Université de Montpellier
M2 Biostat - HAX918X

Année 2023-2024

Examen partiel - 29 septembre 2023
Durée 1h - Documents interdits

Exercise 1 (12 pts) We consider a random phenomenon x distributed according to a Poisson distribution with unknown parameter $\theta \in \Theta$, for all $x \in \mathbb{N}$

$$f(x|\theta) = \exp(-\theta)\frac{\theta^x}{x!}.$$

We observe $x = 8$.

Question 1) We consider the Bayesian paradigm with $\Theta = \{5, 10\}$ and

$$\mathbb{P}^\pi(\theta = 5) = \mathbb{P}^\pi(\theta = 10) = \frac{1}{2}.$$

We consider the loss function $L(\theta, d) = \mathbb{I}_{\theta \neq d}$.

1.1) (3 pts) Give the posterior distribution of θ.

1.2) (3 pts) Knowning that $8\log(2) > 5$, give the Bayesian estimator of θ.

Question 2) We remain in the Bayesian paradigm with $\Theta = \mathbb{R}^+$ and that for all $\theta > 0$

$$\pi(\theta) = \exp(-\theta).$$

We consider the loss function $L(\theta, d) = (\theta - d)^2$.

2.1) (3 pts) Give the posterior distribution of θ.

2.2) (3 pts) Give the Bayesian estimator of θ.

Exercise 2 (4 pts) We consider a random phenomenon x distributed according an exponential distribution with unknown parameter $\theta > 0$, for all $x > 0$

$$f(x|\theta) = \theta\exp(-\theta x).$$

Give the Jeffreys prior distribution on the parameter θ. Is it a proper prior distribution ?

Exercise 3 (4 pts) Lucas and Romain are playing boules and practising their shooting. Romain claims to be able to hit 2 out of 3 balls from 10 metres. Lucas thinks Romain can only hit one ball out of three. **Romain makes 10 attempts and hits five balls.** Model the problem and give the Bayesian answer.

Correction examen
29 septembre 2023
HAX918X

Exercice 1

1.1 $\Theta = \{5, 10\}$

$$f(x|\theta) = e^{-\theta}\frac{\theta^x}{x!} \quad \forall x \in \mathbb{N}$$

on observe $x = 8$

$$\mathbb{P}^\pi(\theta=5) = \mathbb{P}^\pi(\theta=8) = 1/2$$

$$\mathbb{P}^\pi(\theta=5|x=8) \propto (e^{-5}5^8)\left(\frac{1}{2}\right)$$

$$\mathbb{P}^\pi(\theta=10|x=8) \propto (e^{-10}10^8)\left(\frac{1}{2}\right)$$

$$\Rightarrow \mathbb{P}^\pi(\theta=5|x=8) = \left[\frac{e^{-5}5^8}{e^{-5}5^8 + e^{-10}10^8}\right]$$

$$\Leftrightarrow \mathbb{P}^\pi(\theta=5|x=8) = \frac{1}{1+e^{-5}2^8} \# 0{,}37$$

$$\Rightarrow \mathbb{P}^{\pi}(\theta = 10 | x = 8) = \frac{e^{-5} 2^8}{1 + e^{-5} 2^8} \# 0{,}63$$ 2

1.2] We have a symmetric loss function

$$L(\theta, d) = \begin{cases} 1 & \text{if } \theta \neq d \\ 0 & \text{if } \text{n.c.} \end{cases}$$

In that case, the Bayesian estimate of θ is $\hat{\theta} = \arg\max_{z \in \{5, 10\}} \mathbb{P}^{\pi}(\theta = z | x = 8)$

Clearly, $\mathbb{P}^{\pi}(\theta = 10 | x = 8) > \mathbb{P}^{\pi}(\theta = 5 | x = 8)$

$$\Leftrightarrow e^{-10} 10^8 > e^{-5} 5^8$$

$$\Leftrightarrow 2^8 > e^5$$

$$\Leftrightarrow 8 \log(2) > 5 \quad (*)$$

Using the hint in the statement (*) is true, then

$$\hat{\theta} = 10$$

2.1] $\Theta = \mathbb{R}^+$ $\quad \pi(\theta) = e^{-\theta} \mathbb{1}\{\theta > 0\}$

$$\pi(\theta|x) \propto e^{-\theta}\theta^{x} e^{-\theta} \mathbb{1}_{\{\theta > 0\}}$$

$$\Leftrightarrow \pi(\theta|x) \propto e^{-2\theta}\theta^{x} \mathbb{1}_{\{\theta > 0\}}$$

$$\Rightarrow \theta|x \sim \text{Gamma}(x+1, 2)$$

2.2 We consider the quadratic loss function $L(\theta, d) = (\theta - d)^2$

In that case, the Bayesian estimator of θ is the posterior expectation

$$\mathbb{E}^{\pi}(\theta|x) = \left[\frac{x+1}{2}\right]$$

$$\mathbb{E}^{\pi}(\theta|x=8) = 4{,}5$$

<u>Exercise 2</u>

$$f(x|\theta) = \theta e^{-\theta x}\mathbb{1}_{\{x>0\}}$$

$$\theta > 0$$

$\forall x > 0$

$$\log(f(x|\theta)) = \log(\theta) - \theta x$$

$$\frac{d \log f}{d\theta}(x|\theta) = \frac{1}{\theta} - x$$

$$\frac{\partial^2 \log f}{(\partial\theta)^2}(x|\theta) = -\frac{1}{\theta^2}$$

$$\Rightarrow I_x(\theta) = \theta^{-2}$$

$$\Rightarrow \pi^J(\theta) \propto (\theta^{-2})^{1/2}\, \mathbb{1}\{\theta > 0\}$$

$$\Rightarrow \pi^J(\theta) \propto \theta^{-1}\, \mathbb{1}\{\theta > 0\}$$

$\int_0^\infty \theta^{-1}\, d\theta$ is not defim

The Jeffreys prior on θ is improper

<u>Exercise 3</u>

$$x|\theta \sim B(10, \theta)$$

$$\Theta = \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

θ is the probability that Remain hits the ball

$\theta = 1/3$ the assumption of hews

$\theta = 2/3$ the assumption of Remain

We assume $\pi^\pi(\theta = 1/3) = \pi^\pi(\theta = 2/3) = 1/2$

(5)

We observed $x = 5$

$$\mathbb{P}^{\pi}(\theta = 1/3 \mid x = 5) \propto \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^5 C_{10}^5 \left(\frac{1}{2}\right)$$

$$\mathbb{P}^{\pi}(\theta = 2/3 \mid x = 5) \propto \left(\frac{2}{3}\right)^5 \left(1/3\right)^5 C_{10}^5 \left(\frac{1}{2}\right)$$

Then, $\mathbb{P}^{\pi}(\theta = \frac{1}{3} \mid x = 5) = \mathbb{P}^{\pi}(\theta = \frac{2}{3} \mid x = 5) = \frac{1}{2}$

that is very intuitive.

We cannot decide.