University of Montpellier 2024-2025
M2 IDIL - HAX918X

Partial examination - October 2, 2024
Duration 1 hour - Documents not allowed

Exercise 1 (5 pts) The number x of arrivals in a queue follows a negative binomial distribution with parameter (k, θ) for $x \in \mathbb{N}$

$$f(x|k,\theta) = C_x^{k+x-1}\theta^k(1-\theta)^x$$

with $\theta \in]0,1[$. We assume k is fixed and wish to estimate θ. We consider the Bayesian paradigm. What is the family of conjugate distributions for the parameter θ?

Solution The conjugate prior for θ is the Beta distribution

$$\theta \sim \text{Beta}(\alpha, \beta)\,.$$

Exercise 2 (7 pts) The lifetime x of an electrical component follows a lognormal distribution with parameter $(\theta, 1)$ for $x \in \mathbb{R}_+^*$

$$f(x|\theta,1) = \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{1}{2}[\log(x)-\theta]^2\right)$$

with $\theta \in \mathbb{R}$.
We have a n-sample $x_1, \dots, x_n$ of lifetimes. We consider the Bayesian paradigm and assume that θ is distributed according to a standard Gaussian distribution. Calculate the Bayesian estimator associated with the quadratic loss function.

Solution The lifetime follows a lognormal distribution, and we aim to find the Bayesian estimator for θ under the quadratic loss function.
The likelihood function for a single observation x_i is given by the lognormal distribution:

$$f(x_i|\theta) = \frac{1}{x_i\sqrt{2\pi}}\exp\left(-\frac{1}{2}[\log(x_i)-\theta]^2\right)\,.$$

For an n-sample of lifetimes, $x_1, x_2, \dots, x_n$, the likelihood function is the product of the individual likelihoods

$$L(\theta|x_1,\dots,x_n) = \prod_{i=1}^{n}\frac{1}{x_i\sqrt{2\pi}}\exp\left(-\frac{1}{2}[\log(x_i)-\theta]^2\right)$$

$$L(\theta|x_1,\dots,x_n) \propto \exp\left(-\frac{1}{2}\left[n\theta^2 - 2\theta\sum_{i=1}^{n}\log(x_i)\right]\right)\,.$$

We assume a standard normal prior for θ

$$\theta \sim \mathcal{N}(0,1)\,.$$

We get

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\theta^2}{2}\right) ,$$

and using Bayes' theorem, the posterior distribution is such that

$$\pi(\theta|x_1, \ldots, x_n) \propto L(\theta|x_1, \ldots, x_n)\pi(\theta) ,$$

$$\pi(\theta|x_1, \ldots, x_n) \propto \exp\left(-\frac{1}{2}\left[(n+1)\theta^2 - 2\theta \sum_{i=1}^{n} \log(x_i)\right]\right) .$$

This is a quadratic form in θ, so the posterior distribution is Gaussian

$$\theta|x_1, \ldots, x_n \sim \mathcal{N}\left(\frac{\sum_{i=1}^{n} \log(x_i)}{n+1}, \frac{1}{n+1}\right) .$$

This is the Bayesian estimate of θ under the quadratic loss function is

$$\hat{\theta} = \frac{\sum_{i=1}^{n} \log(x_i)}{n+1} .$$

Exercise 3 (8 pts) The waiting time t between two telephone calls in a call centre follows an exponential distribution with parameter $\lambda > 0$. We have an n-sample $t_1, \ldots, t_n$ of waiting times. We consider the Bayesian paradigm.

1) (4 pts) Give the Jeffreys prior distribution for the parameter λ. Is it a proper prior distribution?

Solution For the exponential distribution, the Fisher information is

$$I(\lambda) = \frac{1}{\lambda^2} .$$

The Jeffreys prior is proportional to the square root of the Fisher information

$$\pi(\lambda) \propto \left(\frac{1}{\lambda}\right) \mathbf{1}_{\lambda>0} .$$

This is an improper prior, as it does not integrate to 1 over the positive real line. However, it can still be used in Bayesian analysis.

2) (4 pts) For the Jeffreys prior, calculate the Bayesian estimator of λ associated with the quadratic loss function.

Solution The likelihood function for the exponential distribution is

$$f(t|\lambda) = \lambda \exp(-\lambda t) .$$

For a sample $t_1, t_2, \ldots, t_n$, the likelihood is

$$L(\lambda|t_1, \ldots, t_n) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} t_i\right) .$$

Multiplying this by the Jeffreys prior gives the posterior distribution

$$\pi(\lambda|t_1, \ldots, t_n) \propto \lambda^{n-1} \exp\left(-\lambda \sum_{i=1}^{n} t_i\right) \mathbf{1}_{\lambda>0} \,.$$

This is a Gamma distribution

$$\lambda|t_1, \ldots, t_n \sim \text{Gamma}\left(n, \sum_{i=1}^{n} t_i\right) \,.$$

This is the Bayesian estimate of λ under the quadratic loss function is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i} \,.$$