Département
de Mécanique
Faculté des Sciences

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Département d'enseignement Mécanique de l'UFR de Sciences de l'UMII
Laboratoire de Mécanique et Génie Civil

VERSION_Eng 0.0

- Objectives
  - Present the Finite Element Method (linear case)
  - Understand and uses an FEM code (Ansys)
- Outline
  - Review of Continuum Mechanics
  - Energy Method – Ritz and Galerkin (PPV)
  - Discretization Method
  - Numerical Integration
  - Application to FEM

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- This course unit is evaluated through continuous assessment
  - One on-site written examination (0h30 – 10%)
  - One first report on practical work (20%)
  - One second report on a subject of the student’s choice (20%)
  - One on-site written examination (2h00 - 50%)
- Organization
  - 12 hours Lectures – L.D
  - 15 hours Tutorials – E.A
  - 15 hours Practical Work – N.C. & L.D.

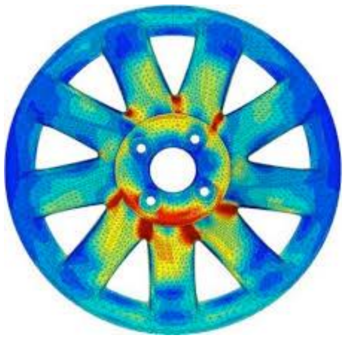
Lexique technique FR → EN

Loi de comportement	→	*Constitutive law*
Champ de déplacement	→	*Displacement field*
Conditions aux limites	→	*Boundary conditions*
Équations d'équilibre	→	*Equilibrium equations*
Équilibre local	→	*Local equilibrium*
Tenseur des contraintes	→	*Stress tensor*
Tenseur de Cauchy	→	*Cauchy stress tensor*
Cinématique de la déformation	→	*Kinematics of deformation*
Mesures de déformation	→	*Strain measures*
Déformation de Green–Lagrange	→	*Green–Lagrange strain*
Loi de Hooke	→	*Hooke's law*
Méthode énergétique	→	*Energy method*
Principe des puissances virtuelles (PPV)	→	*Principle of Virtual Power (PVP)*

Technical Glossary FR → EN

Fonctions d'interpolation	→	*Interpolation functions / Shape functions*
Maillage / Discrétisation	→	*Meshing / Discretization*
Éléments finis	→	*Finite elements*
Système global	→	*Global system*
Matrice de rigidité	→	*Stiffness matrix*
Intégration numérique	→	*Numerical integration*
Quadrature de Gauss	→	*Gauss quadrature*
Résolution du système	→	*System solving / System resolution*
Répartition des contraintes	→	*Stress distribution*
Déformation d'une poutre	→	*Beam deformation*

Why do FE s



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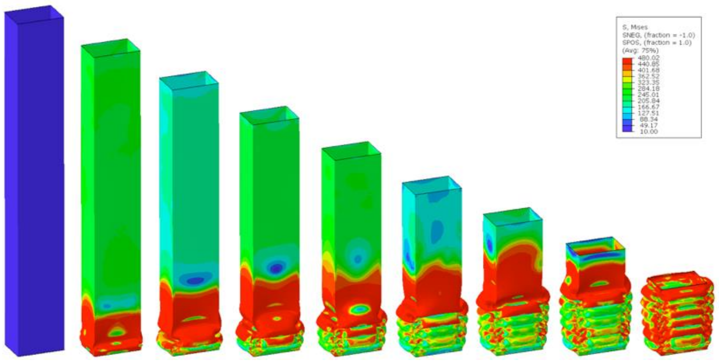
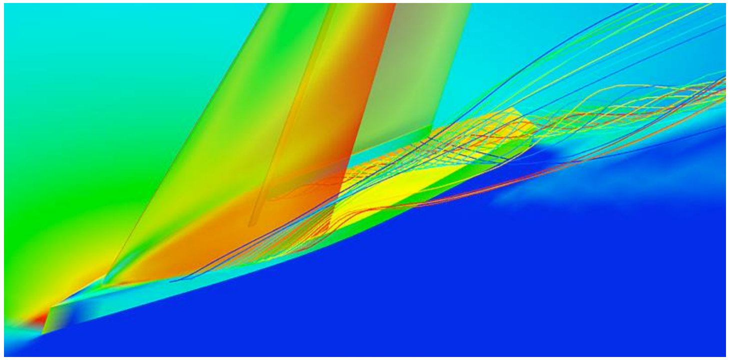
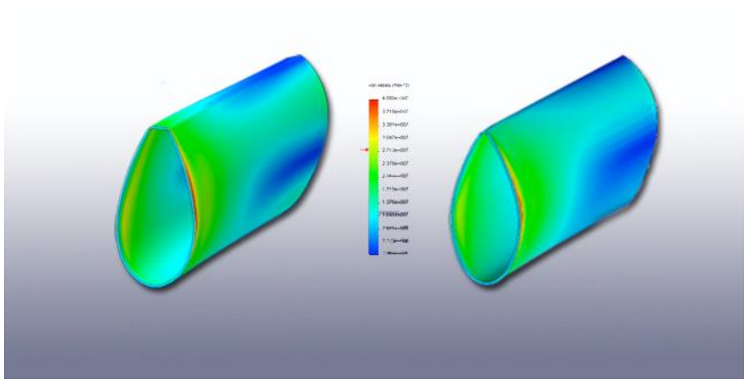


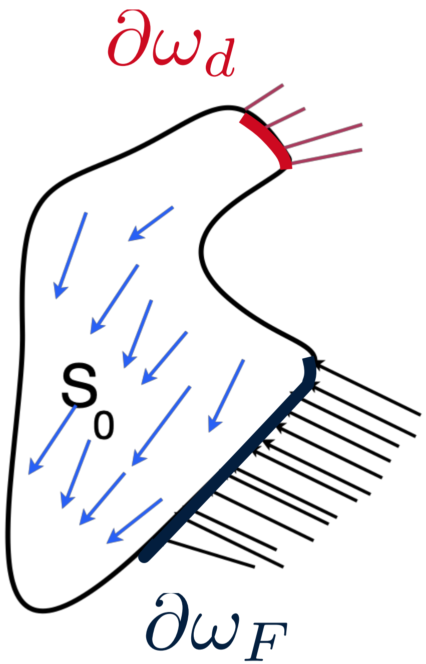
EC2 Modélisation



NORTH DESIGN SUITE™



PRO-FIL Composites



1 Equilibrium equations
MMC

2 Boundary conditions
In effort
In displacement

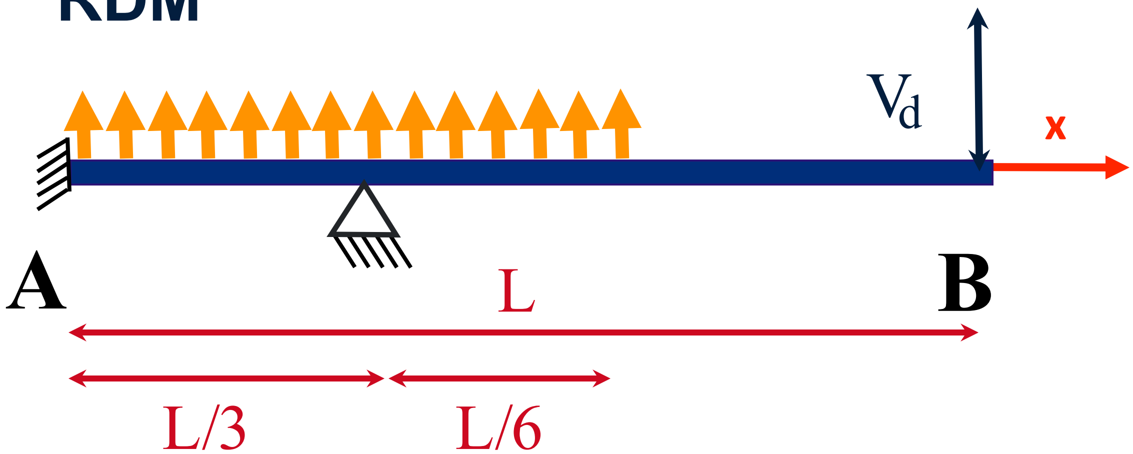
3 Constitutive Behavior
Linear elasticity

Deformation energy of a solid

Work of given external load

What do U remember from last year ?

