HAP708P Modelization and simulation in physics, Faculté des Sciences des Montpellier, 2022 (F. Brümmer)
09/01/2023, duration 90 min. Course documents authorized.

Test 2

Your programs should be **executable** with the command `python3 filename.py` and **display their results**. In particular, it is not acceptable if the results are only accessible through the debugger of your IDE.

Exercise 1. Gauss-Laguerre quadrature

The *Gauss-Laguerre method* can be used to compute integrals of the form

$$I = \int_0^\infty e^{-x}\, f(x)\,\mathrm{d}x \approx \sum_{k=1}^{N} w_k f(x_k)$$

where f is a function which is well approximated by a polynomial. The nodes x_k are the N roots of the Nth *Laguerre polynomial* $L_N(x)$, while the weights are given by $w_k = \frac{x_k}{(N+1)^2 L_{N+1}(x_k)^2}$.

1. Using the recurrence relation

$$L_0(x) = 1\,, \qquad L_1(x) = 1 - x\,, \qquad L_n(x) = \frac{(2n-1-x)L_{n-1}(x) - (n-1)L_{n-2}(x)}{n}$$

   write a Python function `L(n, x)` which returns $L_n(x)$.

2. With the commands
   `import scipy.special;    x, w = scipy.special.roots_laguerre(N)`
   one obtains two one-dimensional NumPy arrays `x` and `w` of length N containing, respectively, the N nodes and weights. Use Gauss-Laguerre quadrature with $N = 10$ to plot the graph of the Gamma function, which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t\,,$$

   between $z = 0.5$ and $z = 5$. Your program should avoid redundant calculations as far as possible.

Exercise 2. Partial differential equations

The *sine-Gordon equation* is the PDE

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = -\sin(\phi)$$

where $\phi = \phi(t, x)$. We would like to solve it on the domain $x \in [-L, L]$ and $t \in [0, T]$, given the following initial and boundary conditions :

$$\phi\Big|_{t=0} = 4\,\arctan\left(e^{\gamma x}\right)\,, \quad \frac{\partial\phi}{\partial t}\Big|_{t=0} = -2\frac{\sqrt{\gamma^2-1}}{\cosh(\gamma x)}\,, \quad \phi\Big|_{x=-L} = 0\,, \quad \phi\Big|_{x=L} = 2\pi\,, \quad \frac{\partial\phi}{\partial t}\Big|_{x=\pm L} = 0\,.$$

Here $\gamma > 1$ is a parameter ; we will take $\gamma = \frac{5}{4}$, $T = 10$ and $L = 20$.

Compute the solution on a lattice of 200 space cells and 2000 time steps. Plot the result as a function of x for $t = 5$ and $t = 10$.

Hints : It may be helpful to start with a pen-and-paper calculation, writing down the discretized second derivative in space, and then rewriting the resulting second-order system of ODEs in time as a first-order system. You can use either the FTCS method (which is stable for this choice of parameters) or the Crank-Nicolson method.