to deduce further implications of this model which may be tested against implications of alternative theories of choice under risk. In the present paper we have taken a step in this direction by developing the notion of comparative risk aversion in the framework of EURDP theory and studying some of its implications for portfolio selection analysis.

The measurements of risk aversion in expected utility with rank-dependent probabilities theory is complicated by the fact that it involves the properties of both the utility function and the probability transformation function. Yaari [12, 13] discussed the problem of risk aversion assuming a linear utility function. Yaari's analysis theorefore identifies the property of the probability transformation function (i.e., convexity, in his presentation), which renders the preference relation risk-averse in the sense that every mean preserving increase in risk is undesirable. Yaari's result, however, is specific to his model.

In this paper we develop the theory of risk aversion for the general EURDP model by defining the relation "more risk averse" and applying this definition to the analysis of various portfolio selection problems. In particular we show that when the EURDP model is Gateaux differentiable then: (a) A preference relation displays risk aversion in the sense that every mean preserving increase in risk reduces the value of the preference functional if and only if the utility function and the probability transformation function are concave. (b) One preference relation is more risk averse than another if and only if the utility function and the probability transformation function of the former are concave transformations of those of the latter. (c) Other things being equal, a more risk averse decision-maker will invest a larger amount of his wealth in a riskless asset. Unlike in the case of expected utility theory, however, the fact that one decision-maker always invests larger amounts of his wealth in the riskless asset does not imply that he is more risk averse. (d) A strictly risk averse person in our model may be a plunger, i.e., when facing a portfolio choice between investing in a risky and a risk-free asset the decision-maker will never strictly prefer an internal solution. This result is also at variance with expected utility analysis where a strictly risk averse person is necessarily a diversifier. (e) Following Machina [7] we define the conditional demand for a risky asset as the optimal proportion of a portfolio to be placed in the risky asset when there is some probability that, for exogenous reasons, the distribution of wealth will be determined independently of the portfolio composition. According to our analysis a decision-maker is more risk-averse if and only if his conditional demand for risky assets is smaller.

Since EURDP is not necessarily Gateaux differentiable we begin our exposition in Section 2 with some technical discussion of the conditions for EURDP to be differentiable in the sense of Gateaux. Section 3 deals with the definition and measurement of risk aversion, and the conditional demand for risky assets. Section 4 contains a comparative statics analysis of a simple portfolio choice problem. To maintain the fluency of the exposition we relegate all proofs to the appendix.

2. The EURDP Model and Differentiability

We consider the following representation of a preference ordering on D_J, the space of probability distributions on an arbitrary interval $J \subset R$,[1]

$$V(F) = \int_J v(z)\, d(g \circ F)(z), \tag{1}$$

where $F \in D_J$, $v: J \to R$ is continuous and strictly increasing and g: $[0, 1] \to [0, 1]$ is continuous, strictly increasing and onto. To avoid the possibility of a super St. Petersburg paradox of the Menger type v must be bounded on J (see Chew [3, Corollary 2]). We shall maintain this assumption throughout this paper.[2]

Next we derive conditions for V to be differentiable in the weak Gateaux and the Gateaux sense.

Definition 1. For F, $G \in D_J$ V: $D_J \to R$ is *weak Gateaux differentiable* at F in the direction $(G-F)$ if

$$\frac{d}{d\theta} V((1-\theta)F + \theta G)\Big|_{0^+} \text{ exists.} \tag{2}$$

We say that V is weak Gateaux differentiable on D_J if its *weak Gateaux differential* (i.e., expression (2)) exists throughout D_J regardless of its direction.

Definition 2. A weakly Gateaux differentiable V: $D_J \to R$ is *Gateaux differentiable* if there is a U_F: $J \to R$ corresponding to each $F \in D_J$ such that for every $G \in D_J$,

$$\frac{d}{d\theta} V((1-\theta)F + \theta G)\Big|_{0^+} = \int_J U_F d(G-F). \tag{3}$$

[1] We do not assume J to coincide necessarily with R because there will be situations when a proper restriction of J is necessary. This arises for example when we impose global risk aversion (v concave) as well as existence of certainty equivalence (v bounded) so that J must be bounded from below in this case. This observation corresponds to the case with expected utility.