RDM

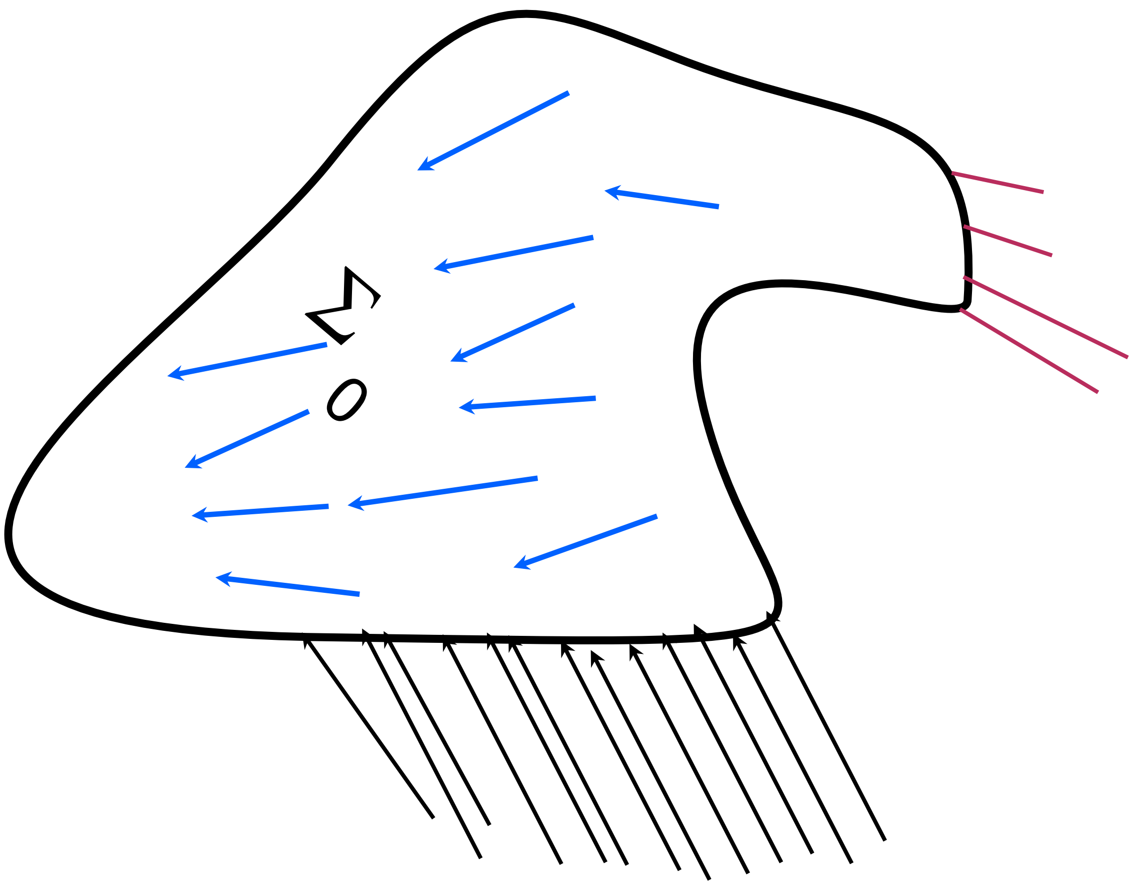


1 Equilibrium equations
MMC
2 Boundary conditions
In effort
In displacement
3 Constitutive Behavior
Linear elasticity

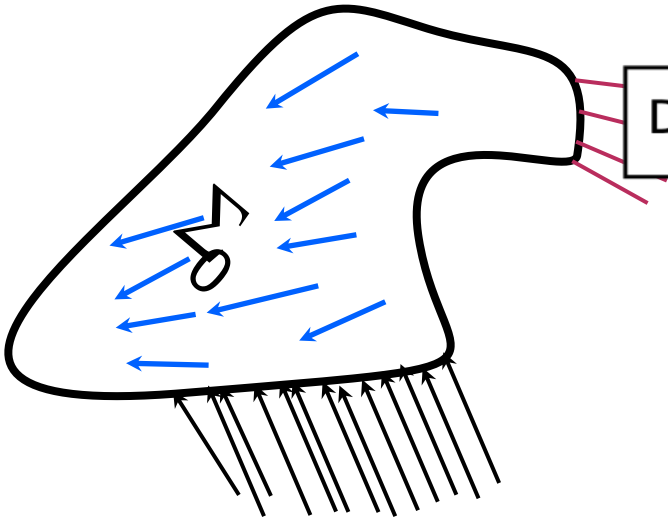
Navier-Bernoulli kinematics

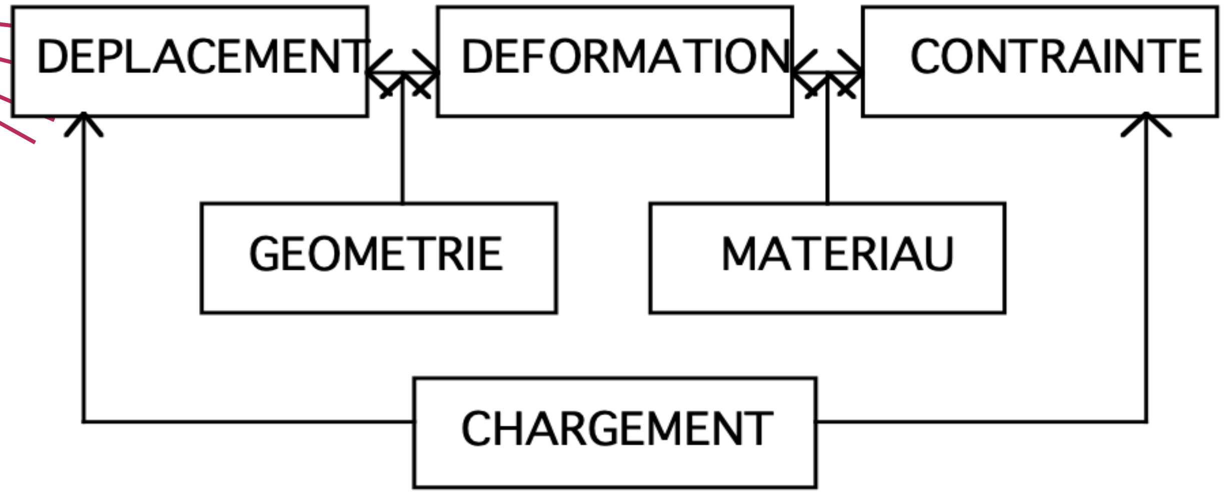
Beam Deformation Energy

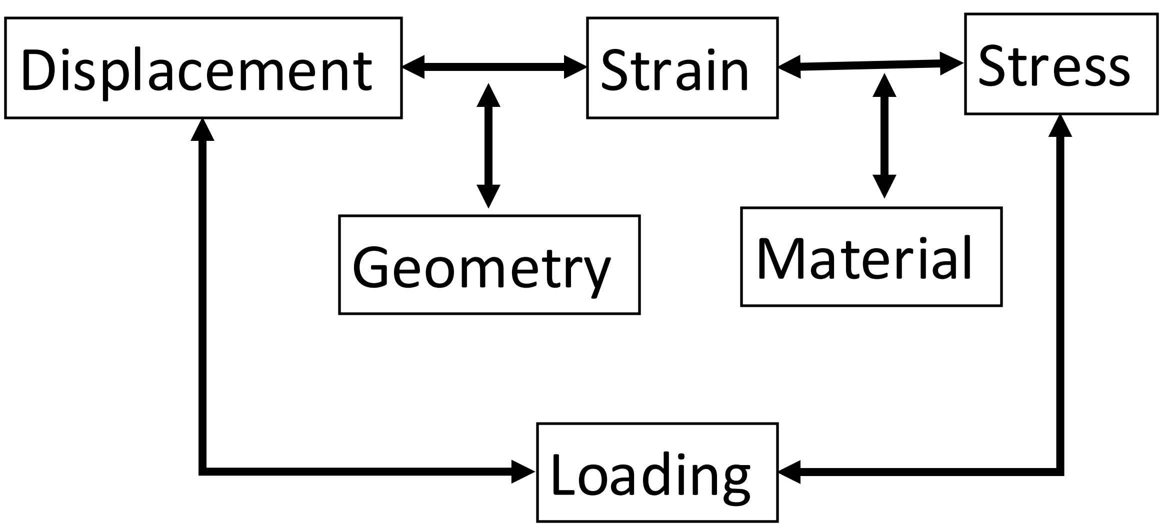
Work of the given external loads

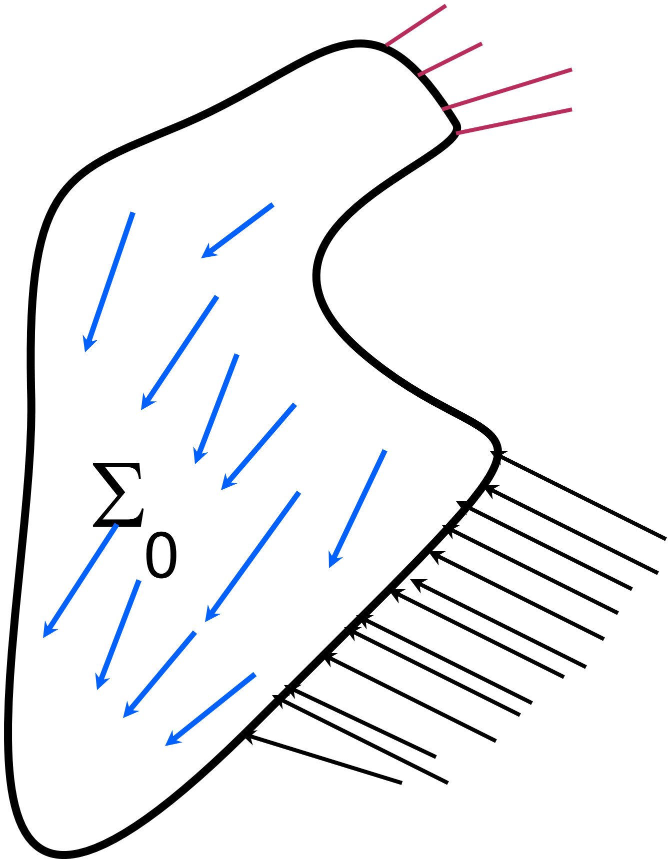


What is a Mechanical Problem









Equilibrium equations

MMC

RDM

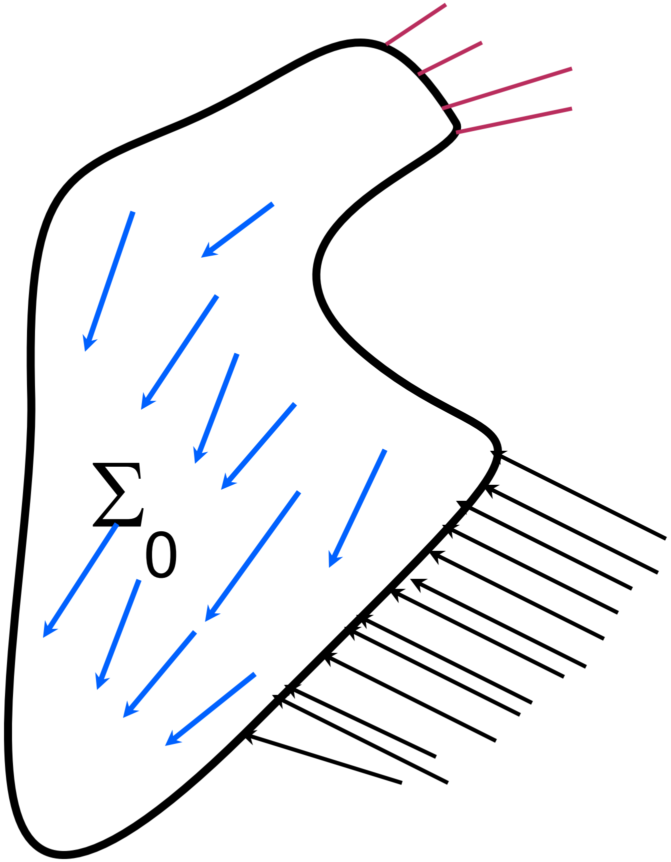
Boundary Conditions

Loading

Displacement

Constitutive Behaviour

The Constitutive behavior of the material is linear elastic (Hooke's law).

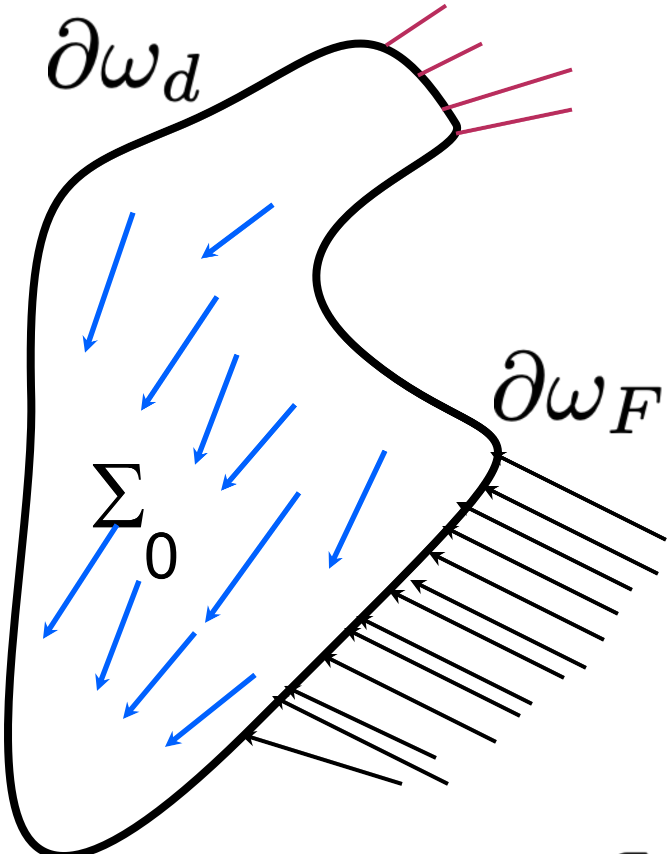


Displacement field

$$U(x,y,z)=\begin{cases} u(x,y,z)=U_1(x_1,x_2,x_3) \\ v(x,y,z)=U_2(x_1,x_2,x_3) \\ w(x,y,z)=U_3(x_1,x_2,x_3) \end{cases}$$

Index Convention

$$i,j,k\in\{1,2,3\}$$
$$\alpha,\beta,\gamma\in\{1,2\}$$



1.Equilibrium equations

$$\mathrm{div}\ \underline{\underline{\sigma}} + \underline{f} = \underline{0}$$

2.Boundary Conditions

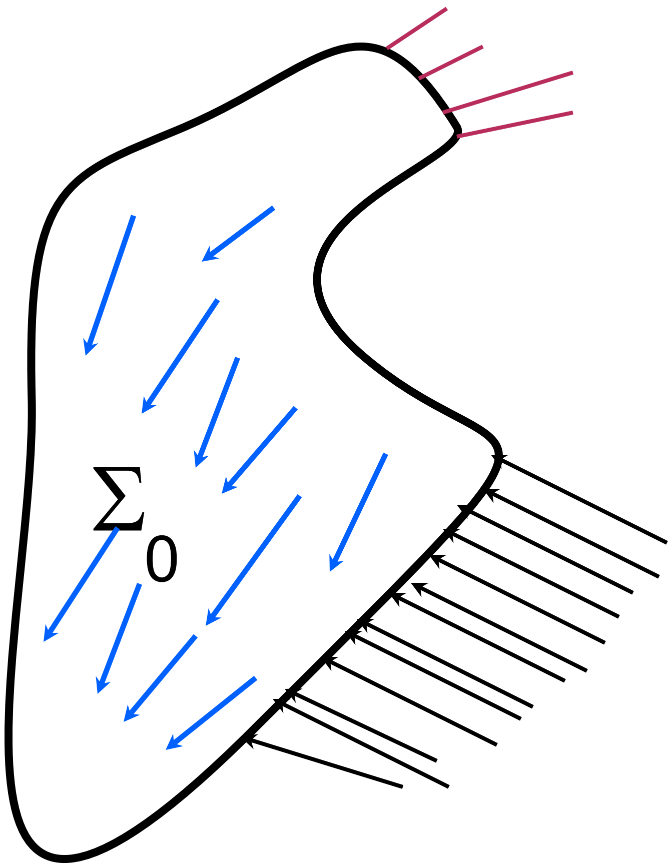
$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3.Constitutive Behaviour

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

The Constitutive behavior of the material is linear elastic (Hooke's law).



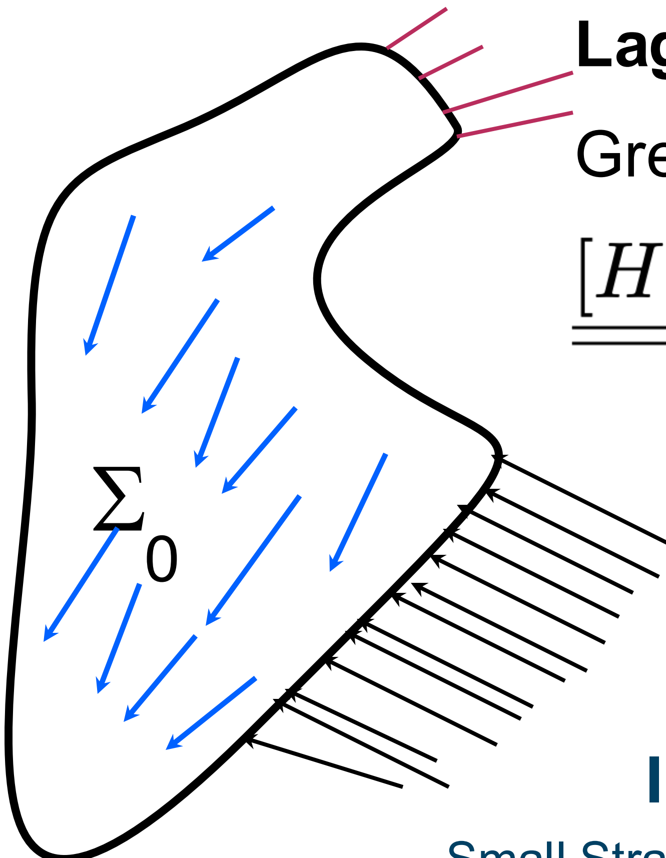
Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \rho\vec{\gamma}$$
$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

Equilibrium equations in Cartesian coordinate

$$\sigma_{ij,j} + f_i = \rho\gamma_i$$

$$[\sigma(x)] = \begin{bmatrix} \sigma 11 & \sigma 12 & \sigma 13 \\ \sigma 21 & \sigma 22 & \sigma 23 \\ \sigma 31 & \sigma 32 & \sigma 33 \end{bmatrix} \approx \begin{bmatrix} \sigma 11 \\ \sigma 22 \\ \sigma 33 \\ \sigma 12 \\ \sigma 13 \\ \sigma 23 \end{bmatrix}$$



Lagrangian finite strain tensor

Green-Lagrange strain tensor

$$\underline{\underline{[H]}} = \underline{\underline{[\varepsilon]}} + \frac{1}{2}\left(\underline{\underline{[\varepsilon]}} - \underline{\underline{[\Omega]}}\right)\left(\underline{\underline{[\varepsilon]}} + \underline{\underline{[\Omega]}}\right)$$

$$\underline{\underline{[\Omega]}} = \frac{1}{2}\left(grad(U) - grad(U)^T\right) = \frac{1}{2}\left(\nabla U - \nabla U^T\right)$$

$$\underline{\underline{[\varepsilon]}} = \frac{1}{2}\left(grad(U) + grad(U)^T\right) = \frac{1}{2}\left(\nabla U + \nabla U^T\right)$$

Infinitesimal Strain Theory

Small Strain, Small Displacement , Small pertubation

$$\boxed{\underline{\underline{[H]}} = \underline{\underline{[\varepsilon]}}}$$

A material is said to be **Isotropic** if its behavior is the same in all the direction.

There are **2** mechanical constants to define the law of behavior

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

Linear elasticity – Constitutive Behavior

A material is said to be **Transverse Isotropic** if it has a plane of symmetry of mechanical behavior, so there is an axis of orthotropy.

There are **6** mechanical constants to define the law of behavior

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_y} & \frac{-\nu_{xy}}{E_y} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_y} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_y} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_y} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu_{yz})}{E_y} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

A material is said to be **orthotropic** if it has two planes of symmetry of mechanical behavior, so there are three axes of orthotropy.

There are **9** mechanical constants to define the law of behavior

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{xy}}{E_y} & \frac{-\nu_{xz}}{E_z} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_y} & \frac{1}{E_y} & \frac{-\nu_{yz}}{E_z} & 0 & 0 & 0 \\ \frac{-\nu_{xz}}{E_z} & \frac{-\nu_{yz}}{E_z} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

https://fr.wikipedia.org/wiki/Loi_de_Hooke

Linear elasticity – Constitutive Behavior

For an **Isotropic** material there are **2** mechanical constants to define the law of behavior

Linear Elastic Constitutive Behavior : Hooke's Law

$$\sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Lamé Constants

$$\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

https://www.youtube.com/watch?v=pGsahq_yUGE

Linear Elastic Constitutive Behavior : Hooke's Law

For an **Isotropic** material there are **2** mechanical constants to define the law of behavior

$$\sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

Bulk Modulus

Module d'élasticité isostatique

$$K = \frac{E}{3(1-2\nu)}$$

Young’s Modulus

Module de Young

$$E = \frac{\mu\left(3\lambda + 2\mu\right)}{\lambda + \mu}$$

Poisson’s Ratio

Coefficient de Poisson

$$\nu = \frac{\lambda}{2\left(\lambda + \mu\right)}$$

Shear modulus

Module de cisaillement

$$G = \mu = \frac{E}{2\left(1 + \nu\right)}$$

Mechanical problem

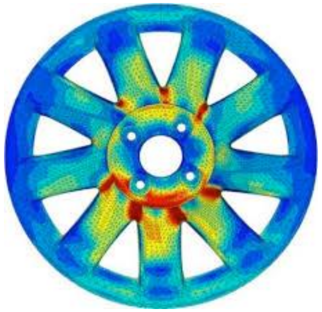
Principal of Virtual Work

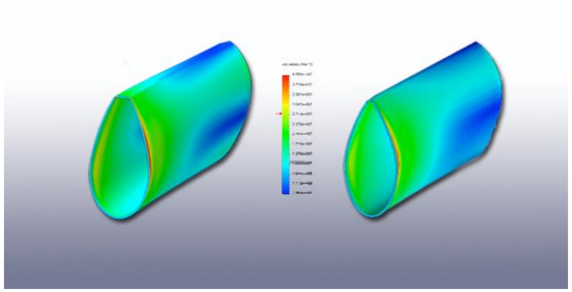
Variational approach

Minimum potential energy principle

$$[\mathbf{K}][Q] = [\mathbf{F}]$$

$$[Q] = [\mathbf{K}]^{-1}[\mathbf{F}]$$



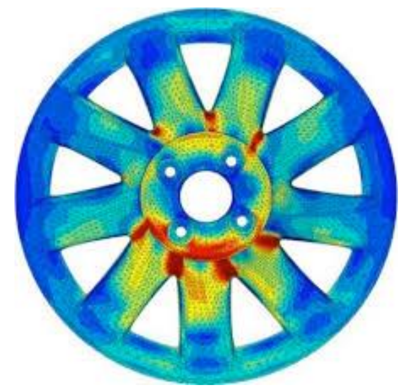


https://www.youtube.com/watch?v=KDCQeXA4KpY

Energy Method

1.Equilibrium equations

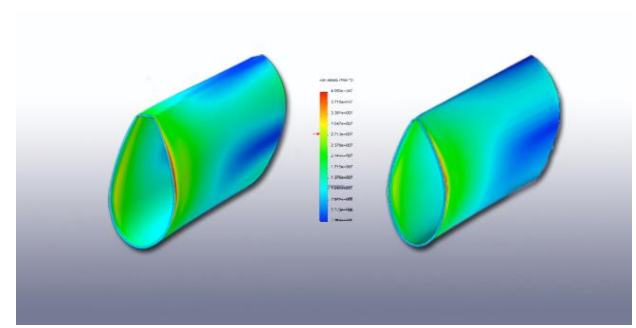
$$\mathrm{div}\ \underline{\underline{\sigma}} + \underline{f} = \underline{0}$$



