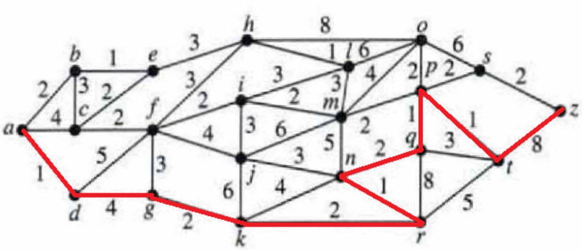
Quelques résultats en optimisation combinatoire robuste

Michael Poss
Marin Bougeret (ALGCo), Francesca Guerriero (Calabria - Italie), Artur Pessoa (UFF - Brésil), Ruslan Sadykov (INRIA Bordeaux)

December 20, 2019

Feasibility: all scenarios matter



Like shortest path but with 2 resources

- Cost
- Time $\leq \ C \Leftrightarrow \sum_{a \in p} u_a \leq C \ \forall u \in \mathcal{U}$

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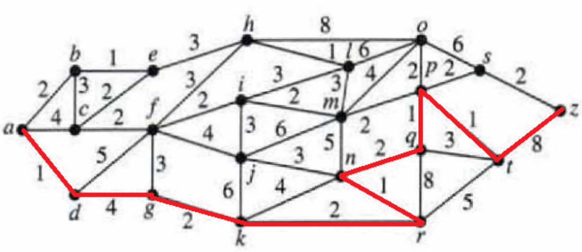

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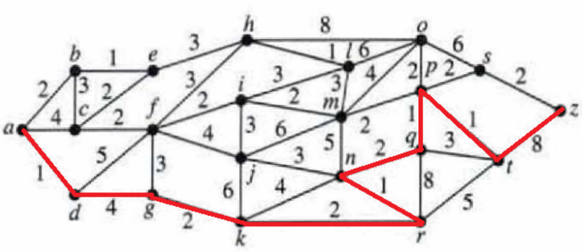

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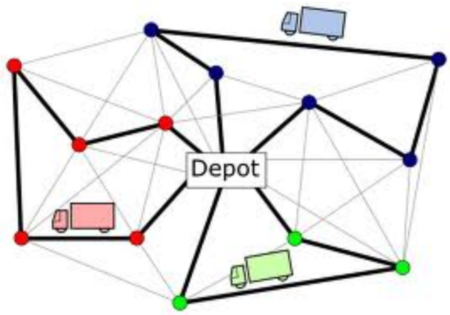
Vehicle routing problem



Different robust counterparts:

- Cost uncertainty
- Demand uncertainty
- Travel time uncertainty

Numerical example with demand uncertainty

- A company needs to be pick up packages of uncertain dimensions
- The company owns 6 vehicles
- Possibility of renting an additional vehicle
- Simulating the failure probability by sampling 10^6 demand values

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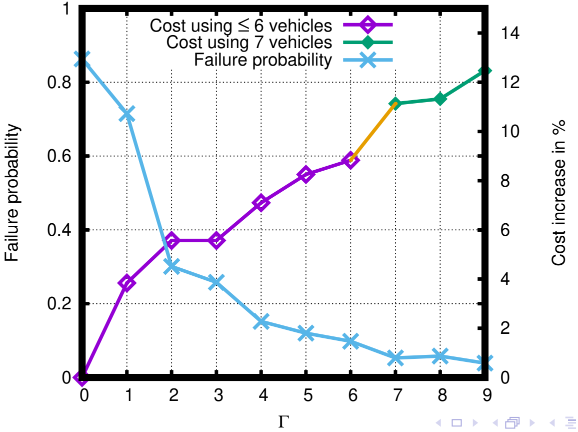
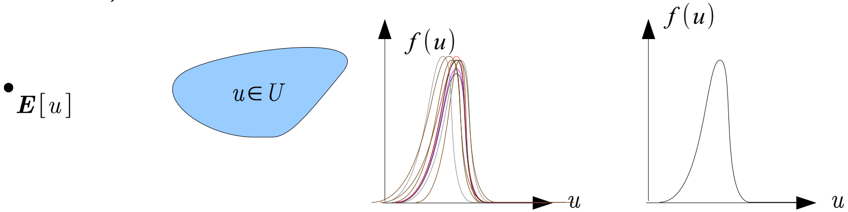


How much do we know ?

Mean value (Deterministic) | *Robust* | *Distributionally robust* | *Stochastic*



static VS adjustable

Static	decisions --→ uncertainty revealed
Complexity	Easy for LP ☺, $\mathcal{NP}$-hard for combinatorial optimization ☹
	MILP reformulation ☺
Two-stages	decisions --→ uncertainty revealed --→ more decisions
Complexity	$\mathcal{NP}$-hard for LP ☹, decomposition algorithms ☺
Multi-stages	decisions --→ uncertainty --→ decisions --→ uncertainty --→ $\cdots$
Complexity	$\mathcal{NP}$-hard for LP ☹, cannot be solved to optimality ☹

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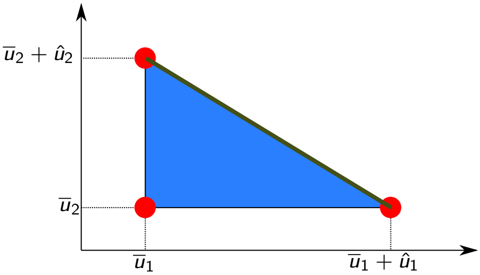
Simpler structure: $\mathcal{U}^\Gamma$-robust combinatorial optimization

- $\mathcal{U} =$ vertices($\mathcal{P}$): good, but need "simpler" $\mathcal{P}$



$$\mathcal{U}^\Gamma = \left\{ \overline{u}_i \leq u_i \leq \overline{u}_i + \hat{u}_i, i = 1, \dots, n, \sum_{i=1}^{n} \frac{u_i - \overline{u}_i}{\hat{u}_i} \leq 1 \right\}$$

Iterative algorithms for $\mathcal{U}^\Gamma$

Theorem (Bertsimas and Sim [2003], Goetzmann et al. [2011], Álvarez-Miranda et al. [2013], Lee and Kwon [2014])