[2] Note that this will always be the case when $J = [A, B]$ is compact.

U_F will be referred to as *Gateaux derivative*, and the expression (3) is the *Gateaux differential* of V.

LEMMA 1. *V in (1) is weakly Gateaux differentiable on D_J if and only if g is Lipschitz continuous on* $[0, 1]$.

COROLLARY 1. *V in (1) is Gateaux differentiable with Gateaux derivative given by $\int_{J^x} g'(F(z))\, dv(z)$, where $J^x = (-\infty, x) \cap J$, if and only if g' exists on* $[0, 1]$.

Note that the boundedness of v is not necessary for Lemma 1 and Corollary 1. Note also that the theory considered here can be extended to probability distributions over non-numerical outcomes, as long as the overall outcome set contains a real interval such that the indifference class of every outcome includes an element in this interval.

A stronger notion of differentiability of V is Fréchet differentiability.

DEFINITION 3. $V\colon D_J \to R$ is *Fréchet differentiable* at F with respect to a metric d on D_J if there is $U_F\colon J \to R$ such that[3]

$$\lim_{d(F,G) \to 0} \frac{|V(G) - V(F) - \int_J U_F d(G-F)|}{d(F, G)} = 0. \tag{4}$$

V is *Fréchet differentiable* if it is Fréchet differentiable at every F.

LEMMA 2. *V in (1) is not Fréchet differentiable with respect to the L^1-metric.*

We denote by Ω the set of all preference relations $\succsim$ on D_J which are representable by a Gateaux differentiable functional V defined in (1).

3. RISK AVERSION AND CONDITIONAL ASSET DEMAND

The notion of risk aversion in EURDP theory is not essentially different from that of expected utility theory. This notion is captured in:

DEFINITION 4. A preference relation $\succsim$ on D_J is said to *display risk aversion (strict risk aversion)* if for all $F, G \in D_J$ $F \succsim G$ ($F \succ G$) whenever G differs from F by a mean preserving increase in risk.

[3] Appropriate metrics for the above definition include those that are norm induced. See Huber [5] for further details regarding Frechet differentiability when the given metric is not necessarily norm induced.

To compare the attitudes toward risk of two preference relations $\succsim$ and $\succsim^*$ on D_J, we define $F \in D_J$ to differ from $G \in D_J$ by a simple compensated spread from the point of view of $\succsim$, if and only if $F \sim G$ and $\exists x^0 \in J$ such that $F(x) \geqslant G(x)$ for all $x < x^0$ and $F(x) \leqslant G(x)$ for all $x \geqslant x^0$.

DEFINITION 5. A preference relation $\succsim^* \in \Omega$ is said to be *more risk averse than* $\succsim \in \Omega$ if $G \succsim^* F$ for every $F, G \in D_J$ such that F differs from G by a simple compensated spread from the point of view of $\succsim$.

Finally, anticipating Theorem 1 we introduce:

DEFINITION 6. A strictly risk-averse individual with preference relation $\succsim \in \Omega$ is said to be *strongly risk averse* if v, the corresponding utility function, is strictly concave.

DEFINITION 7. A risk-averse individual is said to be a *strong diversifier* if for all $H \in D_J$, positive probabilities p, positive constant r, and non-degenerate nonnegative random variables $\tilde{z}$, the individual's preferences over the set of distributions $\{(1-p)H + pF_{(1-\alpha)r+\alpha\tilde{z}} \mid \alpha \in R^1\}$ are strictly quasi-concave in α.

LEMMA 3. *A strongly risk-averse individual is a strong diversifier.*

This result is not trivial since a change in α may change the relative ranking of the outcomes, i.e., the relative ranking of values taken by the r.v. $(1-\alpha)r + \alpha\tilde{z}$ and the values taken by the r.v. corresponding to the distribution H, and the values of the corresponding transformed probabilities.

Theorem 1 provides equivalent characterization of the relation "more risk averse" and a comparative statics result regarding Machina's [7] conditional demand for a risky asset.[4] Below, δ_x denotes the degenerate probability distribution which assigns probability 1 to x in J.