2.Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

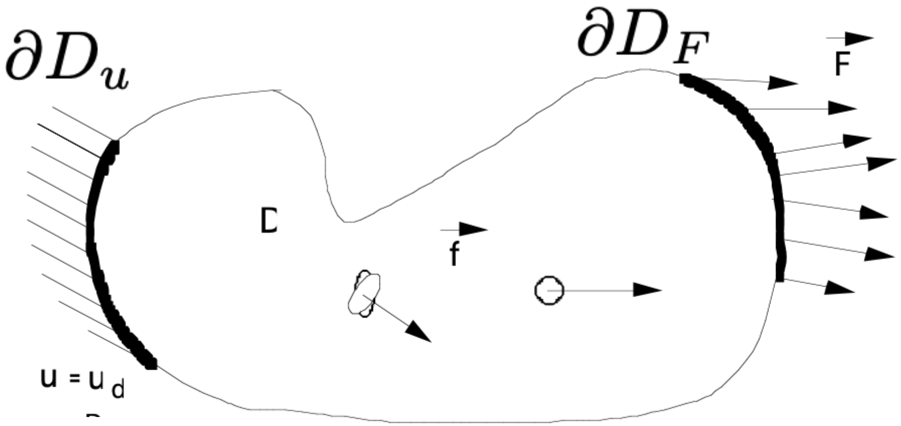
$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$



3.Constitutive Behavior

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

Mechanical Problem – Principal of Virtual Work



∂D_F This is the edge where known efforts are imposed

∂D_u This is the edge where displacement boundary conditions is imposed (Zero or not)

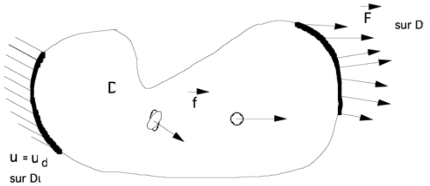
$$u^* \in C.A. \quad <=> \quad u^*(x) = \mathrm{U_d} \quad \forall \mathrm{x} \in \partial D_u$$

$$u^* \in C.A.\{0\} \quad <=> \quad u^*(x) = 0 \quad \forall \mathrm{x} \in \partial D_u$$

A kinematically admissible displacement field needs only to satisfy the displacement boundary conditions

https://en.wikiversity.org/wiki/Elasticity/Kinematic_admissibility

Mechanical Problem – Principal of Virtual Work



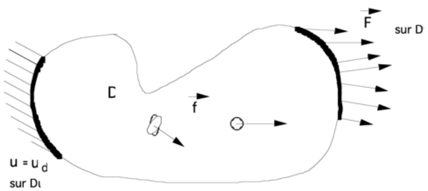
$$\forall u^*(x) \in C.A.\{0\}$$

$$P_{int}(u^*(x)) + P_{ext}(u^*(x)) = P_{acc}(u^*(x))$$

$$P_{int}(u^*(x)) = -\int_D tr(\sigma(x).\varepsilon^*(x))d\Omega$$

$$P_{ext}(u^*(x)) = \int_D f(x).u^*(x)d\Omega + \int_{\partial D} F(x).u^*(x)dS$$

$$P_{acc}(u^*(x)) = \int_D (\rho\frac{d^2(u^*(x))}{d^2(t)})u^*(x)d\Omega$$



$$P_{int}(u^*(x)) = -\int_D tr(\sigma(x).\varepsilon^*(x))d\Omega$$

$\varepsilon_{ij} = M_{ijkl}\sigma_{kl}$ note $\underline{\underline{\varepsilon}} = \underline{\underline{\underline{\underline{M}}}}.\underline{\underline{\sigma}}$

ou

$\sigma_{ij} = L_{ijkl}\varepsilon_{kl}$ note $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$

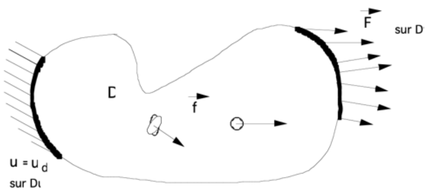
$$P_{int}(u^*(x)) = -\int_D tr(\sigma(x).\varepsilon^*(x))d\Omega$$

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\varepsilon^*_{ij}(x)d\Omega$$

$$P_{int}(u^*(x)) = -\int_D L_{ijkl}\varepsilon_{kl}(x).\varepsilon^*_{ij}(x)d\Omega$$

$$P_{int}(u^*(x)) = -\mathbf{K}(U, U^*)$$

Mechanical Problem – Principal of Virtual Work



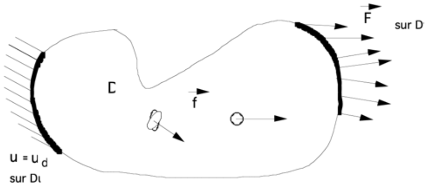
$$\forall u^*(x) \in C.A.\{0\}$$

$$P_{int}(u^*(x)) + P_{ext}(u^*(x)) = P_{acc}(u^*(x))$$

$$P_{ext}(u^*(x)) = \int_D f(x).u^*(x)d\Omega + \int_{\partial D} F(x).u^*(x)dS = P_{don}(u^*(x))$$

$$P_{don}(u^*(x)) = \int_D f(x).u^*(x)d\Omega + \int_{\partial D_F} F(x).u^*(x)dS$$

$$P_{don}(u^*(x)) = -\mathbf{L}(U^*)$$



$$\forall u^*(x) \in C.A.\{0\}$$
$$et$$
$$\forall u(x) \in C.A.$$

$$P_{int}(u^*(x)) + P_{ext}(u^*(x)) = 0$$

Formulation 1

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0 \quad \forall U^* \in C.A.\{0\}$$
$$-\mathbf{K}(U, U^*) + \mathbf{L}(U^*) = 0 \quad et \quad \forall U \in C.A.$$

Formulation 1

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0 \qquad \forall U^* \in C.A.\{0\}$$

$$-\mathbf{K}(U,U^*) + \mathbf{L}(U^*) = 0 \qquad et \quad \forall U \in C.A.$$

Equivalent to

1. Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

2. Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3. Constitutive law

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

Mechanical Problem – Principal of Virtual Work

Pb (1) => PVW

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

Green's theorem or Integration by parts

$$\int_D Div(\underline{\underline{M}}).u(x)d\Omega = -\int_D \underline{\underline{M}}.\nabla u(x)d\Omega + \int_{\partial D} \underline{\underline{M}}.n.u(x)d\omega$$

$$\int_A^B V'(x).U(x)d\Omega = -\int_A^B V(x).U'(x)d\Omega + \left[V(x).U(x)\right]_A^B$$

Pb (1) => PPV

Step 1

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

$$\int_D \left(div\left(\underline{\underline{\sigma}}\right) + \vec{f}\right) U^*(x) d\Omega = \vec{0}$$

Green's theorem

$$-\int_D \underline{\underline{\sigma}}.\nabla U^*(x) d\Omega + \int_{\partial D} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega + \int_D \vec{f}.U^*(x) d\Omega = \vec{0}$$

Mechanical Problem – Principal of Virtual Work

$$
\begin{aligned}
\underline{\underline{\sigma}}.\nabla U^*(x) &= \sigma_{ij}.U^*_{i,j} \\
&= \sigma_{ij}.\frac{1}{2}(U^*_{i,j} + U^*_{i,j}) \\
&= \sigma_{ij}.\frac{1}{2}(U^*_{i,j}) + \sigma_{ij}.\frac{1}{2}(U^*_{i,j}) \\
&= \sigma_{ij}.\frac{1}{2}(U^*_{i,j}) + \sigma_{ji}.\frac{1}{2}(U^*_{i,j}) \\
&= \sigma_{ij}.\frac{1}{2}(U^*_{i,j}) + \sigma_{ij}.\frac{1}{2}(U^*_{j,i}) \\
&= \sigma_{ij}.\frac{1}{2}(U^*_{i,j} + U^*_{j,i})
\end{aligned}
$$

Step 2

Pb (1) => PPV

$$
\boxed{\underline{\underline{\sigma}}.\nabla U^*(x) = \sigma_{ij}.U^*_{ij} \quad = \sigma_{ij}.\epsilon^*_{ij} = \underline{\underline{\sigma}}.\underline{\underline{\epsilon}}^*}
$$

Pb (1) => PPV

Step 3

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

$$\int_D \left(div\left(\underline{\underline{\sigma}}\right) + \vec{f}\right) U^*(x) d\Omega = \vec{0}$$

$$-\int_D \underline{\underline{\sigma}}.\nabla U^*(x) d\Omega + \int_{\partial D} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega + \int_D \vec{f}.U^*(x) d\Omega = \vec{0}$$

$$-\int_D \underline{\underline{\sigma}}.\underline{\underline{\epsilon}}^* d\Omega + \int_{\partial D} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega + \int_D \vec{f}.U^*(x) d\Omega = \vec{0}$$

Pb (1) => PPV

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

$$\int\limits_{\partial D = \partial\omega_F \cup \partial\omega_d} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega = \int\limits_{\partial\omega_F} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega + \int\limits_{\partial\omega_d} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega$$

$$= \int\limits_{\partial\omega_F} \vec{F}.U^* d\omega + \int\limits_{\partial\omega_d} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega$$

$$\text{Si } U^* \in C.A.\{0\} \text{ alors } \int\limits_{\partial\omega_d} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega = 0$$

Mechanical Problem – Principal of Virtual Work

Pb (1) => PPV

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0} \qquad \int_D \left(div\left(\underline{\underline{\sigma}}\right) + \vec{f}\right) U^*(x) d\Omega = \vec{0}$$

$$-\int_D \underline{\underline{\sigma}}.\underline{\underline{\epsilon}}^* d\Omega + \int_{\partial D} (\underline{\underline{\sigma}}.\underline{n}).U^* d\omega + \int_D \vec{f}.U^*(x) d\Omega = \vec{0}$$

$$\underbrace{-\int_D \underline{\underline{\sigma}}.\underline{\underline{\epsilon}}^* d\Omega}_{-\mathbf{K}(U,U^*)} + \underbrace{\int_{\partial\omega_F} \vec{F}.U^* d\omega + \int_D \vec{f}.U^*(x) d\Omega}_{\mathbf{L}(U^*)} = \vec{0}$$

$$-\mathbf{K}(U,U^*) + \mathbf{L}(U^*) = \vec{0}$$

$$\forall U^* \in C.A.\{0\}$$

1. Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

2. Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3. Constitutive law

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0 \qquad \forall U^* \in C.A.\{0\}$$

$$-\mathbf{K}(U, U^*) + \mathbf{L}(U^*) = 0 \qquad et \quad \forall U \in C.A.$$

PVW => Pb (1)

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0$$
$$-\mathbf{K}(U,U^*) + \mathbf{L}(U^*) = 0$$

$\forall U^* \in C.A.\{0\}$
et
$\forall U \in C.A.$

$$P_{int}(u^*(x)) = -\mathbf{K}(U,U^*)$$

$$P_{int}(u^*(x)) = -\int_D L_{ijkl}\varepsilon_{kl}(x).\varepsilon^*_{ij}(x)d\Omega$$

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\varepsilon^*_{ij}(x)d\Omega$$

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\frac{1}{2}\left(\nabla u^*(x) + \nabla^T u^*(x)\right)d\Omega$$

PPV => Pb (1)

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\frac{1}{2}\left(\nabla u^*(x) + \nabla^T u^*(x)\right) d\Omega$$

Step 1

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\frac{1}{2}\left(\nabla u^*(x) + \nabla^T u^*(x)\right) d\Omega$$

$$= \int_D Div\underline{\underline{\sigma}}.u^*(x)d\Omega - \int_{\partial D} \left(\underline{\underline{\sigma}}.\underline{n}\right).\underline{u}^* d\Omega$$

Green's theorem

PPV => Pb (1)

$$P_{don}(u^*(x)) = \int_D f(x).u^*(x)d\Omega + \int_{\partial\omega_F} F(x).u^*(x)dS$$

Step 2

$$= \int_D f(x).u^*(x)d\Omega + \int_{\partial D} F(x).u^*(x)dS$$

$$P_{int}(u^*(x)) = -\int_D \sigma_{ij}(x).\frac{1}{2}\left(\nabla u^*(x) + \nabla^T u^*(x)\right) d\Omega$$

$\forall U^* \in C.A.\{0\}$
et
$\forall U \in C.A.$

$$= \int_D Div\underline{\underline{\sigma}}.u^*(x)d\Omega - \int_{\partial D} \left(\underline{\underline{\sigma}}.\underline{n}\right).\underline{u}^* d\Omega$$

PPV => Pb (1)

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = \int_D \left(Div\underline{\underline{\sigma}} + \vec{f} \right) .u^*(x) d\Omega$$

$$+ \int_{\partial D} \left(\underline{\underline{\sigma}}.\underline{n} - \vec{F} \right) .\underline{u}^* d\Omega$$

$\forall U^* \in C.A.\{0\}$
et
$\forall U \in C.A.$

Step 3

$$\boxed{\sigma_{ij} = L_{ijkl}\varepsilon_{kl}}$$

PPV => Pb (1)

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0 \qquad \forall U^* \in C.A.\{0\}$$

$$et$$

$$-\mathbf{K}(U, U^*) + \mathbf{L}(U^*) = 0 \qquad \forall U \in C.A.$$

1. Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

2. Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3. Constitutive law

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

Pb (1) < = > PPV

1. Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

2. Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$

$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3. Constitutive law

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

$$\forall u^*(x) \in C.A.\{0\} \qquad \forall u^* \in \text{C.A}\{0\} \text{ et } u \in \text{C.A.}$$

$$W_{int}(u^*(x)) + W_{ext_{don}}(u^*(x)) = 0$$

PVW <=> variation of the total energy = 0

$$\forall u^*(x) \in C.A.\{0\}$$
$$W_{int}(u^*(x)) + W_{ext_{don}}(u^*(x)) = 0$$

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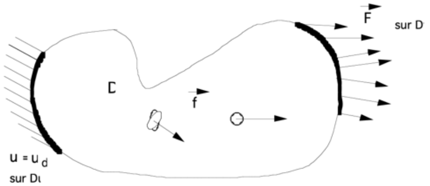
$$u(x) \in C.A\{u_d\}$$
$$\delta\left(\Pi_d(u)\right) = 0$$

This is the standard form used in Finite Element Method (FEM) and structural mechanics textbooks.



Formulation 2

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$$\begin{aligned} & u(x) \in C.A\,\{u_d\} \\ & \delta\left(\Pi_d(u)\right) = 0 \end{aligned}$$

$$\left\{\begin{array}{l} u(x) \in C.A\,\{u_d\} \\ \delta\left(\Pi_d(u)\right) = 0 \end{array}\right. \quad \Leftrightarrow \quad P.P.V \quad où \quad \left\{\begin{array}{l} u(x) \in C.A\,\{u_d\} \\ u^*(x) \in C.A\,\{0\} \end{array}\right.$$

$$\left\{\begin{array}{l} u(x) \in C.A\,\{u_d\} \\ \delta\left(\Pi_d(u)\right) = 0 \end{array}\right. \quad \Leftrightarrow \quad Pb(1)$$

Variation of the total potential energy

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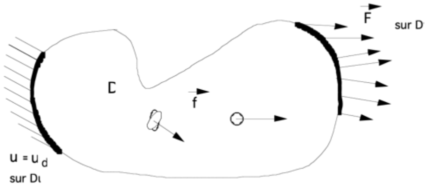

The Total Potential energy is defined by:

$$\Pi_d(u) = \int_D \frac{1}{2} L_{ijkl}\varepsilon_{ij}\varepsilon_{kl} d\omega - \left(\int_D f(x).u(x) d\Omega + \int_{D_F} F(x).u(x) dS \right)$$

$$u(x) \in C.A\left\{u_d\right\}$$

The total potential energy is defined as the difference between the energy of the deformation of the system and the work of the forces imposed, and this for a cinematically permissible field

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$$\Pi_d(u) = W_{def}(u,u) - W_{ext_{don}}(u) = \frac{1}{2}\mathbf{K}(U,U) - \mathbf{L}_{ext_{don}}(U)$$

$$\underline{u}\,(\mathrm{x}) \in C.A.$$

$$W_{def}(u,u) = \frac{1}{2}\int_{S_0} \sigma_{ij}\varepsilon_{ij}d\omega$$

$$= \frac{1}{2}\int_{S_0} L_{ijkl}\varepsilon_{kl}\varepsilon_{ij}d\omega = \frac{1}{2}\mathbf{K}(U,U)$$

$$W_{ext_{don}}(u) = \int_{S_0} \underline{f}(x).\underline{u}(x)d\Omega + \int_{\omega_F} \underline{F}(x).\underline{u}(x)dS = \mathbf{L}(U)$$

Variation of total Potential Energy

Energie potentielle

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$$\partial\left(\Pi_d(u)\right) = \partial\left(\frac{1}{2}\mathbf{K}(U,U) - \mathbf{L}(U)\right) \qquad \forall U \in C.A.$$

$$= \frac{1}{2}\left(\mathbf{K}(\partial U, U) + \mathbf{K}(U, \partial U)\right) - \mathbf{L}(\partial U) \qquad \begin{cases} \forall U \in C.A. \\ \forall \partial U \in C.A.\{0\} \end{cases}$$