Cost uncertainty $\mathcal{U}^\Gamma$-*CO* $\Rightarrow$ *solving* $\sim n/2$ *problems CO.*

Numerical uncertainty $\mathcal{U}^\Gamma$-*CO* $\Rightarrow$ *solving* $\sim (n/2)^m$ *problems CO.*

Generalizations:

Knapsack uncertainty [Poss, 2017])

$$\left\{\overline{u} \leq u \leq \overline{u} + \hat{u}, \sum_{i=1}^{n} a_{ki} u_i \leq b_k,\ k = 1, \dots, s\right\} \Rightarrow \text{solving } O(n^s) \text{ problems } CO$$

Decision-dependent [Poss, 2013, 2014, Nohadani and Sharma, 2016]

$$\left\{\overline{u} \leq u \leq \overline{u} + \hat{u}, \sum_{i=1}^{n} a_i u_i \leq b(x)\right\} \Rightarrow \text{solving } n \text{ problems } CO$$

Cookbook for static problems

Dualization

good easy to apply

bad breaks combinatorial structure (e.g. shortest path)

Iterative algorithms

good good theoretical bounds

bad solving n^s problems can be too much

Let's make budgeted uncertainty great again: the bin backing

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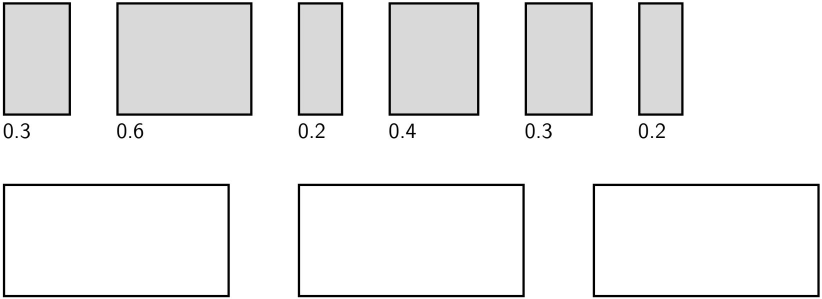
Noam Goldberg *Michael Poss*

LIRMM, University of Montpellier, CNRS, Montpellier, France

Department of Management, Bar-Ilan University, Israel

Bin Packing


0.3
0.6
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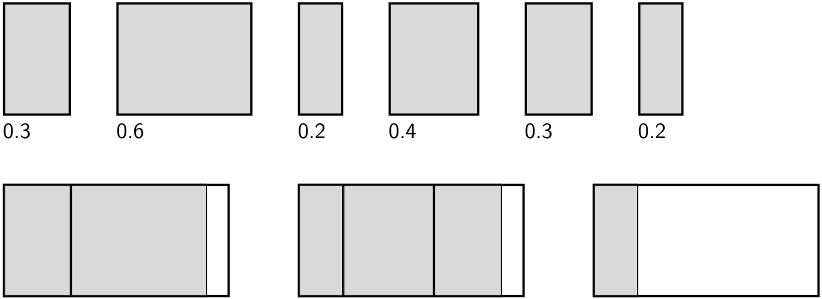
0.4
0.3
0.2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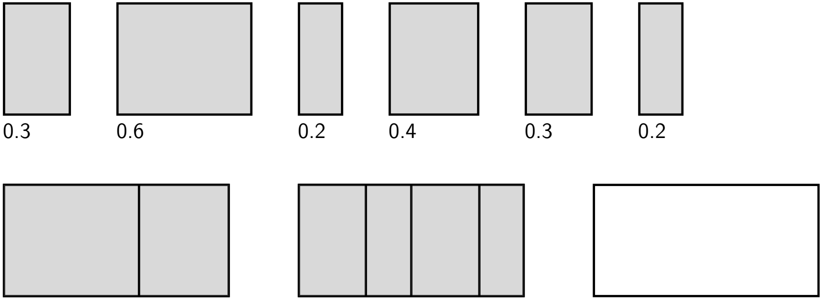
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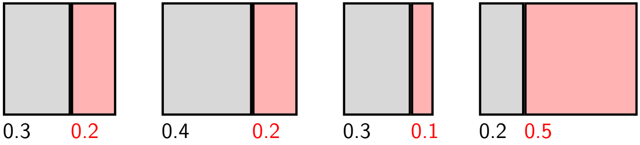
Robust
Bin Packing



ECONOMICS

Anesthesiology
1999; 91:1491–500

Which Algorithm for Scheduling Add-on Elective Cases Maximizes Operating Room Utilization?

Use of Bin Packing Algorithms and Fuzzy Constraints in Operating Room Management

Franklin Dexter, M.D., Ph.D., Alex Macario, M.D., M.B.A.,† Rodney D. Traub, Ph.D.‡*

Background: The algorithm to schedule add-on elective cases that maximizes operating room (OR) suite utilization is unknown. The goal of this study was to use computer simulation to evaluate 10 scheduling algorithms described in the management sciences literature to determine their relative performance at scheduling as many hours of add-on elective cases as

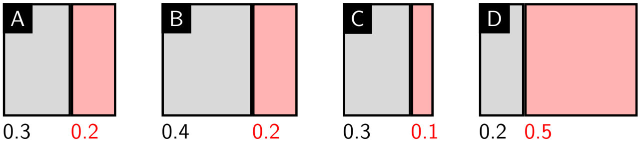
Conclusions: We identified the algorithm for scheduling add-on elective cases that maximizes OR utilization for surgical suites that usually have zero or one add-on elective case in each OR. The ease of implementation of the algorithm, either manually or in an OR information system, needs to be studied. (Key words: operating room economics; staff scheduling; surgical

Γ-uncertainty

Ω-uncertainty

Γ-Uncertainty

At most Γ items can deviate in a bin.