THEOREM 1. *The following conditions on a pair of Gateaux differentiable* EURDP *functionals V and V^* on D_J with respective utility functions v and v^* and probability transformation functions g and g^* are equivalent*:

(i) *For all H, $F \in D_J$ and $p \in (0, 1]$, if c and c^*, respectively, solve $V((1-p)H + pF) = V((1-p)H + p\delta_c)$ and $V^*((1-p)H + pF) = V^*((1-p)H + p\delta_{c^*})$, then $c^* \leqslant c$.*

(ii) *g^* and v^* are concave transformations of g and v, respectively.*

(iii) *If F^* differs from F by a simple compensated spread from the point of view of V then $V^*(F^*) \leqslant V^*(F)$.*

[4] Theorem 1 is analogous to Machina's [7] Theorem 4.

In addition, if both individuals are strongly risk averse then the above conditions are equivalent to:

(iv) *For any* $H \in D_J$, $0 < p \leqslant 1$, $r > 0$, *and nonnegative random variable* $\tilde{z}$ *with* $E(\tilde{z}) > r$, *if* α *and* α^* *represent the most preferred distributions of the form* $(1-p)\,H + pF_{(1-\alpha)r+\alpha\tilde{z}}$ *for* V *and* V^*, *respectively, then* $\alpha^* \leqslant \alpha$.

The properties of the utility function and the probability transformation function that characterize risk aversion are given in

COROLLARY 2. *Let* V^* *represent* $\succsim \in \Omega$ *with a corresponding utility function* v^* *and a probability transformation function* g^*. *Suppose that* v^* *is differentiable, then* V^* *displays risk aversion if and only if both* g^* *and* v^* *are concave.* V^* *displays strict risk aversion if and only if it displays risk aversion and either* v^* *or* g^* *is strictly concave.*

Taking v and g to be linear, then the equivalence of (ii) and (iii) implies the first part of Corollary 2. The second part of Corollary 2 follows almost immediately.

Finally, we note that, since g is independent of x, the definition of decreasing (increasing, constant) risk aversion in this model involves only the properties of v. Therefore a preference relation $\succsim$ is said to display decreasing (increasing, constant) risk aversion if and only if the corresponding utility function v displays decreasing (increasing, constant) risk aversion in the sense of Arrow-Pratt.

4. Portfolio Selection Analysis

Following Arrow's precept that the ultimate justification for any particular measure of risk aversion is its usefulness in theories of specific form of economic behavior under uncertainty we shall apply our measure of risk aversion to some problems involving optimal portfolio composition.

Let there be a risky and a risk-free asset. Assume without essential loss of generality that the rate of return of the riskless asset is zero and that the rate of return of the risky asset, to be denoted $\tilde{x}$, is a random variable with support contained in $J = [-1, \infty)$ and $E\{\tilde{x}\} > 0$. Consider a decision-maker whose preferences are in Ω and whose initial wealth is w. Let α^* denote his optimal investment in the risky asset, i.e., $\alpha^*(w)$ is the solution to

$$\max_{\alpha} \int_J v(w + \alpha x)\, d(g \circ F)(x), \tag{5}$$

where F is the cumulative probability distribution of $\tilde{x}$. The following proposition is an immediate consequence of Arrow [1] and Pratt [8].

THEOREM 2. *$\alpha^*(w) \leqslant \alpha^*(w+a)$ for all w and $a>0$ if and only if v displays decreasing absolute risk aversion in the sense of Arrow–Pratt, i.e., there exists a concave transformation T_a such that $v(w) = T_a \circ v(w+a)$.*

This result involves the comparison of two preference relations in Ω that have the same probability transformation function g. Removing the latter restriction enables the comparison of the portfolio positions of two decision makers with the same initial wealth. Ruling out short sales we assume that $\alpha \geqslant 0$.

THEOREM 3. *Given any random variable $\tilde{x}$ with support in J, let $\alpha^*(\tilde{x})$ and $\bar{\alpha}(\tilde{x})$ denote the optimal portfolio positions of V^* and V, respectively, where V^* and V represent preference relations in Ω. If V^* is more risk averse than V, then $\alpha^*(\tilde{x}) \leqslant \bar{\alpha}(\tilde{x})$.*

Note that this theorem is not directly implied by Theorem 1, since we do not assume here that both individuals are strongly risk averse.