$\mathbf{K}(U,U)$	Semi-Positive definite symmetric quadratic form
$\mathbf{L}(U)$	Linear form

Variation of total Potential Energy

Fv 2 ⇔ PVW

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$$
\begin{aligned}
\delta\left(\Pi_d(u)\right) &= \delta\left(\tfrac{1}{2}\mathbf{K}(U,U) - \mathbf{L}_{ext_{don}}(U)\right)\\
&= \mathbf{K}(U,\delta U) - \mathbf{L}_{ext_{don}}(\delta U)\\
&= -\left(W_{int}(\partial U) + W_{ext}(\partial U)\right)\\
&= 0
\end{aligned}
$$

$$
\begin{cases}
U \in C.A.\\
\forall \delta U \in C.A.\{0\}
\end{cases}
$$

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$$\forall u^*(x) \in C.A.\{0\}$$
$$W_{int}(u^*(x)) + W_{ext_{don}}(u^*(x)) = 0$$

$$W_{int}(u^*(x)) = -\mathbf{K}(U, U^*)$$
$$W_{ext}(u^*(x)) = \mathbf{L}(U^*)$$

where U is the solution of the mechanical problem

and $U^* = \partial U \in \mathrm{C.A.}\{0\}$

$$u(x) \in C.A\{u_d\}$$
$$\delta(\Pi_d(u)) = 0$$

$$\Updownarrow$$

1.Equilibrium equations

$$div\left(\underline{\underline{\sigma}}\right) + \vec{f} = \vec{0}$$

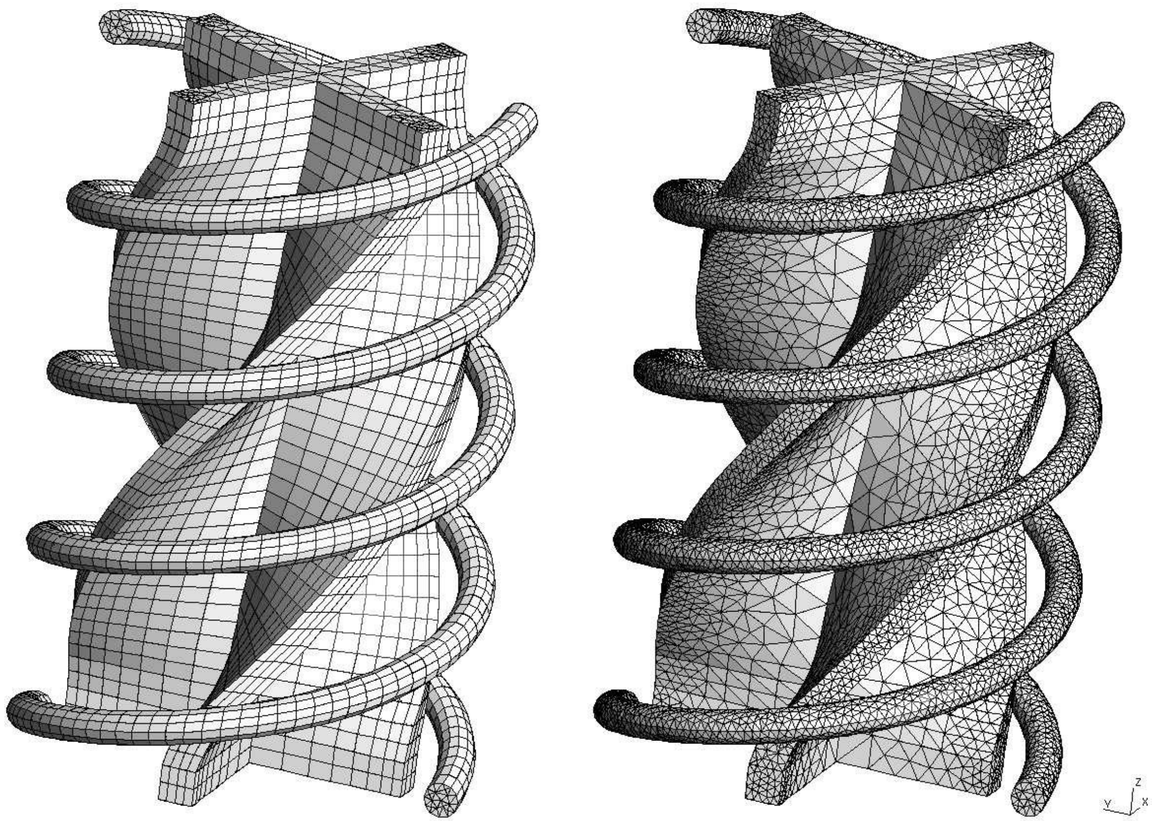
2.Boundary Conditions

$$\underline{\underline{\sigma}}.\vec{n} = \vec{F} \quad sur \quad \partial\omega_F$$
$$\vec{U} = \vec{U}_d \quad sur \quad \partial\omega_d$$

3. Behavior law

$$\sigma_{ij} = L_{ijkl}\varepsilon_{kl} \quad \text{noté} \quad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{L}}}}.\underline{\underline{\varepsilon}}$$

Discretisation



Mesh – Finite Element

GALLERKIN'S METHOD — PVW

$$u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \in C.A.$$

$$u^*(x) = P_i(x) \in C.A.\{0\}$$

RITZ'S METHOD — Potential energy

$$u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \in C.A.$$

Energy Approach - Discretization

$$u(x) \in C.A\,\{u_d\}$$

An unknown continuous function has to be found

$$\forall u^*(x) \in C.A.\{0\}$$
$$W_{int}(u^*(x)) + W_{ext_{don}}(u^*(x)) = 0$$

Or

$$u(x) \in C.A\,\{u_d\}$$
$$\delta\left(\Pi_d(u)\right) = 0$$

Discretization:

Determining a finite number of unknowns

Gallerkin's method

Gallerkin's method is adapted to the variational formulation resulting from the P.V.W

Principle: ***We give ourselves n shape or basis functions {Pi(x)} belonging to the kinematically admissible to zero and we look for the solution of Pb(I) as a linear combination of these functions, in the case where there are imposed displacement we add a function realizing them.***

$$u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \qquad \in C.A.$$

$$u^*(x) = P_i(x) \qquad \in C.A\{0\}$$

Gallerkin's method

The unknown ns of Pb are: $Q_i \quad i \in \{1, \cdots, n\}$

PVW :

$$P_{int}(u^*(x)) + P_{don}(u^*(x)) = 0$$
$$-\mathbf{K}(U, U^*) + \mathbf{L}(U^*) = 0$$

$$u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \qquad \in C.A.$$

$$u^*(x) = P_i(x) \qquad \in C.A.\{0\}$$

Gallerkin's method

$$\mathbf{K}(U, U^*) = \mathbf{L}(U^*)$$

$$\begin{aligned} U &= u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \in C.A. \\ U^* &= u^*(x) = P_i(x) \in C.A.\{0\} \end{aligned} \qquad i \in \{1, \cdots, n\}$$

$$\mathbf{K}\left(\sum_{i=1}^{n} Q_i P_i(x) + u_d(x), U^*\right) = \mathbf{L}(U^*)$$

One equation , **n** Unknowns:

$$\sum_{i=1}^{n} Q_i \mathbf{K}(P_i(x), U^*) = \mathbf{L}(U^*) - \mathbf{K}(u_d(x), U^*)$$

Gallerkin's method

$$\mathbf{K}(U, U^*) = \mathbf{L}(U^*)$$

$$U = u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \in C.A.$$

$$i \in \{1, \cdots, n\}$$

$$\mathbf{K}\Big(\sum_{i=1}^{n} Q_i P_i(x) + u_d(x), U^*\Big) = \mathbf{L}(U^*)$$

One equation , **n** Unknowns:

$$\sum_{i=1}^{n} Q_i \mathbf{K}(P_i(x), U^*) = \mathbf{L}(U^*) - \mathbf{K}(u_d(x), U^*)$$

Gallerkin's method

n equations , **n** Unknowns :

$$U^* = u^*(x) = P_i(x) \in C.A.\{0\} \qquad \forall i \in \{1, \cdots, n\}$$

$$\sum_{i=1}^{n} Q_i \mathbf{K}(P_i(x), U^*) = \mathbf{L}(U^*) - \mathbf{K}(u_d(x), U^*)$$

$$\sum_{i=1}^{n} Q_i \mathbf{K}(P_i(x), P_j(x)) = \mathbf{L}(P_j(x)) - \mathbf{K}(u_d(x), P_j(x))$$

$$\forall j \in \{1, \cdots, n\}$$

Gallerkin's method

n equations , **n** Unknowns :

$$\sum_{i=1}^{n} Q_i \mathbf{K}(P_i(x), P_j(x)) = \mathbf{L}(P_j(x)) - \mathbf{K}(u_d(x), P_j(x))$$

$$\boxed{[\mathbf{K}][Q] = [\mathbf{F}]}$$

$[\mathbf{K}]$ STIFFNESS Matrix

$[Q]$ Unknowns

$[\mathbf{F}]$ Load Vector

Gallerkin's method

$$[\mathbf{K}][Q] = [\mathbf{F}]$$

$$[\mathbf{K}] = [K_{ij}] = [\mathbf{K}(P_i(x), P_j(x))]$$

$$\mathbf{K}(P_m(x), P_n(x)) = -\int_D L_{ijkl}\varepsilon_{kl}^m(x).\varepsilon_{ij}^n(x)d\Omega$$

$$\varepsilon_{kl}^m(x) = \frac{1}{2}(\nabla U_m + \nabla^T U_m)$$

$\boxed{[\mathbf{K}] \text{ is symmetrical}}$

Property of a symmetrical matrix : https://math.univ-cotedazur.fr/~lemahieu/Fond3/amphi5.pdf

Gallerkin's method

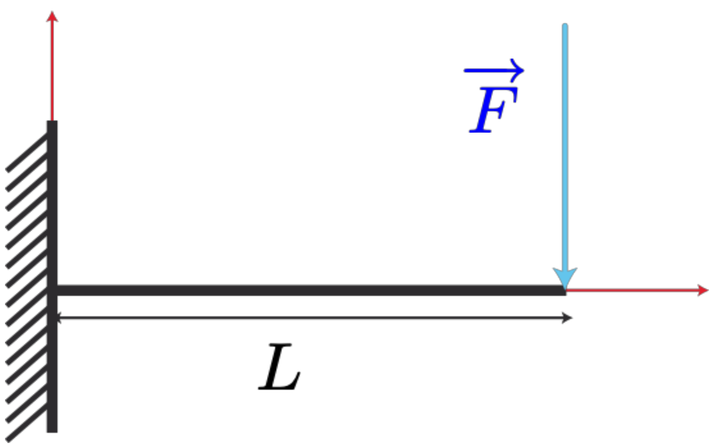
$$F_i = \mathbf{L}(P_i(x)) - [\mathbf{K}(u_d(x), P_i(x))]$$
$$= \int_D f(x).P_i(x)d\Omega + \int_{D_F} F(x).P_i(x)dS - \int_D L_{mnkl}\varepsilon^d_{kl}(x).\varepsilon^i_{mn}(x)d\Omega$$

$$[\mathbf{F}] = [F_i]$$

At the end, You have to solve

$$\boxed{[\mathbf{K}][Q] = [\mathbf{F}]}$$

Example 1-D



hpp $\underline{\underline{[\varepsilon]}} = \frac{1}{2}\left(\nabla U + \nabla U^T\right)$

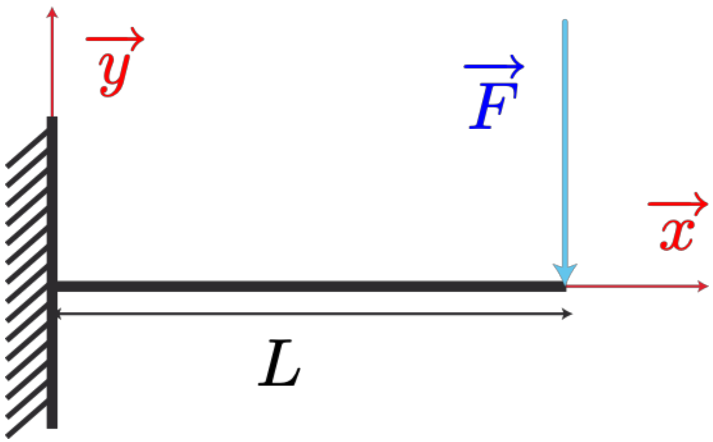
Euler-Bernouilli

$$U(x,y) = \begin{cases} u(x) - (y - y_G)v'(x) \\ v(x) \end{cases}$$

$$\mathbf{K}(U,U^*) = \int_D L_{ijkl}\varepsilon_{ij}\varepsilon^*_{kl}d\omega$$

$$= \int_O^L ESu'(x)u'^*(x) + EIv''(x)v''^*(x)dx$$

Example 1-D   Bending problem in hpp in a plan $u(x) = 0$



$$\mathbf{K}(U, U^*) = \int_O^L EIv''(x)v''^*(x)dx$$

$$\mathbf{L}(U^*) = -Fv^*(L)$$

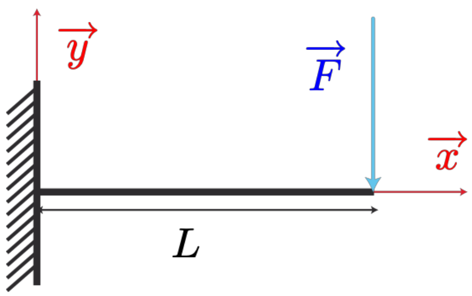
$$\begin{cases} v(0) = 0 \\ v'(0) = 0 \end{cases}$$

$$\{v(x), v^*(x)\} \in C.A\{0\}$$

2 shape or basis functions

$$\begin{cases} P_1(x) = cos(\frac{\pi x}{L}) - 1 \\ P_2(x) = x^2 \end{cases}$$

Example 1-D



$$v^*(x) \in \{P_1(x), P_2(x)\}$$

2 shape or basis functions

$$v(x) = Q_1 P_1(x) + Q_2 P_2(x)$$

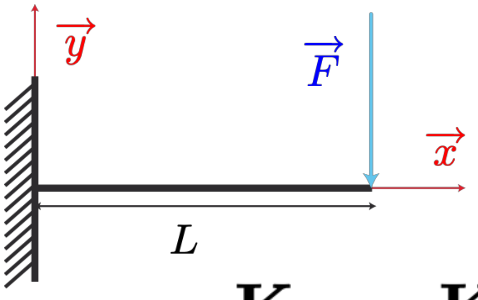
=> 2 unknowns

$$[\mathbf{K}] = [K_{ij}] = [\mathbf{K}(P_i(x), P_j(x))]$$

$$\begin{cases} P_1(x) = cos(\frac{\pi x}{L}) - 1 \\ P_2(x) = x^2 \end{cases}$$

Example 1-D

$$P_1(x) = cos(\frac{\pi x}{L}) - 1 \qquad P_2(x) = x^2$$



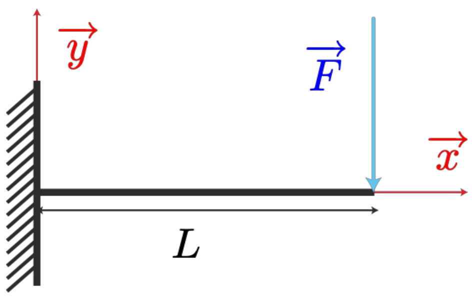
$$[\mathbf{K}] = [K_{ij}] = [\mathbf{K}(P_i(x), P_j(x))]$$

$$\mathbf{K}_{11} = \mathbf{K}(P_1(x), P_1(x)) = \int_O^L EIP_1"(x)P_1"(x)dx$$

$$\mathbf{K}_{12} = \mathbf{K}(P_1(x), P_2(x)) = \int_O^L EIP_1"(x)P_2"(x)dx$$

$$\mathbf{K}_{22} = \mathbf{K}(P_2(x), P_2(x)) = \int_O^L EIP_2"(x)P_2"(x)dx$$

Example 1-D



$$P_1(x) = cos(\frac{\pi x}{L}) - 1 \qquad P_2(x) = x^2$$

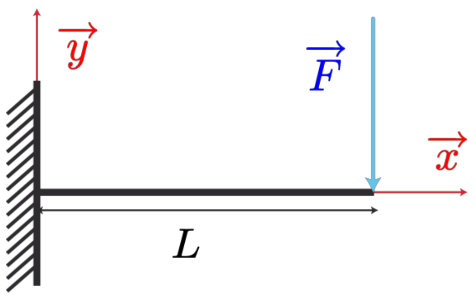
$$[\mathbf{F}] = [F_i]$$

$$F_1 = -FP_1(L) = 2F$$

$$F_2 = -FP_2(L) = -FL^2$$

Example 1-D

$$P_1(x) = cos(\frac{\pi x}{L}) - 1 \qquad P_2(x) = x^2$$



$$F_1 = -FP_1(L) = 2F$$

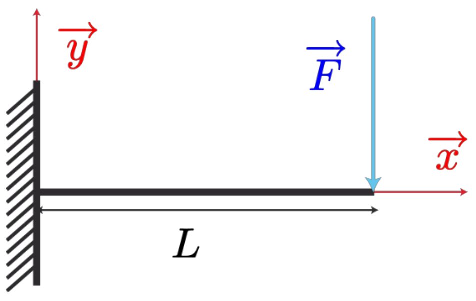
$$F_2 = -FP_2(L) = -FL^2$$

$$\left[\begin{array}{cc} EI\frac{1}{2}\frac{\Pi^4}{L^3} & 0 \\ 0 & 4LEI \end{array}\right] \left[\begin{array}{c} Q_1 \\ Q_2 \end{array}\right] = \left[\begin{array}{c} 2F \\ -FL^2 \end{array}\right]$$

$$\left[\begin{array}{c} Q_1 \\ Q_2 \end{array}\right] = \left[\begin{array}{c} 4\frac{FL^3}{EI\Pi^4} \\ \frac{-FL}{4EI} \end{array}\right]$$