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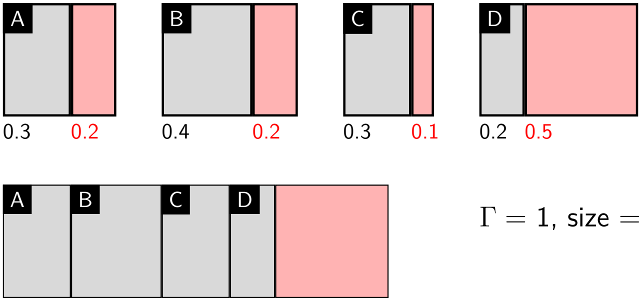

$\Gamma = 1$, size $= 1.7$

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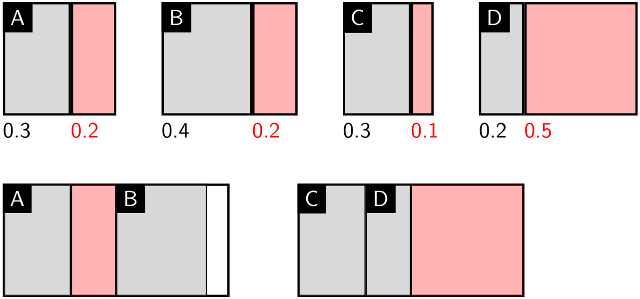

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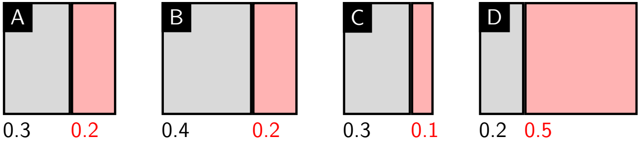
In a bin, the total deviation can be at most Ω.



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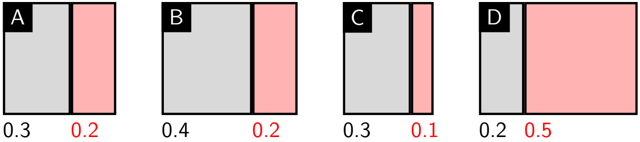

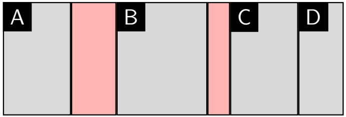

$\Omega = 0.3$, size $= 1.5$

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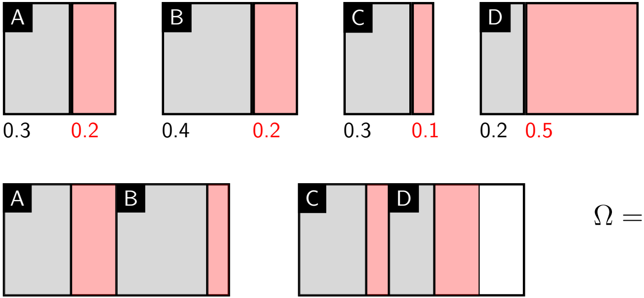


$\Omega = 0.3$, 2 bins

Act II

The story so far

Γ-uncertainty

Knapsack problem

- Bertsimas and Sim (03)

Makespan minimization

- Bougeret, Jansen, P., Rohwedder (19)

One-machine scheduling

- Bougeret, Pessoa, P. (19)
- Tadayon, Cole Smith (15)

Ω-uncertainty

Knapsack problem

- Bertsimas and Sim (03)

One-machine scheduling

- Tadayon, Cole Smith (15)

CVRP

- Floudas, Gounaris, Wiesemann (15)

Act III

Our Contribution

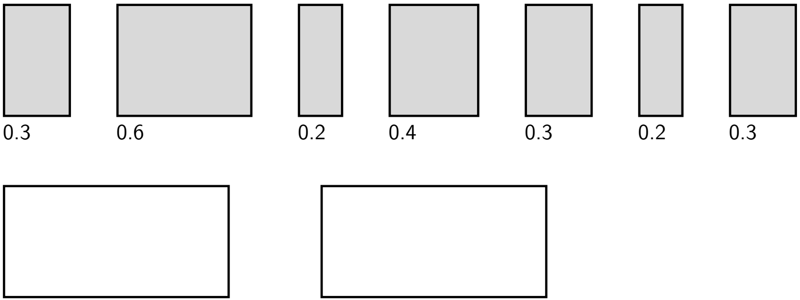
Γ-uncertainty

- 2Γ-APX using Next Fit
- 4.5-APX using DP

Ω-uncertainty

- 2-APX using Next Fit

Next Fit Algorithm



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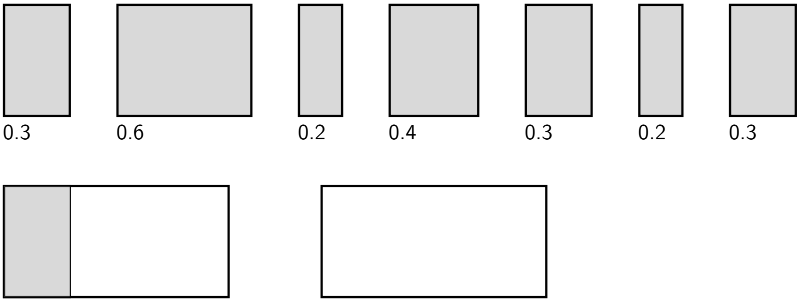

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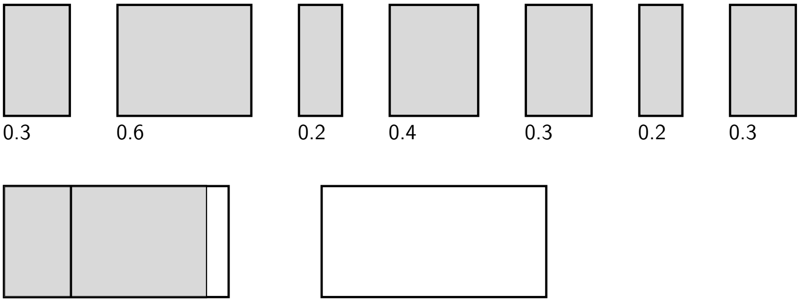