It is interesting to note that Yaari's [12] analysis based on a linear v function of the aforementioned portfolio problem leads to somewhat different results. First, in Yaari's comparative risk aversion analysis a necessary and sufficient condition for one decision-maker to invest larger amounts in the risk-free asset than another is that his probability transformation function be *higher* (lower in his representation). This is a consequence of the constancy of the marginal utility of wealth which implies that the decision hinges on the expected return of the *transformed* random variable. In our analysis this condition is not sufficient. Instead, the sufficient condition on the probability transformation for one decision-maker to invest larger amounts in the risk-free asset is that his probability transformation function is a *concave* transformation of the other. Moreover, the condition in Theorem 3 is not necessary. To verify this, consider two decision-makers with linear utility functions and suppose that the probability transformation function of one is above that of the other. Applying Yaari's result, the individual with the higher probability transformation function always invests more in the risk-free asset although he is not risk averse according to Definition 5.

Second, we note that although decision-makers in Yaari's framework are unconditional plungers, they may be conditional diversifiers. This follows from the proof of Theorem 1. When the intersection of the supports of H and $F_{\tilde{z}}$ is of positive measure, then as α, the proportion of the portfolio invested in the risky asset, tends to 1 there will be a crossing of ranks between the random variables represented by H and $F_{(1-\alpha)r+\alpha\tilde{z}}$. Hence, if g

is strictly concave, Yaari's functional is strictly concave in α at that range and the individual may be a conditional diversifier.

Appendix

Proof of Lemma 1. (i) (Necessity). Since g is continuous and strictly increasing if it is not Lipschitz on [0, 1], then there is $p \in [0, 1]$, where $g'_+(p)$ or $g'_-(p)$ does not exist. Let $F = p\delta_x + (1-p)\delta_y$ $(x < y)$, where $\delta_x \in D_J$ assigns probability 1 to $x \in J$.

$$V((1-\theta)F + \theta\delta_y) = v(y) + [v(x) - v(y)]\, g((1-\theta)p). \qquad \text{(A1)}$$

Observe that (A1) cannot admit both upper and lower derivatives in θ at $\theta = 0$.

(ii) (Sufficiency). If g is Lipschitz on [0, 1], then $g'_+(p)$ and $g'_-(p)$ exist on [0, 1]. Given $F \in D_J$, for any $G \in D_J$, define

$$\tilde{g}'_G(F(x)) = \begin{cases} g'_+(F(x)), & F(x) \leq G(x), \\ g'_-(F(x)), & F(x) > G(x). \end{cases} \qquad \text{(A2)}$$

Let $F_\theta = (1-\theta)F + \theta G$ then,

$$\frac{d}{d\theta} V((1-\theta)F + \theta G)\bigg|_{0+} = \lim_{\theta \downarrow 0} \int_J v d\left\{\frac{1}{\theta}[g \circ F_\theta - g \circ F]\right\}. \qquad \text{(A3)}$$

Since g is Lipschitz, there is $K > 0$ such that

$$\left|\frac{1}{\theta}[g \circ F_\theta(x) - g \circ F(x)]\right| < K\,|G(x) - F(x)| \qquad \text{(A4)}$$

so that $(g \circ F_\theta(x) - g \circ F(x))/\theta$ is of uniformly bounded variation. Noting that it also converges pointwise to $\tilde{g}'_G(F(x))[G(x) - F(x)]$, the above argument implies

$$\frac{d}{d\theta} V(F_\theta)\bigg|_{0+} = \int_J v d(\tilde{g}'_G \circ F)[G - F]. \qquad \text{(A5)}$$

Q.E.D.