Example 1-D

$$P_1(x) = cos(\frac{\pi x}{L}) - 1 \qquad P_2(x) = x^2$$



$$v(x) = Q_1(cos(\frac{\pi x}{L}) - 1) + Q_2 x^2$$

$$v(x) = 4\frac{FL^3}{\pi^4 EI}(cos(\frac{\pi x}{L}) - 1) - \frac{FL}{4EI}x^2$$

$$v(L) = -\frac{FL^3}{EI}\left(\frac{8}{\pi^4} + \frac{1}{4}\right)$$
$$= -\frac{FL^3}{EI}(0.3321278580)$$

Theoretical value

$$v(L) = -\frac{FL^3}{3EI}$$

Ritz's method is adapted to the variational formulation resulting from of the total potential energy

Principle: We give ourselves n shape or basis functions {Pi(x)} belonging to the kinematically admissible to zero and we look for the solution of Pb(I) as a linear combination of these functions, in the case where there are imposed displacements we add a function realizing them.

$$u(x) = \sum_{i=1}^{n} Q_i P_i(x) + u_d(x) \in C.A.$$

$$\delta u(x) = \sum_{i=1}^{n} \delta Q_i P_i(x) \in C.A.\{0\}$$

Total Potential energy

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$$\boxed{\Pi_p = \frac{1}{2}\mathbf{K}\left(u\left(x\right), u\left(x\right)\right) - \mathbf{L}\left(u\left(x\right)\right)}$$

$$\delta\Pi_p = 0$$

$$\delta\left(\tfrac{1}{2}\mathbf{K}\left(u\left(x\right), u\left(x\right)\right) - \mathbf{L}\left(u\left(x\right)\right)\right) = 0$$

$$\tfrac{1}{2}\left(\mathbf{K}\left(\delta u\left(x\right), u\left(x\right)\right) + \mathbf{K}\left(u\left(x\right), \delta u\left(x\right)\right)\right) - \mathbf{L}\left(\delta u\left(x\right)\right) = 0$$

$$\mathbf{K}\left(u\left(x\right), \delta u\left(x\right)\right) - \mathbf{L}\left(\delta u\left(x\right)\right) = 0$$

Ritz's method

as $\mathbf{K}(u,u)$ is a bilinear symetrical form

$$\begin{aligned}
\delta\Pi_p = \quad & \mathbf{K}\left(\sum_{i=1}^{n} Q_i P_i(x) + u_d(x), \sum_{i=1}^{n} \delta Q_i P_i(x)\right) - \mathbf{L}\left(\sum_{i=1}^{n} \delta Q_i P_i(x)\right) = 0 \quad \forall \delta\,[Q] \\
& \mathbf{K}\left(\sum_{i=1}^{n} Q_i P_i(x), \sum_{i=1}^{n} \delta Q_i P_i(x)\right) + \\
& \mathbf{K}\left(u_d(x), \sum_{i=1}^{n} \delta Q_i P_i(x)\right) - \mathbf{L}\left(\sum_{i=1}^{n} \delta Q_i P_i(x)\right) = 0 \qquad \forall \delta\,[Q]
\end{aligned}$$

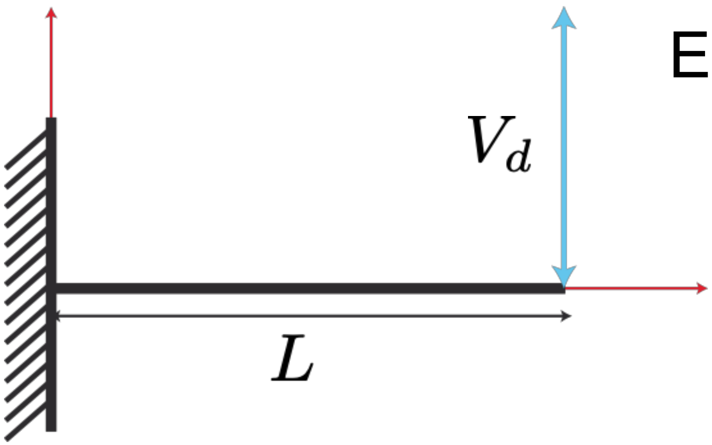
Which can be rewritten in matrix form using the previously intruded notations

$$\begin{aligned}
\delta\Pi_p \quad = \quad & \delta\,[Q]\,[\mathbf{K}]\,[Q] - \delta\,[Q]\,[\mathbf{F}] = 0 \quad \forall \delta\,[Q] \\
& \delta\,[Q]\left(\,[\mathbf{K}]\,[Q] - [\mathbf{F}]\,\right) = 0 \quad \forall \delta\,[Q]
\end{aligned}$$

$$\text{et donc} \qquad [\mathbf{K}]\,[Q] = [\mathbf{F}] \;\; \text{où} \left\{ \begin{array}{lcl} \mathbf{K}_{ij} & = & \mathbf{K}(P_i(x), P_j(x)) \\ \mathbf{F}_i & = & \mathbf{L}(P_i(x)) - \mathbf{K}(ud(x), P_i(x)) \end{array} \right.$$

Example 1-D

hpp $\underline{\underline{[\varepsilon]}} = \frac{1}{2}\left(\nabla U + \nabla U^T\right)$



Euler-Bernouilli

$$U(x,y) = \begin{cases} u(x) - (y - y_G)v'(x) \\ v(x) \end{cases}$$

1 shape or basis function

$$\begin{cases} v(0) = 0 \\ v'(0) = 0 \end{cases} \qquad v(L) = V_d$$

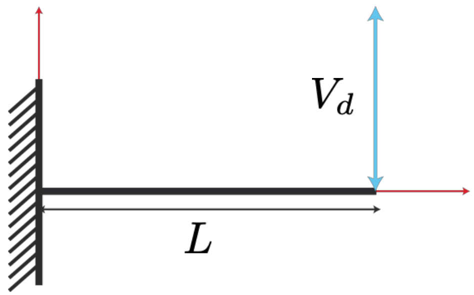
$$v_1(x) = (cos(\frac{\pi x}{L}) - 1)$$

OU

$$v_1(x) = (cos(\frac{2\pi x}{L}) - 1)$$

$$v(x) = Q_1\, v_1(x) + \frac{x^2}{L^2} V_d$$

Example 1-D



$$v(x) = Q_1\, v_1(x) + \frac{x^2}{L^2} V_d$$

$$\delta v(x) = \delta Q_1 v_1(x)$$

$$\Pi_p = \frac{1}{2}\mathbf{K}\left(u\left(x\right), u\left(x\right)\right)$$

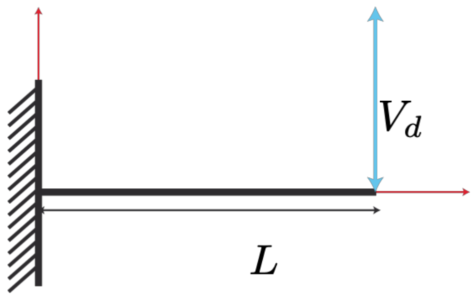
$$\Pi_d = \frac{1}{2}\int_O^L EIv"(x)v"(x)dx$$

$$\delta\Pi_p = 0$$

$$\frac{\partial \Pi_p}{\partial Q_i} = 0$$

Example 1-D

$$\Pi_d = \frac{1}{2}\int_O^L EIv"(x)v"(x)dx$$



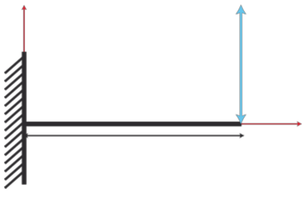
$$\begin{aligned} \delta\Pi_p &= 0 \\ \delta\left(\tfrac{1}{2}\mathbf{K}\left(v\left(x\right), v\left(x\right)\right)\right) &= 0 \end{aligned}$$

$$v(x) = Q_1(cos(\frac{2\,\pi x}{L}) - 1) + \frac{x^2}{L^2}V_d$$

$$v"(x) = -Q_1(\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L}) + 2\frac{V_d}{L^2}$$

$$\delta v"(x) = -\delta Q_1(\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L})$$

Example 1-D



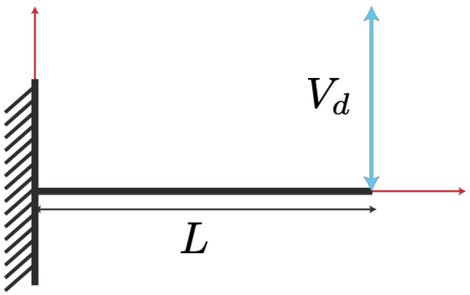
$$\begin{aligned}\delta\Pi_p &= 0\\ \delta\left(\frac{1}{2}\mathbf{K}\left(v\left(x\right),v\left(x\right)\right)\right) &= 0\end{aligned}$$

$$\mathbf{K}\left(v\left(x\right),\delta v\left(x\right)\right)=\int_{O}^{L} EIv"(x)\delta v"(x)dx=0$$

$$\int_{O}^{L} EI\left(-Q_1(\frac{2\,\pi}{L})^2cos(\frac{2\,\pi x}{L})+2\frac{V_d}{L^2}\right)\left(-\delta Q_1(\frac{2\,\pi}{L})^2cos(\frac{2\,\pi x}{L})\right)dx=0 \qquad \forall\delta Q_1$$

$$\delta Q_1\left(\int_{O}^{L} EI\left(-Q_1(\frac{2\,\pi}{L})^2cos(\frac{2\,\pi x}{L})+2\frac{V_d}{L^2}\right)\left(-(\frac{2\,\pi}{L})^2cos(\frac{2\,\pi x}{L})\right)dx\right)=0 \qquad \forall\delta Q_1$$

Example 1-D



$$
\begin{aligned}
\delta \Pi_p &= 0 \\
\delta \left(\tfrac{1}{2} \mathbf{K} \left(v(x), v(x) \right) \right) &= 0
\end{aligned}
$$

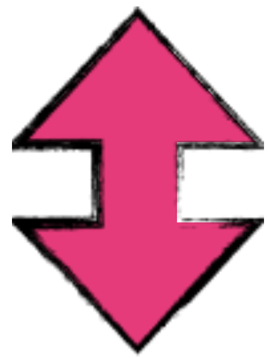
$$v(x) = Q_1\, v_1(x) + \frac{x^2}{L^2} V_d$$

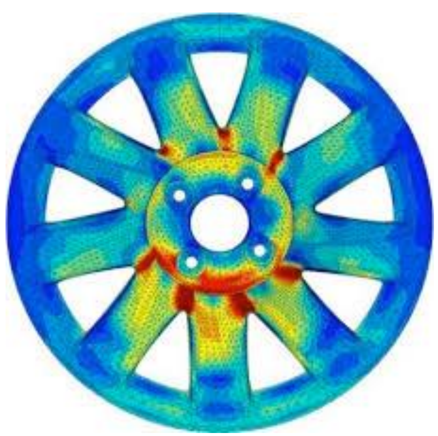
$$Q_1 = \frac{\int_O^L 2 \frac{V_d}{L^2} (\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L}) dx}{\int_O^L EI \left((\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L}) \right)^2 dx}$$

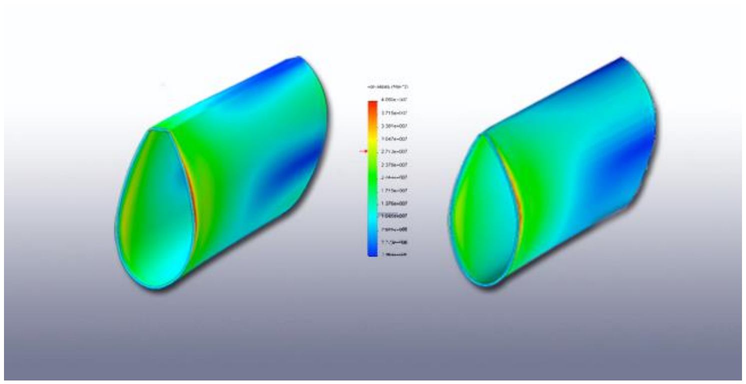
$$v(x) = Q_1\, v_1(x) + \frac{x^3}{6\,L^3} V_d$$

$$Q_1 = \frac{\int_O^L x \frac{V_d}{L^3} (\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L}) dx}{\int_O^L EI \left((\frac{2\,\pi}{L})^2 cos(\frac{2\,\pi x}{L}) \right)^2 dx}$$

$$[\mathbf{K}][Q] = [\mathbf{F}]$$







It is clear that in order to solve the

$$[\mathbf{K}][Q] = [\mathbf{F}]$$

, there are integrations to be made.

Numérical Integration

Newton-Cotes method

Gauss's method

GAUSS's method integrates exactly a polynomial of order 2n-1 with n points.

$$I = \int_{-1}^{1} (ax+b)\,dx = 2b = w_1 f\left(x_1\right) + w_2 f\left(x_2\right)$$

$$\Leftrightarrow \begin{cases} w_1 + w_2 = 2 \\ x_1 w_1 + x_2 w_2 = 0 \Rightarrow \text{deux types de solutions} \\ x_1 = -x_2 \end{cases}$$

$$\begin{cases} w_1 = w_2 = 1 \\ x_1 = -x_2 \end{cases} \quad \text{ou} \quad \begin{cases} w_1 = 2 \\ x_1 = -x_2 = 0 \end{cases}$$

GAUSS's method integrates exactly a polynomial of order 2n-1 with n points.

$$I = \int_{-1}^{1} (ax^3 + bx^2 + cx + d)\,dx = \frac{2}{3}b + 2d = w_1 f\left(x_1\right) + w_2 f\left(x_2\right)$$

$$\Leftrightarrow \begin{cases} w_1 + w_2 = 2 \\ (w_1 - w_2)x_1 = 0 \\ x_1^2(w_1 + w_2) = \dfrac{2}{3} \\ (w_1 - w_2)x_1^3 = 0 \end{cases} \Rightarrow \begin{cases} w_1 = w_2 = 1 \\ x_1 = -x_2 = \dfrac{1}{\sqrt{3}} \end{cases}$$