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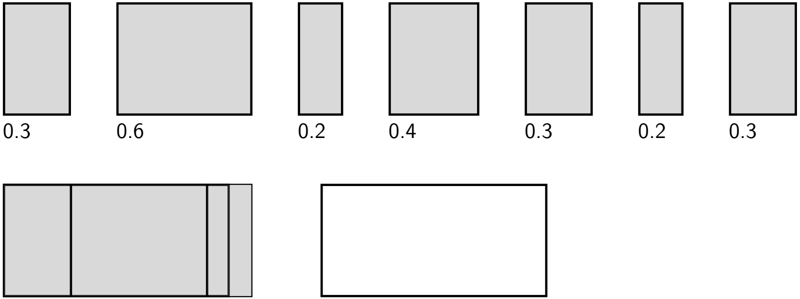

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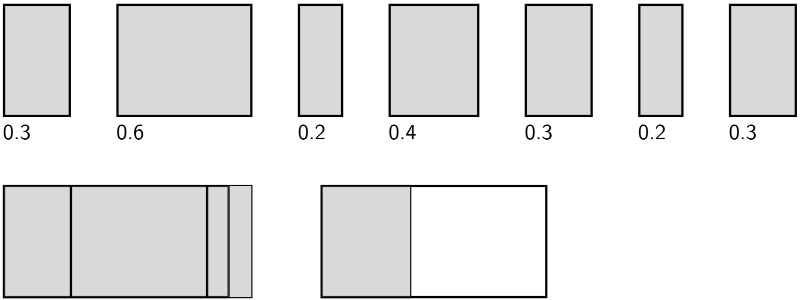

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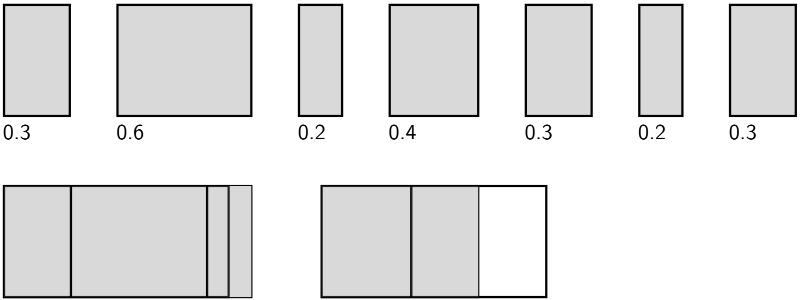

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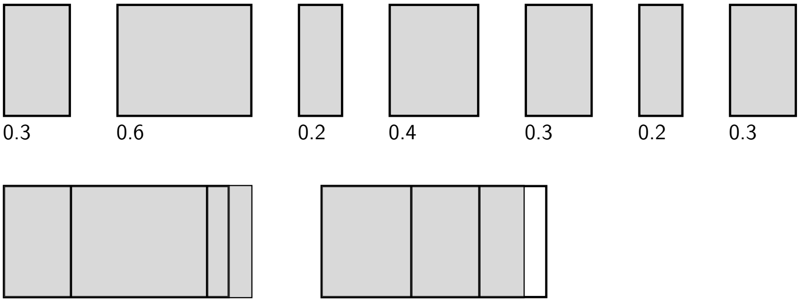

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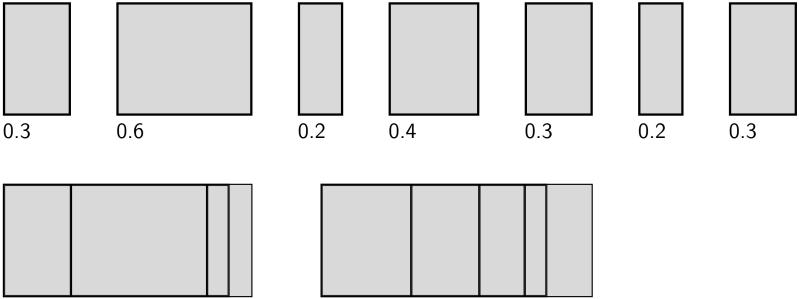

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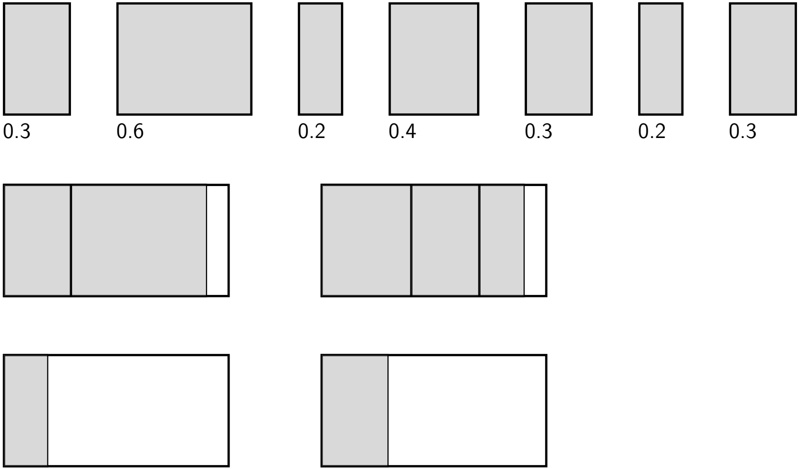
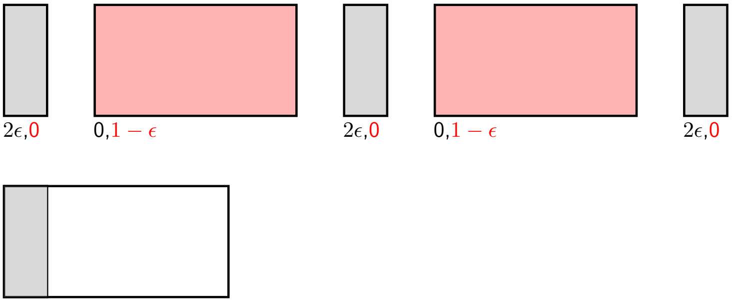


