INTEGRATION OF PARTIAL DIFFERENTIAL EQUATIONS (2)

Objectives:

- Integrate a Poisson-like partial differential equation numerically;
- Implement the Jacobi method;
- Implement the Gauss-Seidel method;
- Implement the overrelaxation method;
- Set up Dirichlet boundary conditions.

No list manipulation is allowed in this tutorial!

I. Introduction

In this tutorial, we determine the electrostatic potential inside a square domain, 1 meter on each side, bounded by four conductors at fixed potential, see Fig. 1. We assume that the space between the conductors is empty.

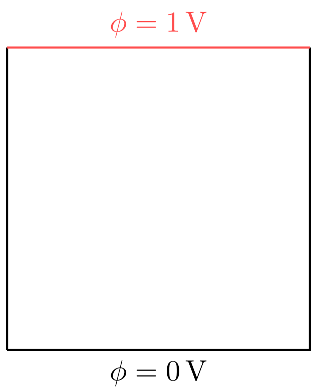

Figure 1: **Electrostatic problem in vacuum to solve.** An empty space is delimited by 4 conductors. Three of them (in black) are at zero potential, the last one (in pink) is at a potential of 1 volt.

The boundary-value problem to solve is thus (with distances expressed in meters, and the potential expressed in volts):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad \phi(0, y) = 0, \quad \phi(1, y) = 0, \quad \phi(x, 0) = 0, \quad \phi(x, 1) = 1. \tag{1}$$

The objective is to compare the speed of resolution and the accuracy of different methods: the Jacobi method, the Gauss-Seidel method, and the overrelaxation method. For all methods, we discretize space as follows: $x_j = j\delta$, $y_k = k\delta$, with δ the discretization step size, $j, k \in [\![0, M]\!]$, and $M\delta = 1$.

II. The Jacobi method

Question 1: We recall that all methods described in the lecture notes to integrate Eq. (1) are based on the fact that the solution to the diffusion-like partial differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \tag{2}$$

converges to the solution to Eq. (1) when $t \to +\infty$.

a. Within the FTCS scheme, derive the recurrence relation between ϕ_{jk}^{n+1} and the $\{\phi_{j'k'}^{n}\}_{j',k' \leq M}$'s for $j,\, k \in [\![1,\, M-1]\!]$ and for a discretization timestep h.

b. The above scheme is stable as long as $h \leq \delta^2/4$. By taking $h = \delta^2/4$, show that the recurrence relation for $j,\, k \in [\![1,\, M-1]\!]$ simplifies to:

$$\phi_{jk}^{n+1} = \frac{1}{4}\left(\phi_{j+1,k}^{n} + \phi_{j-1,k}^{n} + \phi_{j,k+1}^{n} + \phi_{j,k-1}^{n}\right). \tag{3}$$

c. What are the values of $\phi_{j,0}^{n}$, $\phi_{j,M}^{n}$, $\phi_{0,k}^{n}$ and $\phi_{M,k}^{n}$ for all $j, k \in [\![0,\, M]\!]$?

Question 2: We first propose to solve Eq. (1) using the Jacobi method. The latter consists in directly applying the recurrence relation given by Eq. (3).

a. When should you stop to iterate this recurrence relation? Propose a quantitative stopping criterion based on how much the solution changes between successive iterations..

b. Iterate the recurrence relation given by Eq. (3) and plot a heat map of the solution. You can take $\delta = 0.01$ (in meters).

c. How many steps are required for the method to converge? How long does it take (in seconds)?

III. The Gauss-Seidel and the overrelaxation methods

Question 3: We now propose to solve Eq. (1) using the Gauss-Seidel method.

a. How is the recurrence relation given by Eq. (3) modified?

b. Implement the Gauss-Seidel method and plot a heat map of the solution. Use the same value of δ as used for the Jacobi method.

c. How many steps are required for the method to converge? How long does it take (in seconds)?

Question 4: We finally propose to solve Eq. (1) using the overrelaxation method.

a. How is the recurrence relation given by Eq. (3) modified?

b. Implement the overrelaxation method and plot a heat map of the solution. Use the same value of δ as used for the two previous methods. You can take $\omega = 1.8$.

c. How many steps are required for the method to converge? How long does it take (in seconds)?

IV. A quantitative comparison between the different methods

Question 5: The exact solution to Eq. (1) is known and is given by:

$$\phi(x, y) = \frac{4}{\pi} \sum_{m=0}^{+\infty} \frac{\sin[(2m+1)\pi x] \sinh[(2m+1)\pi y]}{(2m+1) \sinh[(2m+1)\pi]}. \tag{4}$$

We define the relative error between the numerical solution and the exact solution as

$$e = \frac{\sum_{j,k} |\phi_{jk} - \phi(x_j, y_k)|}{\sum_{j,k} |\phi(x_j, y_k)|}. \tag{5}$$

a. For the three methods implemented above, compute e.

b. Which solution is a good compromise between computation time and accuracy?

Question 6 (bonus): Explore how the above conclusion is sensitive to the choice of ω in the overrelaxation method.