Extrait Doc ANSYS

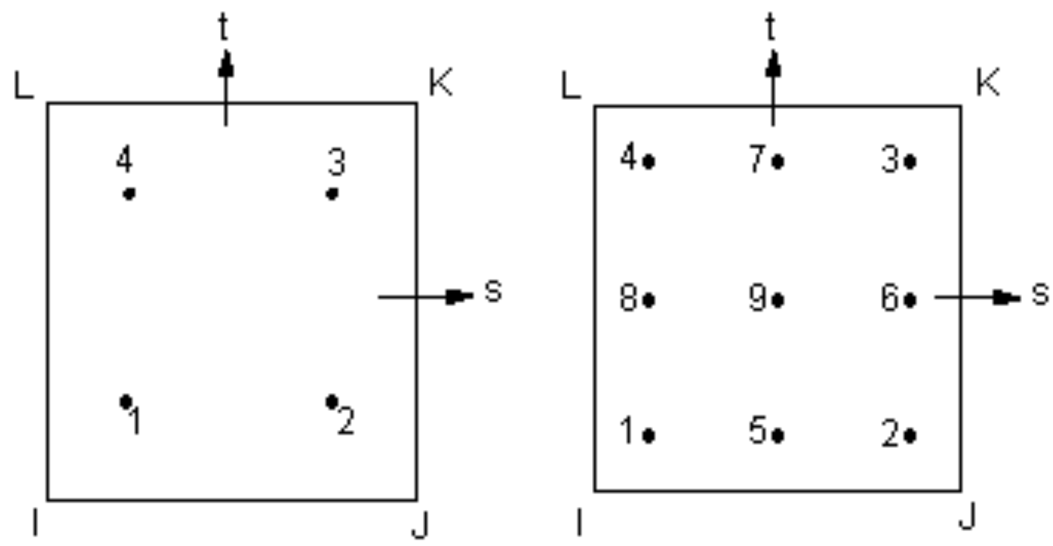
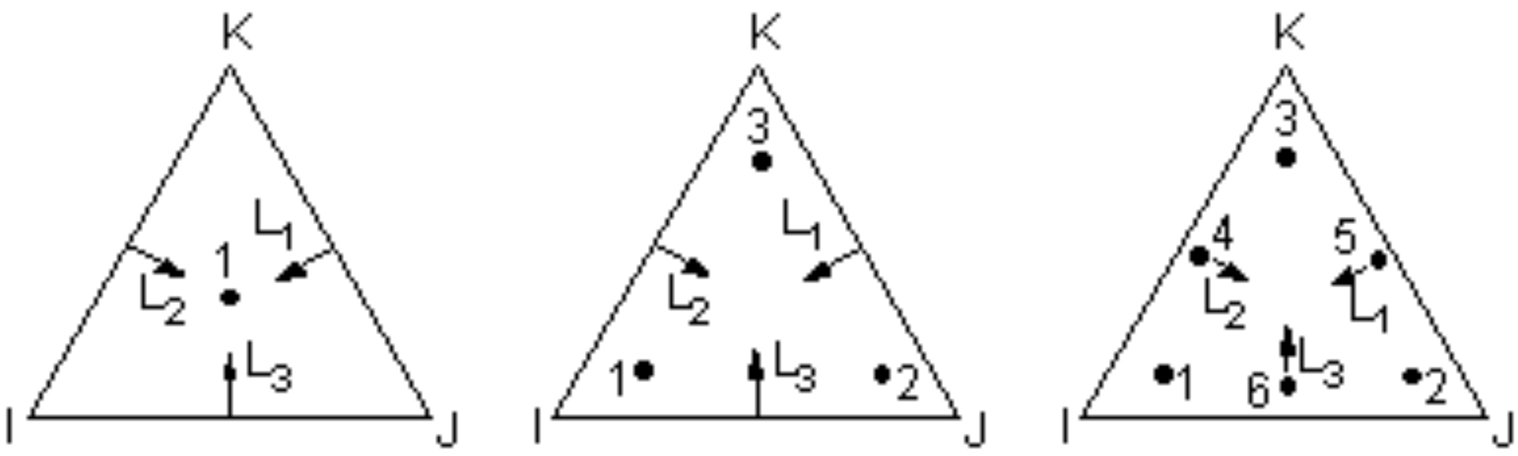


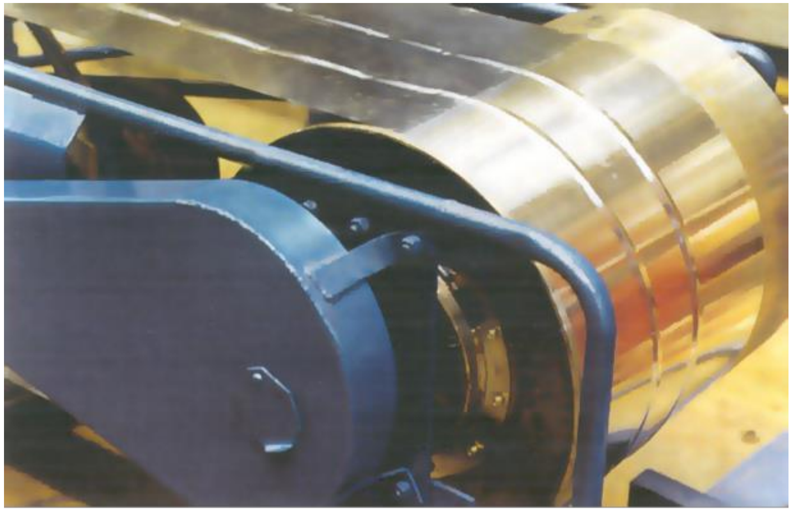
Figure 13.1 Integration Point Locations for Quadrilaterals

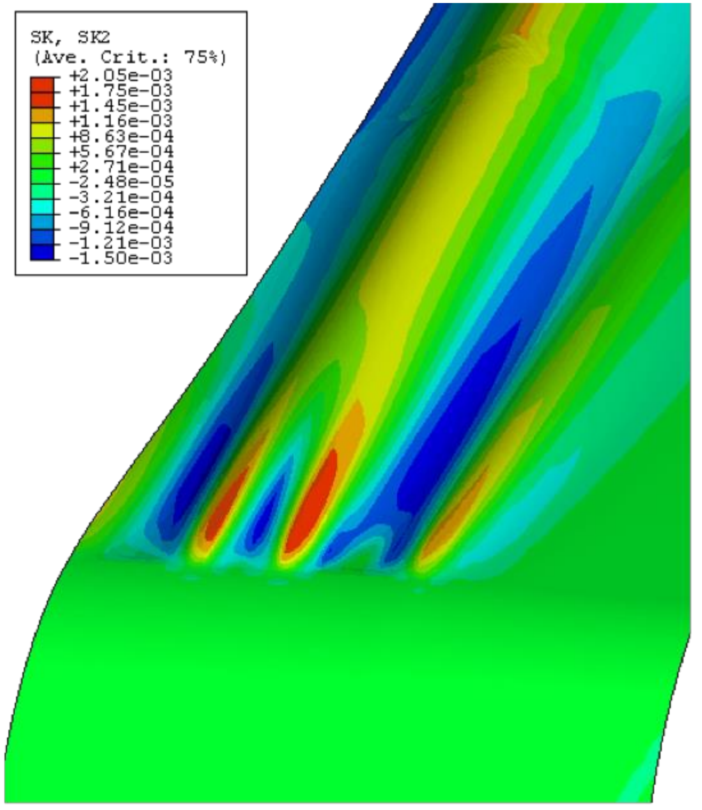
One element models with midside nodes (e.g., PLANE82) using a 2 x 2 mesh of integration points have been seen to generate spurious zero energy (hourglassing) modes.

Excerpt from ANSYS documentation

The integration points used for these triangles are given in Table 13.2: "Numerical Integration for Triangles" and appear as shown in Figure 13.3: "Integration Point Locations for Triangles". L varies from 0.0 at an edge to 1.0 at the opposite vertex.







Folding of sheet metal

Courbure dans le sens travers

Principle of RESOLVING FE

Going from a continuous to discrete problem (Make the mesh)

Determine the equations to be solved (Choose the formulation)

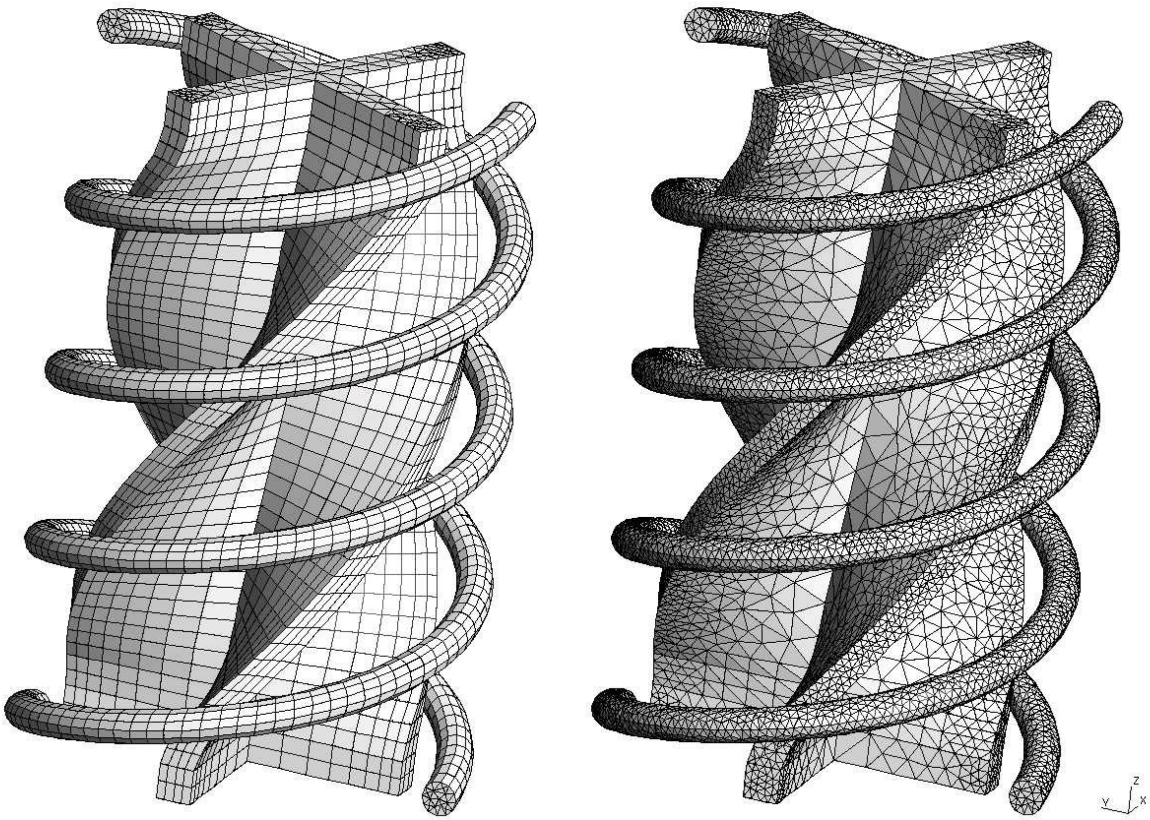
Determine the unknowns of the problem (Choose the element)

Solve the problem (Choose the numerical method)

Analyze the results (Post-process files)

Principe de résolution des EF

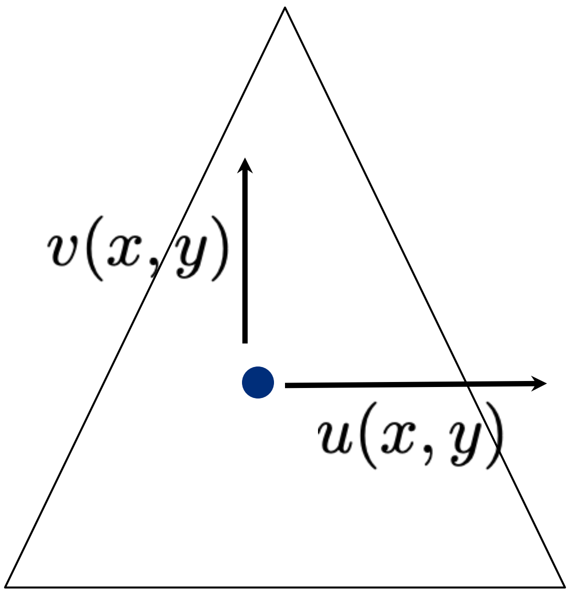
Passer d'un problème continu à discret (faire le maillage)



(http ://www.geuz.org/gmsh/)

Approximation des déplacements 2D

$$\begin{cases} u(x,y) = a_1 + a_2 x + a_3 y \\ v(x,y) = a_4 + a_5 x + a_6 y \end{cases}$$

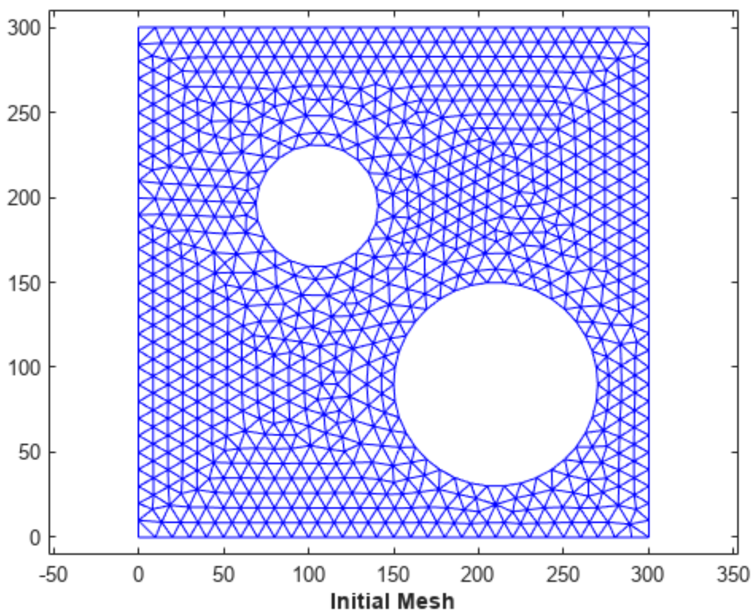


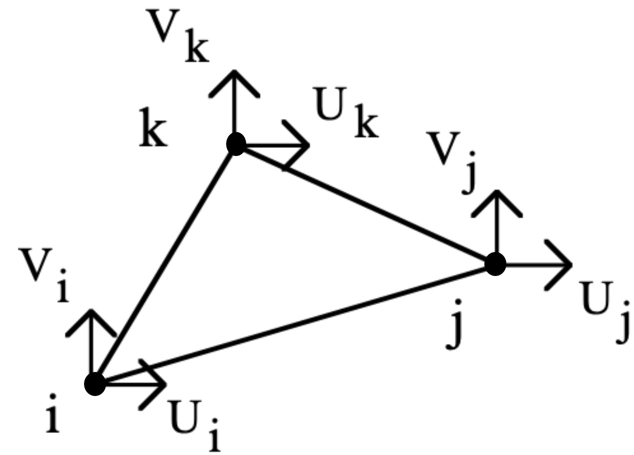
$$[U] = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

$$[U]_e = [P(x,y)][a]_e$$

$[a]_e$ Pas très physique

Approximation nodale 2D



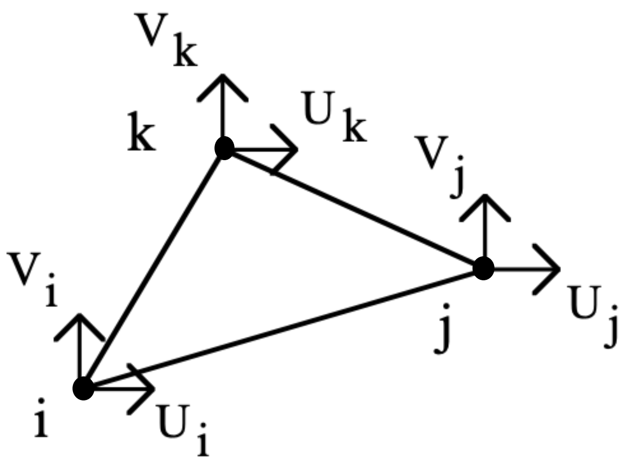


(i, j, k) sont les noeuds

Inconnues NODALES (ddl)

$$[Q] = [q_i] = \begin{bmatrix} U_i & U_j & U_k & V_i & V_j & V_k \end{bmatrix}$$

Approximation nodale 2D



Inconnues NODALES (ddl)

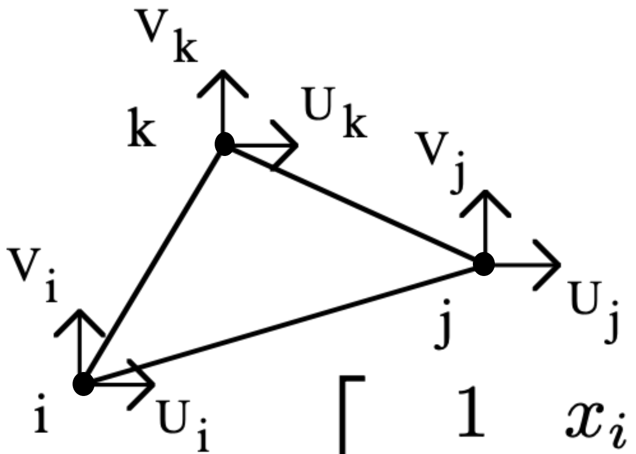
$$[Q] = [q_i] = \begin{bmatrix} U_i & U_j & U_k & V_i & V_j & V_k \end{bmatrix}$$