Next Fit Algorithm

n items, $\epsilon = \frac{1}{2n}$

$\Omega = 1 - \epsilon$ or $\Gamma = 1$



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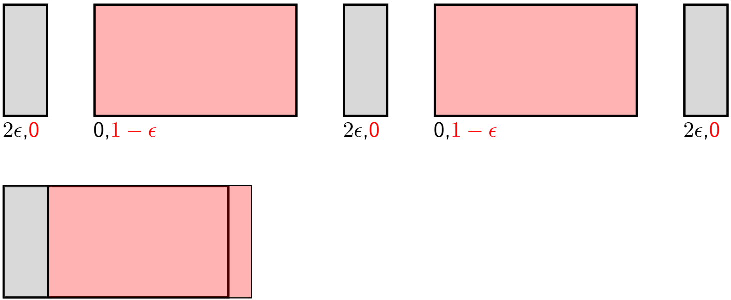

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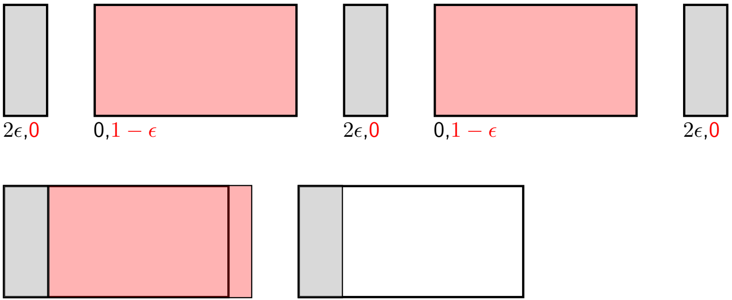

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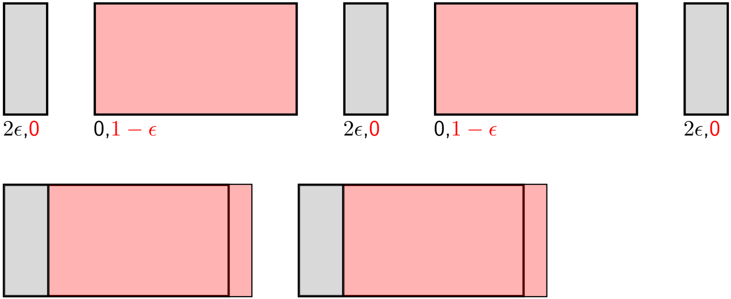

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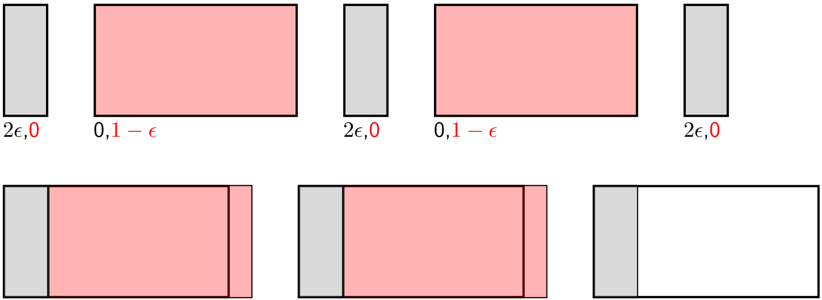

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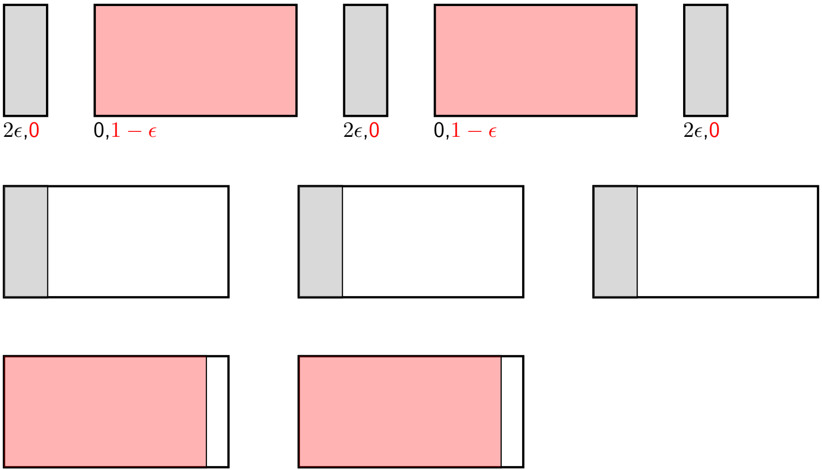

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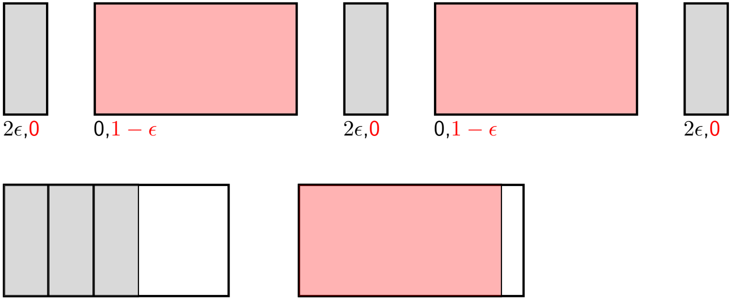

Optimum

Next Fit Algorithm

However, if we order the items as below

$$\frac{\hat{a}_1}{\bar{a}_1} \geq \frac{\hat{a}_2}{\bar{a}_2} \geq \cdots \geq \frac{\hat{a}_n}{\bar{a}_n}$$

then the Next Fit algorithm yields a 2-approximate solution for the Ω-uncertainty model.