Proof of Corollary 1. It is clear that if g is differentiable on [0, 1] then the weak Gateaux differential derived in the proof of Lemma 1 becomes

$$\int_J v d(g' \circ F)(G - F).$$

Integrating by parts twice, this expression may be written as

$$\int_J U_F d(G-F), \tag{A6}$$

where

$$U_F(x)=\int_{J^x} g'(F(z))\, dv(z), \qquad J^x=(-\infty, x]\cap J. \qquad \text{Q.E.D.}$$

Proof of Lemma 2. The proof is by a counterexample. Let $v(x)=x$, $g(p)=2p-p^2$, and assume without loss of generality that $J=[0,1]$. Let $F_n(x)=x^n$ and $F\equiv\delta_1$. Then

$$d(F_n, F)=\int_0^1 x^n\, dx=\frac{1}{n+1},$$

$$V(F_n)=1-\int_0^1 (2x^n-x^{2n})\, dx=1-\frac{2}{n+1}+\frac{1}{2n+1}$$

$$V(F)=1,$$

$$\int_J U_F d(F_n-F)=-\int_0^1 2x^n\, dx=-\frac{2}{n+1}.$$

Substituting in (4) we get

$$\frac{|V(F_n)-V(F)+\int_J U_F d(F_n-F)|}{d(F_n, F)}=\frac{n+1}{2n+1}\to\frac{1}{2}. \qquad \text{Q.E.D.}$$

Proof of Lemma 3. Denote $H_\alpha(x)=(1-p)\, H(x)+pF_{r+\alpha(z-r)}(x)$. Then

$$\frac{d}{d\alpha}V(H_\alpha)=\lim_{\Delta\downarrow 0}\frac{1}{\Delta}\int_J vd(g\circ H_{\alpha+\Delta}-g\circ H_\alpha)$$

$$=-\lim_{\Delta\downarrow 0}\frac{1}{\Delta}\int_J G(x)(H_{\alpha+\Delta}(x)-H_\alpha(x))\, dv(x) \tag{A7}$$

after integrating by parts and defining $G(x)=(g(H_{\alpha+\Delta}(x))-g(H_\alpha(x)))/(H_{\alpha+\Delta}(x)-H_\alpha(x))$ whenever it is possible. After another integration by parts and changes of variables (A7) becomes

$$\lim_{\Delta\downarrow 0}\frac{1}{\Delta}p\int_{-r}^{\infty}\left(\int_{r+\alpha(z-r)}^{r+(\alpha+\Delta)(z-r)} G(y)\, dv(y)\right) dF_z(z) \tag{A.8}$$

$$=p\int_{-r'}^{\infty} g'((1-p)\, H(r+\alpha(z-r))$$

$$+pF_z(z)(z-r)\, v'(r+\alpha(z-r))\, dF_z(z) \qquad \text{a.e.,} \tag{A9}$$

and it is clear that this expression is strictly declining if g is concave and v is strictly concave. Q.E.D.

Proof of Theorem 1 (Outline). It can be shown that condition (ii) is equivalent to

(ii′) For all $F \in D_J$, $U_F^* \circ U_F^{-1}$ is concave.

(For details see Chew *et al.* [4].)

Next we show how Machina's method of proving his [7] Theorem 4 can be applied here with some modifications in view of the fact that EURDP is not Fréchet differentiable and that over a smooth path (in the given metric), $\{F_\alpha \in D_J \mid \alpha \in [0, 1]\}$, the path derivative, which for Fréchet differentiable V takes the form

$$\frac{d}{d\alpha} V(F_\alpha)\bigg|_{\alpha^*} = \frac{d}{d\alpha}\left(\int U_F \, dF_\alpha(x)\right)\bigg|_{\alpha^*}, \tag{A10}$$

does not obtain for merely Gateaux differentiable V.

Nevertheless, we show that for linear enough paths, (A10) will still be true. Assume that the path is given by $F_\alpha = F + \alpha(G-F) + \beta(\alpha)(H-F)$ where F, G, $H \in D_J$, for all x, $(G-F)(x) \cdot (H-F)(x) = 0$, β is increasing and differentiable $\beta(0)=0$ and $\beta(1)=1$ (taking $H=F$ will give a simple linear path). We have