Déplacement sur l'élément

$$\begin{cases} u(x,y) = a_1 + a_2 x + a_3 y \\ v(x,y) = a_4 + a_5 x + a_6 y \end{cases}$$

$$[U] = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = [P(x,y)][a]_e$$

Approximation nodale 2D

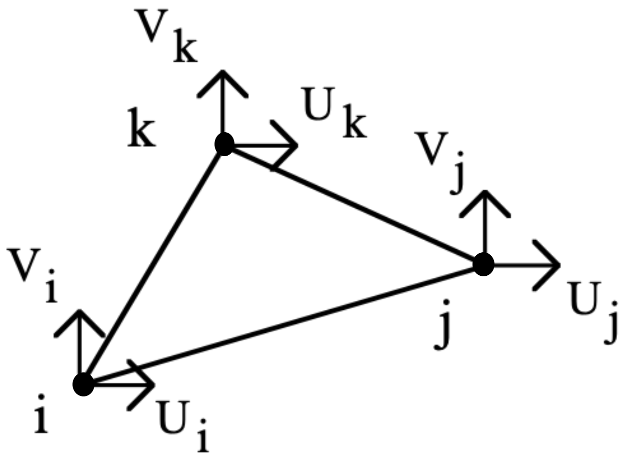


Inconnues NODALES (ddl)

$$[Q] = [q_i] = \begin{bmatrix} U_i & U_j & U_k & V_i & V_j & V_k \end{bmatrix}$$

$$^t[Q] = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 1 & x_k & y_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 0 & 0 & 0 & 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

Approximation nodale 2D



Inconnues NODALES (ddl)

$$[Q]=[q_i]=\begin{bmatrix}U_i & U_j & U_k & V_i & V_j & V_k\end{bmatrix}$$

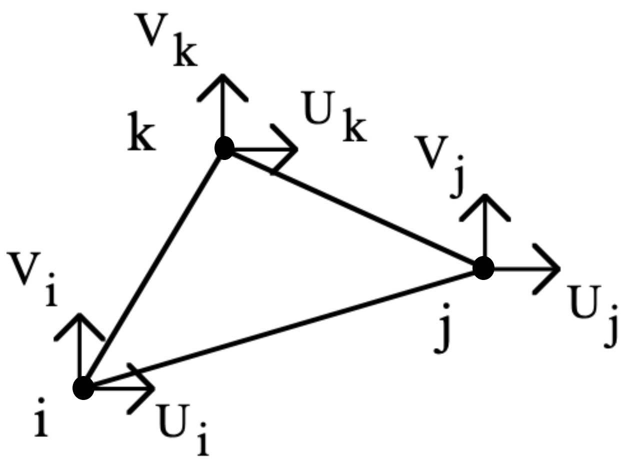
$$^{t}[Q]=\left[\ P_{node}\ \right][a]_e$$

Approximation des déplacements

Approximation nodale 2D

Inconnues NODALES (ddl)

$$[Q] = [q_i] = \begin{bmatrix} U_i & U_j & U_k & V_i & V_j & V_k \end{bmatrix}$$



$$
\begin{aligned}
[U]_e &= [P(x,y)]\,[P_n]^{-1}\,{}^t[Q]_e \\
&= [N(x,y)]\,{}^t[Q]_e \\
&= \begin{bmatrix} N_x \\ Ny \end{bmatrix}{}^t[Q]_e
\end{aligned}
$$

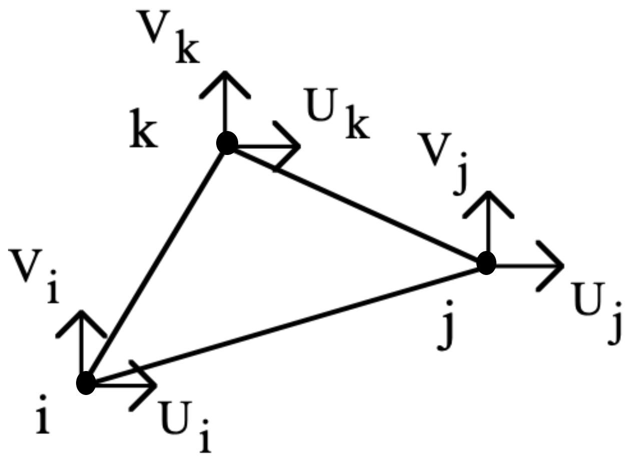
$[U]_e$ = Déplacement dans l'élément

$[N(x,y)]$ = Fonctions de forme

Approximation nodale 2D hpp Déformation plane

$$
\begin{aligned}
[\varepsilon_{\alpha\beta}] &= \begin{bmatrix} \dfrac{\partial u(x,y)}{\partial x} \\ \dfrac{\partial v(x,y)}{\partial y} \\ \dfrac{1}{2}\left(\dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial v(x,y)}{\partial x}\right) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial u(x,y)}{\partial x} \\ \dfrac{\partial u(x,y)}{\partial y} \\ \dfrac{\partial v(x,y)}{\partial x} \\ \dfrac{\partial v(x,y)}{\partial y} \end{bmatrix}
\end{aligned}
$$

Approximation nodale 2D hpp Déformation plane



$$[\varepsilon_{\alpha\beta}] \quad = [C] \begin{bmatrix} N_{x,x} \\ N_{x,y} \\ N_{y,x} \\ N_{y,y} \end{bmatrix}^t [Q]_e$$

$$= [B]_e{}^t\, [Q]_e$$

Approximation nodale 2D hpp Déformation plane

$$\sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\lambda = \frac{\nu\, E}{(1-2\nu)(1+\nu)}$$

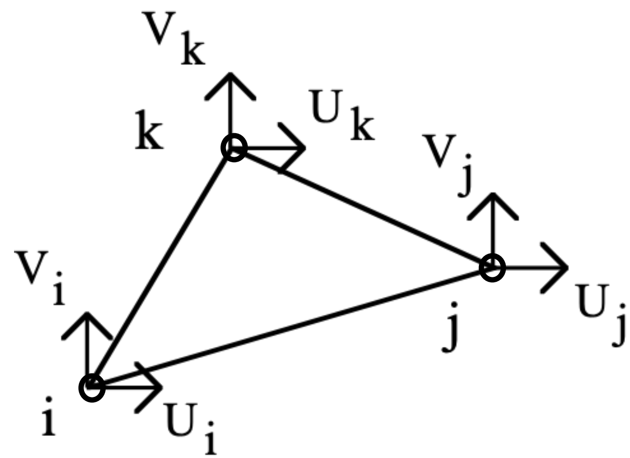
$$\mu = \frac{E}{2(1+\nu)} = G$$

$$[\sigma_{\alpha\beta}]_e = [D]\, [\, \varepsilon_{\alpha\beta}]_e$$

$$= [D]\, [B]_e{}^t\, [Q]_e$$

$$[D] = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix}$$

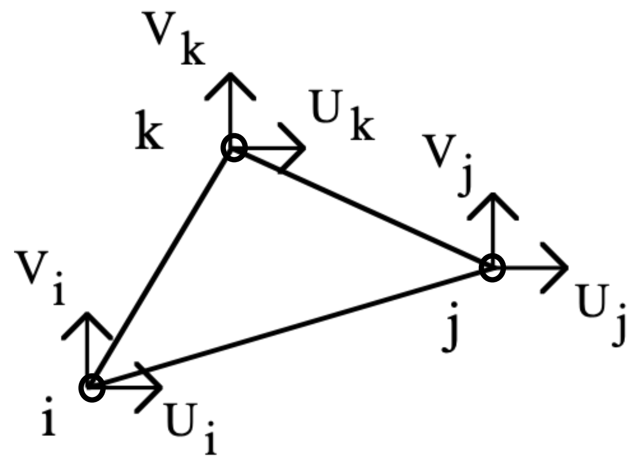
Approximation de l'énergide de Déf. dans un élément



$$
\begin{aligned}
W_e &= \frac{1}{2}\int_e \varepsilon L \varepsilon d\omega_e \\
&= \frac{1}{2}\int_e {}^t\left[\varepsilon\right]_e \left[\sigma\right]_e d\omega_e \\
&= \frac{1}{2}\int_e {}^t\left[\varepsilon\right]_e \left[D\right] \left[\varepsilon\right]_e d\omega_e \\
&= \frac{1}{2}\int_e {}^t\left[Q\right]_e {}^t\left[\mathbf{B}\right]_e \left[D\right] \left[B\right]_e \left[Q\right]_e d\omega_e \\
&= \frac{1}{2} {}^t\left[Q\right]_e \int_e {}^t\left[\mathbf{B}\right]_e \left[D\right] \left[B\right]_e d\omega_e \left[Q\right]_e \\
&= \frac{1}{2} {}^t\left[Q\right]_e \left[K\right]_e \left[Q\right]_e
\end{aligned}
$$

Approximation de l’énergide de Déf. dans un élément

Matrice de rigidité élémentaire



$$[K]_e = \int_e {}^t\,[\mathbf{B}]_e\,[D]\,[B]_e\,d\omega_e$$

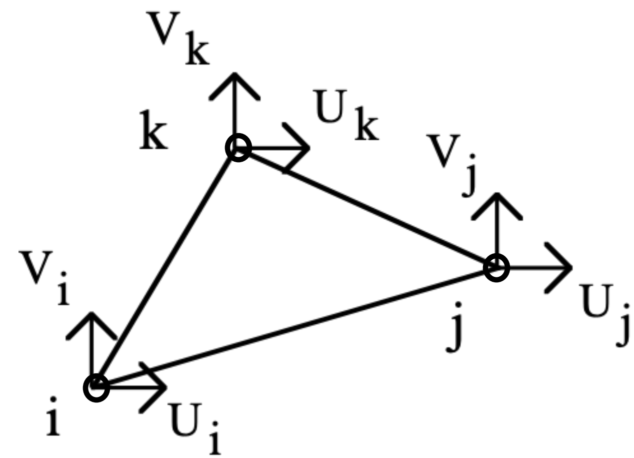
$[K]_e$ est symétrique positive

$[K]_e$ a trois Valeurs propres nulles et trois positives

Approximation de l’énergide de Déf. dans un élément

Matrice de rigidité élémentaire

$$[K]_e = \int_e {}^t\,[\mathbf{B}]_e\,[D]\,[B]_e\,d\omega_e$$



Energie de déformation élémentaire

$$W_e = \frac{1}{2}\,{}^t\,[Q]_e\,[K]_e\,[Q]_e$$

Rq: Dans le cas des interpolations linéaires l'énergie de déformation élémentaire est directement proportionnelle à la surface de l'élément considéré.

Cas d'un élément qui est en contact avec la frontière où les efforts volumique donnés sont non-nuls

$$\begin{aligned}
\mathrm{P_{d_e}(u)} &= \int_{\partial e} F_d u ds_e \\
&= \int_{\partial e} {}^t\left[F_d(x,y)\right]\left[U\right]_e ds_e \\
&= \int_{\partial e} {}^t\left[F_d(x,y)\right]\left[N\right] ds_e \left[Q\right]_e \\
&= \langle \mathrm{F_{d_e}}, \mathrm{Q} \rangle \\
&= {}^t\left[\mathrm{F_{d_e}}\right] \cdot \left[Q\right]_e
\end{aligned}$$

Énergie de Déformation élémentaire

$$\begin{aligned} \Pi_{de} &= \frac{1}{2}\; K_e\left(Q_e, Q_e\right) - \langle F_e, Q_e\rangle \\ &= \frac{1}{2}\; {}^t[Q]_e[K]_e[Q]_e - \langle F_e, Q_e\rangle \end{aligned}$$

Assemblage matrice de rigidité

$$W = \sum_e W_e = \frac{1}{2}\sum_e {}^t[Q]_e[K]_e[Q]_e$$

Matrice de rigidité totale

$$[K]_T = \sum_{\substack{e \\ au\ sens\ de\ l'assemblage}} [K]_e$$

$$u(x) \in C.A\{u_d\} / \delta(\Pi_d(u)) = 0 \text{ est solution du problème}$$

$$\Pi_d = \frac{1}{2}{}^t[Q]\Big[K\Big]_t[Q] - \langle \mathrm{F}_t, \mathrm{Q} \rangle$$

$$\begin{aligned}
\delta\left(\Pi_d\right) &= \delta\left(\tfrac{1}{2}{}^t\left[Q\right]\left[K\right]_t\left[Q\right] - \langle \mathrm{F}_t, \mathrm{Q} \rangle\right) \\
&= \tfrac{1}{2}\delta\left({}^t\left[Q\right]\right)\left[K\right]_t\left[Q\right] + \tfrac{1}{2}{}^t\left[Q\right]\left[K\right]_t\delta\left(\left[Q\right]\right) - \delta\left({}^t\left[Q\right]\right)\left[F\right]_t \\
&= \tfrac{1}{2}\delta\left({}^t\left[Q\right]\right)\left[K\right]_t\left[Q\right] + \tfrac{1}{2}{}^t\left({}^t\left[Q\right]\left[K\right]_t\delta\left(\left[Q\right]\right)\right) - \delta\left({}^t\left[Q\right]\right)\left[F\right]_t \\
&= \tfrac{1}{2}\delta\left({}^t\left[Q\right]\right)\left[K\right]_t\left[Q\right] + \tfrac{1}{2}\delta\left({}^t\left[Q\right]\right){}^t\left[K\right]_t\left[Q\right] - \delta\left({}^t\left[Q\right]\right)\left[F\right]_t \\
&= \delta\left({}^t\left[Q\right]\right)\left[K\right]_t\left[Q\right] - \delta\left({}^t\left[Q\right]\right)\left[F\right]_t
\end{aligned}$$

Variation de l’énergie potentielle

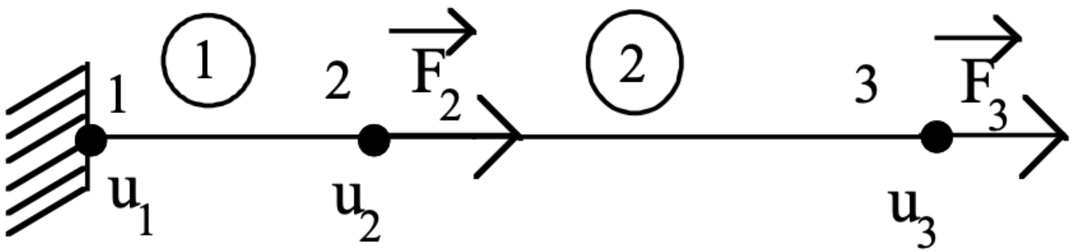
$u(x) \in C.A\{u_d\} / \delta(\Pi_d(u)) = 0$ est solution du problème

$$\Pi_d = \frac{1}{2}\,{}^t[Q]\big[K\big]_t[Q] - \langle \mathrm{F}_t, \mathrm{Q} \rangle$$

$$\delta\,(\Pi_d) \quad = \delta\,({}^t\,[Q])\,([K]_t\,[Q] - [F]_t)\ \forall\,\delta\left({}^t\,[Q]\right) \in CA\{0\}$$

$$\boxed{[K]_t\,[Q] - [F]_t = 0}$$

Exemple d'assemblage pour une structure simple

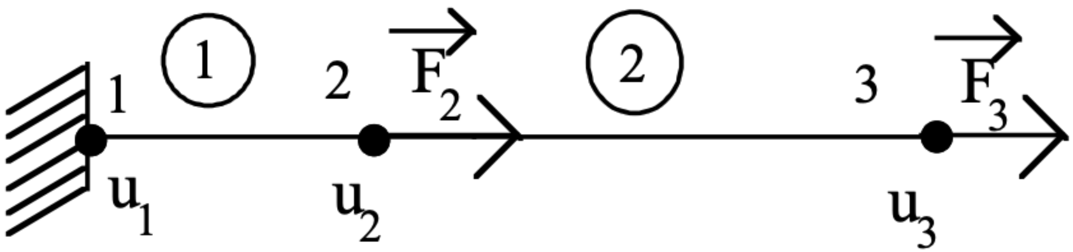


$$W_e = \frac{1}{2}{}^t[Q]_e [K]_e [Q]_e$$

$$[K]_1 = \begin{bmatrix} k_{11}^1 & k_{12}^1 \\ k_{21}^1 & k_{22}^1 \end{bmatrix} \quad \text{et} \quad [Q]_1 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

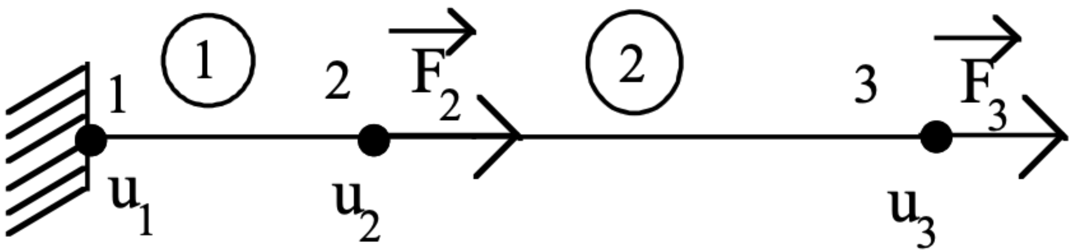
$$[K]_2 = \begin{bmatrix} k_{11}^2 & k_{12}^2 \\ k_{21}^2 & k_{22}^2 \end{bmatrix} \quad \text{et} \quad [Q]_2 = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix}$$

Exemple d'assemblage pour une structure simple



$$[K]_t = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix} \quad \text{et} \quad [Q] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Exemple d'assemblage pour une structure simple



$$\langle F_t, Q \rangle = {}^t[Q][F]_t$$

$$[Q] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ et } [F]_t = \begin{bmatrix} 0 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\langle \mathrm{F}_{\mathrm{d}_e}, \mathrm{Q} \rangle = {}^t\left[\mathrm{F}_{\mathrm{d}_e}\right].[Q]_e$$

Propriété de la Matrice de rigidité élémentaire

Que sont les dof ?