Next Fit Algorithm

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Sktech of the proof

- $G^* = \{j \in [k^*] \mid \hat{a}(b_j^*) > \Omega\}$

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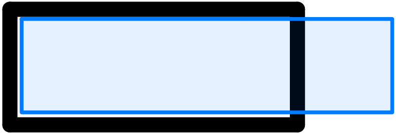

$size(b_l) > 1$

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$$size(b_\ell) = \min(\Omega, \hat{a}(b_\ell)) + \tilde{a}(b_\ell) > 1$$

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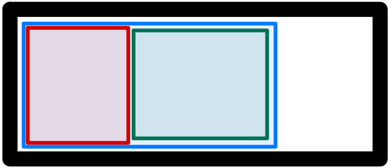

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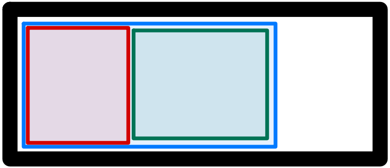

$$\text{size}(b_l) = \min(\Omega, \hat{a}(b_l)) + \bar{a}(b_l) \leq 1$$

Next Fit Algorithm

However, if we order the items as below

$$\frac{\hat{a}_1}{\bar{a}_1} \geq \frac{\hat{a}_2}{\bar{a}_2} \geq \cdots \geq \frac{\hat{a}_n}{\bar{a}_n}$$

then the Next Fit algorithm yields a 2-approximate solution for the Ω-uncertainty model.

Sktech of the proof

- $G^* = \{j \in [k^*] \mid \hat{a}(b_j^*) > \Omega\}$
- G denote the first $|G^*|$ bins opened by Next-Fit
- $\bar{a}(b_l) > 1 - \Omega, \ \forall l \in G$ and $\bar{a}(b_l) \leq 1 - \Omega, \ \forall l \in G^*$

$$\Rightarrow \sum_{j \in G} \bar{a}(b_j) > \sum_{j \in G^*} \bar{a}(b_j^*)$$

Next Fit Algorithm

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Next Fit Algorithm

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- $\bar{G} = [k'] \setminus G$ and $\bar{G}^* = [k^*] \setminus G^*$

Next Fit Algorithm

However, if we order the items as below

$$\frac{\hat{a}_1}{\bar{a}_1} \geq \frac{\hat{a}_2}{\bar{a}_2} \geq \cdots \geq \frac{\hat{a}_n}{\bar{a}_n}$$

then the Next Fit algorithm yields a 2-approximate solution for the Ω-uncertainty model.

Sktech of the proof

- $G^* = \{j \in [k^*] \mid \hat{a}(b_j^*) > \Omega\}$
- G denote the first $|G^*|$ bins opened by Next-Fit
- $\bar{a}(b_l) > 1 - \Omega,\ \forall l \in G$ and $\bar{a}(b_l) \leq 1 - \Omega,\ \forall l \in G^*$

$$\Rightarrow \sum_{j \in G} \bar{a}(b_j) > \sum_{j \in G^*} \bar{a}(b_j^*)$$

$$\Rightarrow \sum_{j \in G} \hat{a}(b_j) > \sum_{j \in G^*} \hat{a}(b_j^*)$$

- $\bar{G} = [k'] \setminus G$ and $\bar{G}^* = [k^*] \setminus G^*$

$$\Rightarrow |\bar{G}| \leq |\bar{G}^*|$$

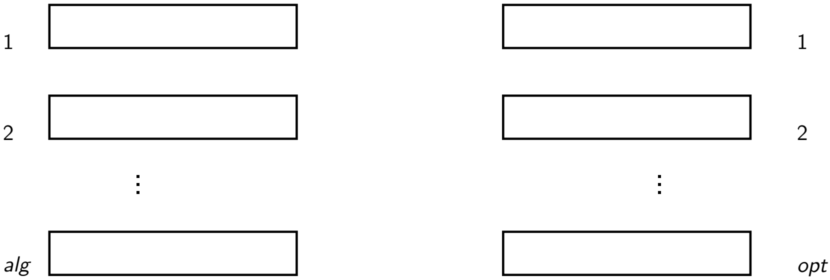
Next Fit Algorithm

Similarly, if we order the items as below

$$\hat{a}_1 \geq \hat{a}_2 \geq \cdots \geq \hat{a}_n$$

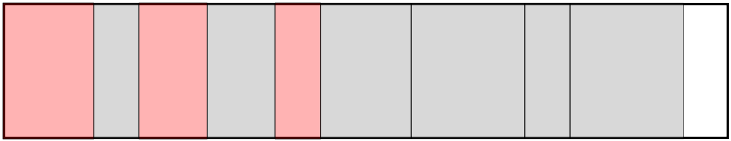
then the Next Fit algorithm yields a 2Γ-approximate solution for the Γ-uncertainty model.

