Théorie de la décision

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1 Decreasing risk premium and DARA

We want to prove that the risk premium (i.e. the cost of an additive risk) is decreasing with wealth if and only if absolute risk aversion is decreasing (DARA).

Suppose that the DM has sure wealth w and an additive zero-mean risk $\widetilde{z}$. The risk premium, written as a function of wealth, is implicitly defined by

$$u\left(w-\pi_u\left(w\right)\right)=Eu\left(w+\widetilde{z}\right) \tag{1}$$

Differentiating with respect to wealth w we get

$$\pi_u'\left(w\right)=\frac{u'\left(w-\pi_u\left(w\right)\right)-Eu'\left(w+\widetilde{z}\right)}{u'\left(w-\pi_u\left(w\right)\right)} \tag{2}$$

Therefore, recalling that $u'>0$ by assumption, we have

$$\pi_u'\left(w\right)\leq 0\Leftrightarrow u'\left(w-\pi_u\left(w\right)\right)\leq Eu'\left(w+\widetilde{z}\right) \tag{3}$$

Defining $v=-u'$, we equivalently have

$$\pi_u'\left(w\right)\leq 0\Leftrightarrow v\left(w-\pi_u\left(w\right)\right)\geq Ev\left(w+\widetilde{z}\right) \tag{4}$$

By definition of the risk premium $\pi_v\left(w\right)$, for the utility function v, we have

$$Ev\left(w+\widetilde{z}\right)=v\left(w-\pi_v\left(w\right)\right) \tag{5}$$

Therefore:

$$\pi_u'\left(w\right)\leq 0\Leftrightarrow v\left(w-\pi_u\left(w\right)\right)\geq v\left(w-\pi_v\left(w\right)\right) \tag{6}$$

Recalling that $u''\leq 0$ under risk aversion, we know that $v=-u'$ is increasing because $v'=-u''\geq 0$. Thus we have

$$v\left(w-\pi_u\left(w\right)\right)\geq v\left(w-\pi_v\left(w\right)\right)\Leftrightarrow \pi_u\left(w\right)\leq \pi_v\left(w\right) \tag{7}$$

and finally:

$$\pi_u'\left(w\right)\leq 0\Leftrightarrow \pi_u\left(w\right)\leq \pi_v\left(w\right) \tag{8}$$

Using Proposition 1.5, we know that

$$\begin{aligned}\pi_u\left(w\right) &\leq \pi_v\left(w\right)\Leftrightarrow v \text{ is more risk averse than } u \\ \Leftrightarrow & \quad A_v=\frac{-v''}{v'}=\frac{-u'''}{u''}=P_u\geq\frac{-u''}{u'}=A_u\end{aligned} \tag{9}$$

We conclude that

$$\pi_u'\left(w\right)\leq 0\Leftrightarrow P_u\geq A_u \tag{10}$$

Observe also that

$$A_u'\leq 0\Leftrightarrow DARA \tag{11}$$

where

$$A'_u(w) = \frac{\partial\left(\frac{-u''(w)}{u'(w)}\right)}{\partial w} = \frac{-u'''.u' + u''.u''}{u'.u'} \tag{12}$$

Thus, we have

$$\begin{aligned} DARA \quad &\Leftrightarrow \quad \frac{-u'''.u' + u''.u''}{u'.u'} \leq 0 \\ &\Leftrightarrow \quad \frac{-u'''}{u'} + \frac{u''}{u'}\frac{u''}{u'} \leq 0 \\ &\Leftrightarrow \quad \frac{u''}{u'}\left[\frac{-u'''}{u''} + \frac{u''}{u'}\right] \leq 0 \\ &\Leftrightarrow \quad \frac{-u''}{u'}\left[\frac{u'''}{u''} - \frac{u''}{u'}\right] \leq 0 \\ &\Leftrightarrow \quad A_u\left[-P_u + A_u\right] \leq 0 \end{aligned} \tag{13}$$

Because $A_u \geq 0$ under risk aversion, we have

$$DARA \Leftrightarrow A'_u \leq 0 \Leftrightarrow P_u \geq A_u \tag{14}$$

Finally, we conclude that

$$\pi'_u(w) \leq 0 \Leftrightarrow P_u \geq A_u \Leftrightarrow DARA \tag{15}$$

Q.E.D.