$$\begin{aligned}\lim_{\alpha \downarrow 0} \frac{V(F_\alpha) - V(F)}{\alpha} &= \frac{d}{d\alpha} V(F + \alpha(G-F) + \beta(\alpha)(H-F))\bigg|_{0^+} \\ &= \int_J v \, dg'(F)[(G-F) + \beta'(0)(H-F)] \\ &= \int_J U_F d[(G-F) + \beta'(0)(H-F)] \end{aligned} \tag{A11}$$

as in (A6). Thus

$$V(F_\alpha) - V(F) = \int_J U_F d(F_\alpha - F) + o(\alpha). \tag{A12}$$

With the above it is straightforward to see how the theorem is proved using Machina's proof of his Theorem 4. The implication (i) $\Rightarrow$ (ii′) was proved by Machina using simple linear path. By the aforementioned argument it holds true for our case, and we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (ii′). The implication (ii′) $\Rightarrow$ (iii) was proved there by using a path of the form F_α that was discussed earlier, and we thus have (ii) $\Rightarrow$ (iii). The proof of the implication (iii) $\Rightarrow$ (i) was based on consistency with stochastic dominance and it immediately holds in our case.

For the fourth condition, note that if individuals are strong diversifiers, then they are diversifiers in Machina's sense (he requires quasi concavity which is implied by concavity here). The equivalence (ii) $\Leftrightarrow$ (iv) follows from Machina's proof.

Proof of Theorem 3. Let $\tilde{x}$ be a random variable with support in J. Define

$$W(\alpha)=\frac{1}{v'(w)\,g'(F(0))}\int v(w+\alpha x)\,g'(F(x))\,dF(x)$$

$$\begin{aligned}W^*(\alpha)&=\frac{1}{v^{*\prime}(w)\,g^{*\prime}(F(0))}\int v^*(w+\alpha x)\,g^{*\prime}(F(x))\,dF(x)\\&=\frac{1}{T'(v(w))\,v'(w)\,H'(g(F(0)))\,g'(F(0))}\\&\quad\times\int T(v(w+\alpha x))\,H'(g(F(x)))\,g'(F(x))\,dF(x),\end{aligned}$$

where $v^*=T(v)$, $g^*=H(g)$, and T and H are concave and increasing. Now,

$$\begin{aligned}\frac{d}{d\alpha}\left[W(\alpha)-W^*(\alpha)\right]=&\int\frac{v'(w+\alpha x)\,g'(F(x))}{v'(w)\,g'(F(0))}\\&\times\left\{1-\frac{T'(v(w+\alpha x)\,H'(g(F(x)))}{T'(v(w))\,H'(g(F(0)))}\right\}dF(x).\end{aligned}$$

Since $\alpha\geqslant 0$,

$$x\geqslant 0\Leftrightarrow w+\alpha x\geqslant w\Leftrightarrow v(w+\alpha x)\geqslant v(w)$$

$$x\geqslant 0\Leftrightarrow F(x)\geqslant F(0)\Leftrightarrow g(F(x))\geqslant g(F(0)).$$

Thus

$$x\geqslant 0\Leftrightarrow 1-\frac{T'(v(w+\alpha x))\,H'(g(F(x)))}{T'(v(w))\,H'(g(F(0)))}\geqslant 0,$$

which implies

$$\frac{d}{d\alpha}\left[W(\alpha)-W^*(\alpha)\right]\geqslant 0.$$

Next we show that if W attains a global minimum at $\bar{\alpha}$ then W^* attains a global maximum at some $\alpha^* \leqslant \bar{\alpha}$. In other words, we show that

$$[\forall\alpha,\ W(\bar{\alpha}) \geqslant W(\alpha)] \Rightarrow [\forall\alpha \geqslant \bar{\alpha},\ W^*(\bar{\alpha}) \geqslant W^*(\alpha)].$$

Let $\alpha \geqslant \bar{\alpha}$, then $W(\alpha) - W^*(\alpha) \geqslant W(\bar{\alpha}) - W^*(\bar{\alpha})$, so that $W^*(\bar{\alpha}) \geqslant W^*(\alpha) + W(\bar{\alpha}) - W(\alpha) \geqslant W^*(\alpha)$. Since V and W, (and V^* and W^*), attain global maximum at the same α, (α^*), this completes the proof. Q.E.D.

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