Next Fit (phase 1) Optimum



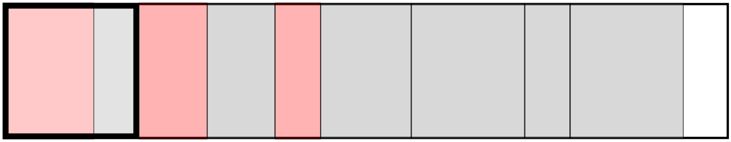
A bin in the optimum solution

$\Gamma = 3$



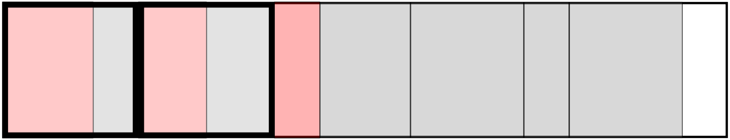
A bin in the optimum solution

$\Gamma = 3$



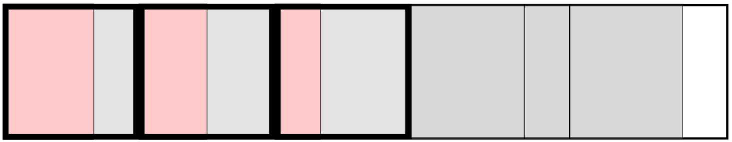
A bin in the optimum solution

$\Gamma = 3$



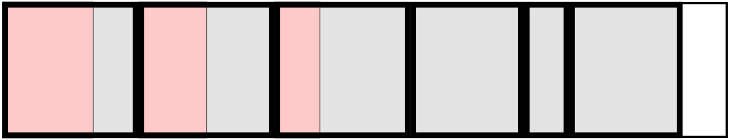
A bin in the optimum solution

$\Gamma = 3$



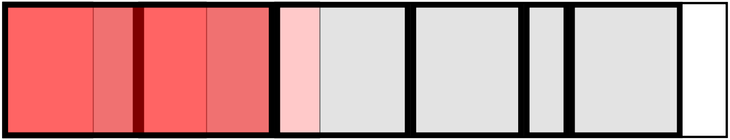
A bin in the optimum solution

$\Gamma = 3$



A bin in the optimum solution

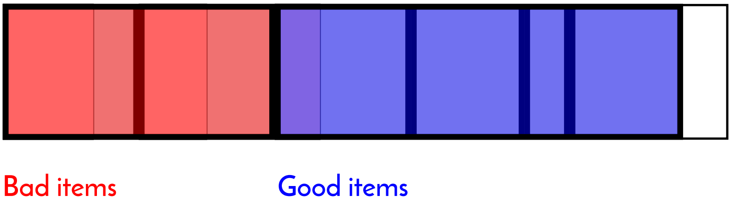
$\Gamma = 3$



Bad items

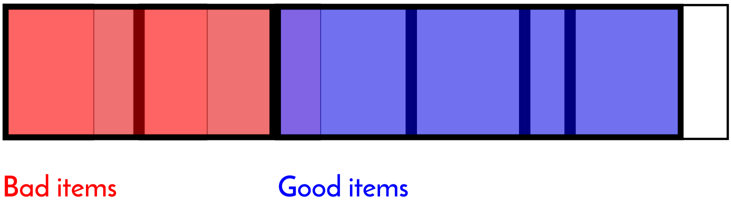
A bin in the optimum solution

$\Gamma = 3$



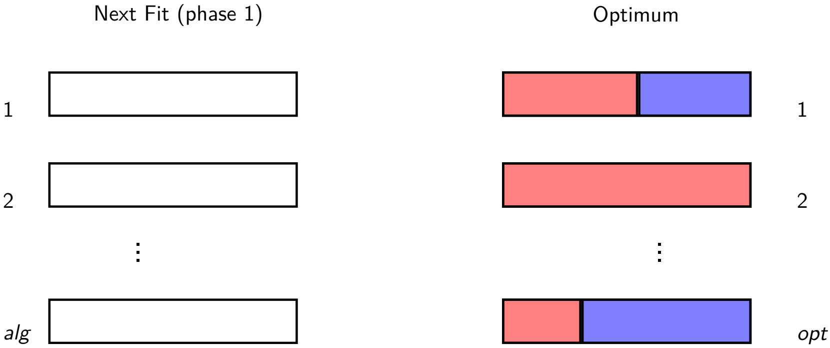
A bin in the optimum solution

$\Gamma = 3$

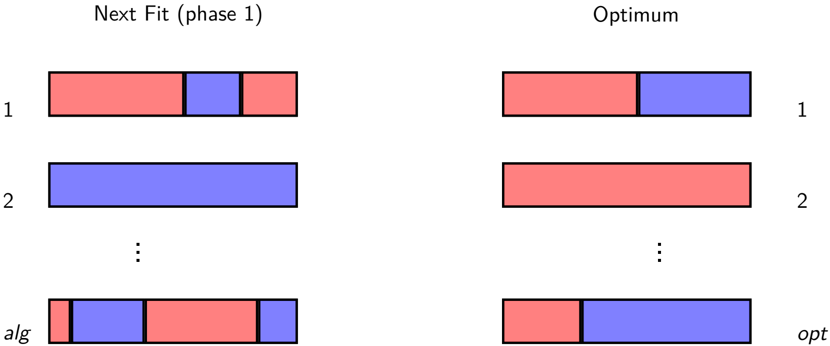


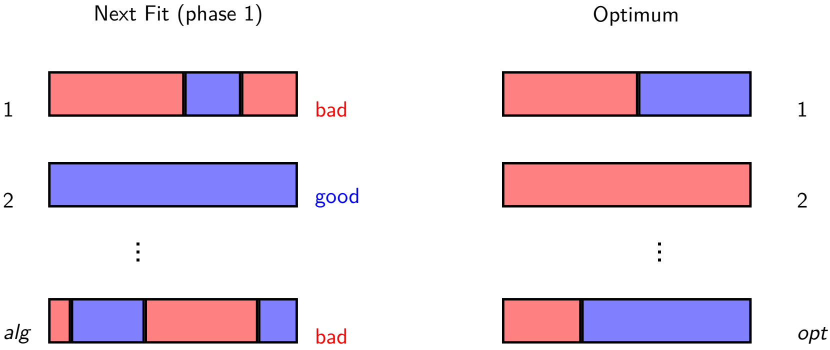
A Next Fit bin is called bad if it contains at least one bad item.

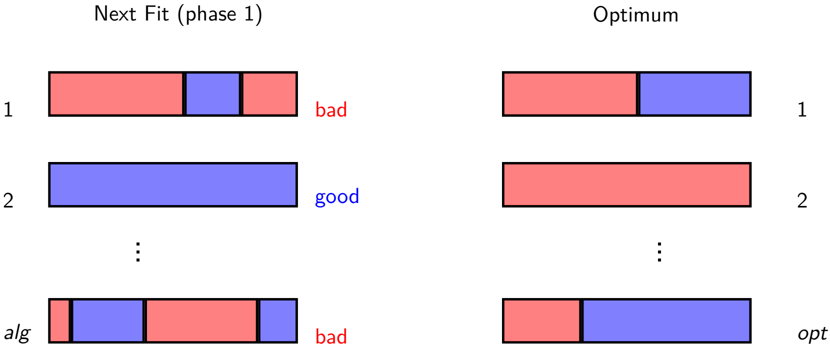
Otherwise it is called good.



Next Fit (phase 1)
Optimum
1
2
alg
1
2
opt


Next Fit (phase 1)
Optimum
1
2
alg
1
2
opt

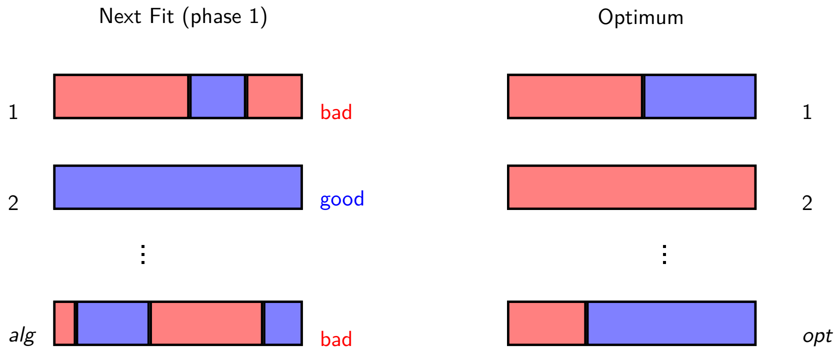

Next Fit (phase 1)
Optimum
1
bad
1
2
good
2
alg
bad
opt



$$alg = \#\text{bad} + \#\text{good}$$

$$\#\text{bad} \leq (\Gamma - 1)opt, \;\; \#\text{good} \leq opt$$

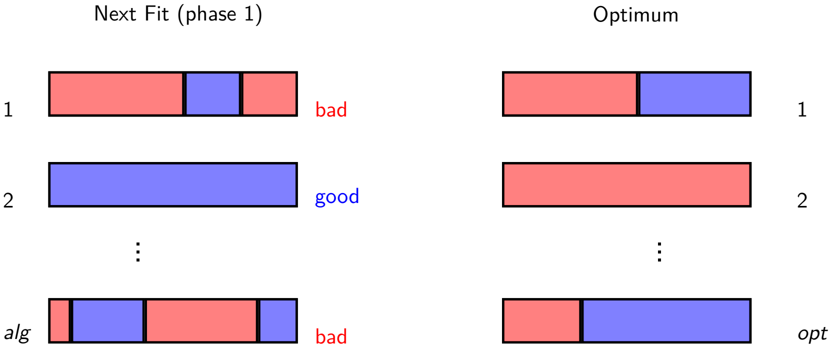
$$\text{alg} \leq \Gamma \cdot opt \Rightarrow \text{Next Fit (phase 2)} \leq 2\Gamma \cdot opt$$



$alg = \#\text{bad} + \#\text{good}$

$\#\text{bad} \le (\Gamma - 1) opt$, $\#\text{good} \le opt$

$\text{alg} \le \Gamma \cdot opt \Rightarrow \text{Next Fit (phase 2)} \le 2\Gamma \cdot opt$



$alg = \#\text{bad} + \#\text{good}$

$\#\text{bad} \leq (\Gamma - 1)opt$, $\#\text{good} \leq opt$

alg $\leq \Gamma \cdot opt \Rightarrow$ Next Fit (phase 2) $\leq 2\Gamma \cdot opt$

Next Fit Algorithm

Lower Bound for Γ-uncertainty

$$
\begin{array}{llll}
(\epsilon,\ 1/\Gamma - \delta_1) & (0,\ 1/\Gamma - \delta_1) & \dots & (0,\ 1/\Gamma - \delta_1) \\
(\epsilon,\ 1/\Gamma - \delta_2) & (0,\ 1/\Gamma - \delta_2) & \dots & (0,\ 1/\Gamma - \delta_2) \\
\vdots & \vdots & \ddots & \vdots \\
(\epsilon,\ 1/\Gamma - \delta_{\Gamma-1}) & (0,\ 1/\Gamma - \delta_{\Gamma-1}) & \dots & (0,\ 1/\Gamma - \delta_{\Gamma-1}) \\
(\epsilon,\ 1/\Gamma - \delta_\Gamma) & (0,\ 1/\Gamma - \delta_\Gamma) & \dots & (0,\ 1/\Gamma - \delta_\Gamma)
\end{array}
$$

$\epsilon \le 1/\Gamma$ and $\delta_1 \le \cdots \le \delta_\Gamma < \epsilon/\Gamma$

Next Fit Algorithm

Lower Bound for Γ-uncertainty

$$\begin{array}{cccc}
(\epsilon,\ 1/\Gamma-\delta_1) & (0,\ 1/\Gamma-\delta_1) & \dots & (0,\ 1/\Gamma-\delta_1) \\
(\epsilon,\ 1/\Gamma-\delta_2) & (0,\ 1/\Gamma-\delta_2) & \dots & (0,\ 1/\Gamma-\delta_2) \\
\vdots & \vdots & \ddots & \vdots \\
(\epsilon,\ 1/\Gamma-\delta_{\Gamma-1}) & (0,\ 1/\Gamma-\delta_{\Gamma-1}) & \dots & (0,\ 1/\Gamma-\delta_{\Gamma-1}) \\
(\epsilon,\ 1/\Gamma-\delta_\Gamma) & (0,\ 1/\Gamma-\delta_\Gamma) & \dots & (0,\ 1/\Gamma-\delta_\Gamma)
\end{array}$$

$\epsilon \leq 1/\Gamma$ and $\delta_1 \leq \cdots \leq \delta_\Gamma < \epsilon/\Gamma$

Next Fit Algorithm

Lower Bound for Γ-uncertainty

$(\epsilon, 1/\Gamma - \delta_1)$	$(0, 1/\Gamma - \delta_1)$	$\dots$	$(0, 1/\Gamma - \delta_1)$
$(\epsilon, 1/\Gamma - \delta_2)$	$(0, 1/\Gamma - \delta_2)$	$\dots$	$(0, 1/\Gamma - \delta_2)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\epsilon, 1/\Gamma - \delta_{\Gamma-1})$	$(0, 1/\Gamma - \delta_{\Gamma-1})$	$\dots$	$(0, 1/\Gamma - \delta_{\Gamma-1})$
$(\epsilon, 1/\Gamma - \delta_\Gamma)$	$(0, 1/\Gamma - \delta_\Gamma)$	$\dots$	$(0, 1/\Gamma - \delta_\Gamma)$

$\epsilon \leq 1/\Gamma$ and $\delta_1 \leq \dots \leq \delta_\Gamma < \epsilon/\Gamma$