Matrice de comportement

Méthodes d’Intégration numérique

Méthode brutale

$$[K]_t [Q] = [F]_t \qquad q_i = q_d$$

$[K]_t = [\mathrm{kij}]_t$ devient :

$$[K]_t = \begin{bmatrix} k11 & \dots & k1\,(i-1) & k1i & k1\,(i+1) & \dots & k1n \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ k\,(i-1)\,1 & \dots & k\,(i-1)\,(i-1) & k\,(i-1)\,(i) & k\,(i-1)\,(i+1) & \dots & k\,(i-1)\,n \\ 0 & \dots & 0 & kii & 0 & \dots & 0 \\ k\,(i+1)\,1 & \dots & k\,(i+1)\,(i-1) & k\,(i+1)\,(i) & k\,(i+1)\,(i+1) & \dots & k\,(i+1)\,n \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ kn1 & \dots & kn\,(i-1) & kn\,(i) & kn\,(i+1) & \dots & knn \end{bmatrix}$$

Méthode brutale

$$q_i = q_d$$

$$[K]_t [Q] = [F]_t$$

$[F]_t = [\mathrm{fi}]_t$ devient :

$$[F] = \begin{bmatrix} f1 \\ \vdots \\ f(i-1) \\ fi = kii * q_d \\ f(i+1) \\ \vdots \\ fn \end{bmatrix}$$

Méthode de pénalisation

$$q_i = q_d$$

$[K]_t = [\mathrm{kij}]_t$ devient :

$$[K] = \begin{bmatrix} k11 & \cdots & k1n \\ \vdots & kii + k & \vdots \\ kn1 & \cdots & knn \end{bmatrix}$$

$[F]_t = [\mathrm{fi}]_t$ devient

$$[F] = \begin{bmatrix} f1 \\ \vdots \\ f(i-1) \\ fi + k * q_d \\ f(i+1) \\ \vdots \\ fn \end{bmatrix}$$

où $k = 10^{10} * \max(kij)$:

Méthode de pénalisation

$$q_i = q_d$$

$$q_i = q_d \left(\frac{k}{k + kii} - \sum_{\text{l prive de i}} \frac{q_l}{q_d} \frac{kil}{kii + k} + \frac{fi}{(k + kii) q} \right) \cong q_d (1 + \mathrm{e})$$

Méthode Lagrangienne

Cette méthode est la plus élégante car elle consiste à relaxer les r conditions aux limites

$$[R]_{r\times n}[Q]=[S] \qquad q_i = q_d$$

$${}^{t}[\lambda]=[\lambda_1\ldots\lambda_r]$$

Multiplicateur de Lagrange

Méthode Lagrangienne

$$Fv = \frac{1}{2}{}^{t}[Q]\Big[K\Big]_{t}[Q] - \langle \mathrm{F}_{t}, \mathrm{Q}\rangle + {}^{t}\Big[\lambda\Big]\left([R]_{r\times n}[Q]-[S]\right)$$

$$\begin{aligned}\delta\left(FV\right) \quad &= \delta\left(\tfrac{1}{2}{}^{t}\left[Q\right]\left[K\right]_{t}\left[Q\right] - \langle \mathrm{F}_{t}, \mathrm{Q}\rangle + {}^{t}\left[\lambda\right]\left(\left[R\right]_{r\times n}\left[Q\right]-\left[S\right]\right)\right)\\ &= \delta\left({}^{t}\left[Q\right]\right)\left(\left[K\right]_{t}\left[Q\right]-\left[F\right]_{t}\right) + \delta\left({}^{t}\left[\lambda\right]\right)\left(\left[R\right]_{r\times n}\left[Q\right]\right)\\ &\quad +{}^{t}\left[\lambda\right]\left[R\right]_{r\times n}\delta\left(\left[Q\right]\right) - \delta\left({}^{t}\left[\lambda\right]\right)\left[S\right]\end{aligned}$$

$$\delta(FV) \quad = 0 \ \forall \delta\left({}^{t}\left[Q\right]\right), \delta\left({}^{t}\left[\lambda\right]\right) \in C.A\left(0\right)$$

Méthode Lagrangienne

$$Fv = \frac{1}{2}{}^{t}[Q]\left[K\right]_{t}[Q] - \langle \mathrm{F}_{t}, \mathrm{Q} \rangle + {}^{t}\left[\lambda\right]\left([R]_{r\times n}[Q] - [S]\right)$$

$$\delta(FV) \quad = 0 \;\forall\delta\left({}^{t}[Q]\right), \delta\left({}^{t}[\lambda]\right) \in C.A\,(0)$$

$$\begin{bmatrix} [K] & {}^{t}[R] \\ [R] & [0] \end{bmatrix} \begin{bmatrix} Q \\ \lambda \end{bmatrix} - \begin{bmatrix} F \\ S \end{bmatrix} = 0 \quad \forall\delta\left({}^{t}[Q]\right), \delta\left({}^{t}[\lambda]\right) \in C.A\,(0)$$

Méthode Lagrangienne

$$\begin{bmatrix} [K] & {}^{t}[R] \\ [R] & [0] \end{bmatrix} \begin{bmatrix} Q \\ \lambda \end{bmatrix} - \begin{bmatrix} F \\ S \end{bmatrix} = 0 \quad \forall \delta \left({}^{t}[Q] \right), \delta \left({}^{t}[\lambda] \right) \in C.A(0)$$

L'inconvénient principal est d'avoir introduit des inconnues supplémentaires.

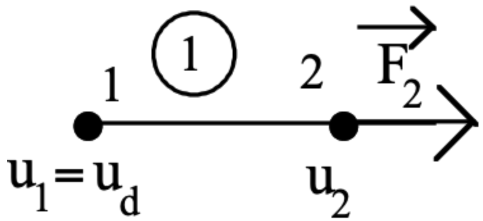
L'avantage c'est que la condition est réalisée exactement tout en conservant la symétrie de la matrice à inverser.

Les inconnues supplémentaires donnent des informations supplémentaires.

Prise en compte des Cdts aux limites

Méthode Lagrangienne

Exemple



$$[R] = [1 \quad 0] \quad [K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad [Q] = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad [F] = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} k & -k & 1 \\ -k & k & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \\ u_d \end{bmatrix}$$

A quelles conditions peut on affirmer que

$$\Pi_{dt} = \sum_{elt} \Pi_e \quad \text{ou} \quad W_t = \sum_{elt} W_e$$

Considérons le cas 1-D (Poutre) Flexion

$$W_{fl} = \frac{EI}{2}\int_{x_1}^{x_2} \left(v''(x)\right)^2 dx = \frac{1}{2}\int_{x_1}^{x_2} M(x)v''(x)dx$$

$$= \frac{1}{2}\left(\left[M(x)v'(x)\right]_{x_1}^{x_2} - \left[M'(x)v(x)\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} M''(x)v(x)dx \right)$$

Assemblons deux éléments, et donc leurs énergies de déformation

$$2W^1{}_{fl} + 2W^2{}_{fl} = \left[M^1(x)v'^1(x)\right]_{x_1}^{x_2} - \left[M'^1(x)v^1(x)\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} M''^1(x)v^1(x)dx$$

$$+\left[M^2(x)v'^2(x)\right]_{x_2}^{x_3} - \left[M'^2(x)v^2(x)\right]_{x_2}^{x_3} + \int_{x2}^{x_3} M''^2(x)v^2(x)dx$$

M(x) et M'(x) sont continus s'il n'existe pas d'efforts ou de moments ponctuels

En regroupant les termes de raccord, on observe que :

$$M^1(x_2)\left(v'^1(x_2) - v'^2(x_2)\right) \quad \text{et} \quad M'^1(x_2)\left(v^1(x_2) - v^2(x_2)\right)$$

$$M^1(x_2)\left(v'^1(x_2)-v'^2(x_2)\right) \quad \text{et} \quad M'^1(x_2)\left(v^1(x_2)-v^2(x_2)\right)$$

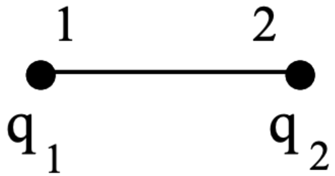
SI v(x) et v'(x) ne sont pas continue il y a stockage d'une énergie finie à chaque raccord.

On demande donc que v(x) soit deux fois intégrables sur l'élément et seulement qu'elle soit à dérivée continue aux extrémités

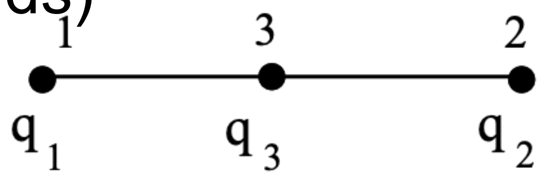
Def : Un élément vérifiant ces propriétés de continuité est dit conforme.

Rq : Dans le cas d'énergie dépendant d'un gradient, élasticité, il suffit que u(x) soit continue.

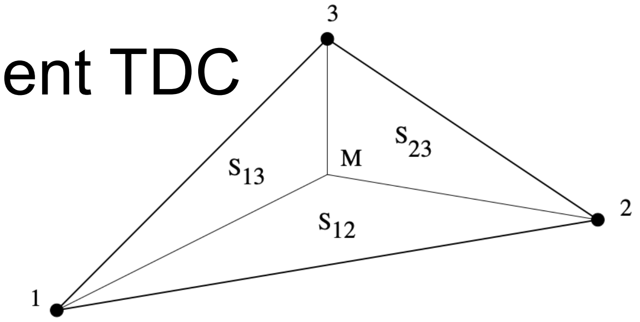
Barre à champ linéaire (2 noeuds)



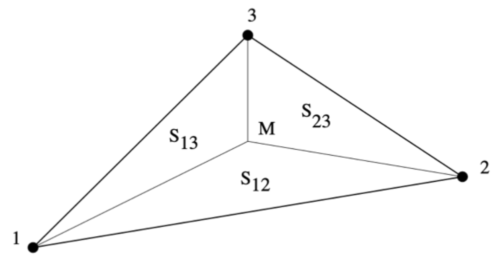
Barre à champ quadratique (3noeuds)

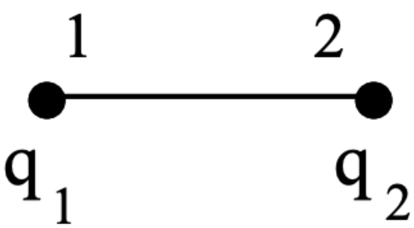


Matrice de rigidité élémentaire de l'élément TDC

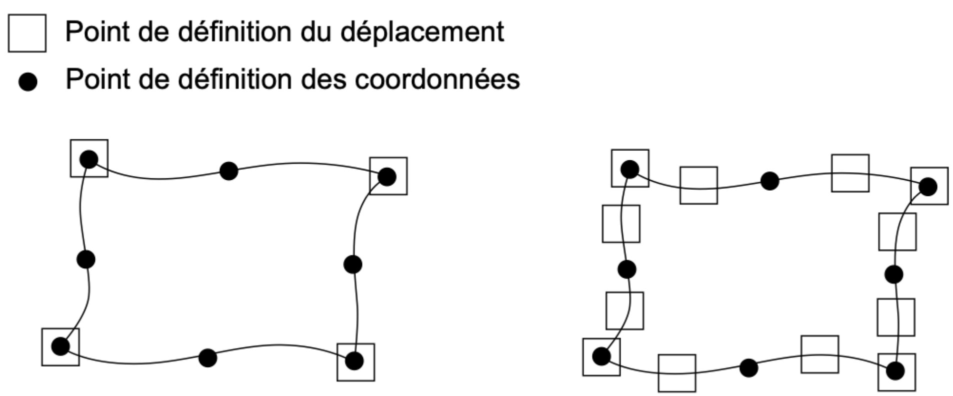


Un élément est dit isoparamétrique si on prend les mêmes fonctions d'interpolation pour le déplacement et la géométrie.





On définit également les éléments super et sous-paramétriques

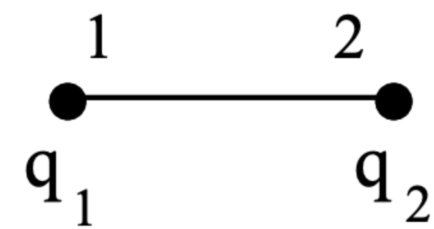


Elément superparamétrique

Elément sous-paramétrique

Elément est utilisé pour traiter les problèmes de traction-compression dans une barre

$$W_e = \frac{1}{2}\int_{x_1}^{x_2} ES\left(u'(x)\right)^2 dx$$



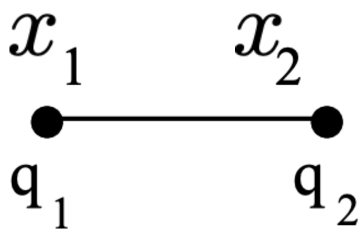
Interpolation

$$u(x) = a_1 + a_2 x$$

$$u(x) = N_1(x)q_1 + N_2(x)q_2 = N_i(x)q_i$$

Elément Barre (2ddl)

Elément réel

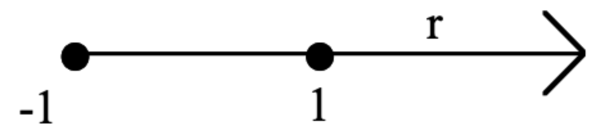


Fonction d'interpolation en déplacement

$$\begin{cases} N_1(x) = 1 - \dfrac{x - x_1}{L} \\ N_2(x_1) = \dfrac{x - x_1}{L} \end{cases}$$

$$u(x) = N_i(x) q_i$$

Elément de référence



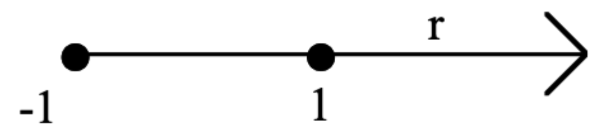
Fonction d'interpolation géométrique

$$\begin{aligned} x &= x_1 \frac{1-r}{2} + x_2 \frac{1+r}{2} \\ &= x_1 G_1(r) + x_2 G_2(r) \end{aligned}$$

Elément Barre (2ddl)

Calcul de la matrice de rigidité élémentaire sur l'Elément de référence

$$W_{ref}^{elem} = \frac{1}{2}\int_{-1}^{1} ES\left(\frac{du(r)}{dr}\right)^2 dr$$



$$W_{ref}^{elem} = \frac{1}{2}\,[Q]^T\,[K]_{ref}\,[Q]$$

$${}^t[B] = \begin{bmatrix} N_{1,r} \\ N_{2,r} \end{bmatrix}$$

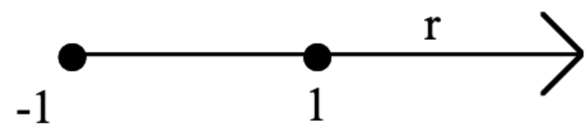
$$[K]_{ref} = \int_{-1}^{1} {}^t[B]\,[D]\,[B]\,dr$$

$$[D] = [ES]$$

Elément Barre (2ddl)

Calcul de la matrice de rigidité sur l'Elément de référence

$$[K]_{ref} = \int_{-1}^{1} {}^{t}[B]\,[D]\,[B]\,dr$$



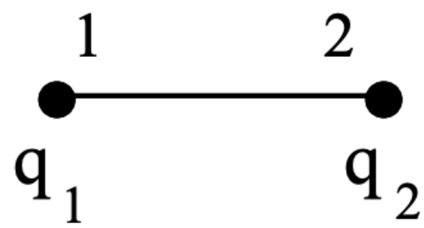
$$N_1(r) = \frac{(1-r)}{2}$$

$$N_2(r) = \frac{(1+r)}{2}$$

$$[K]_{ref} = \int_{-1}^{1} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} [ES] \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} dr$$

$$= ES \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Calcul de la matrice de rigidité sur l'Elément réel



$$[K]_e = \int_{x_1}^{x_2} {}^t[B]_e\,[D]\,[B]_e\,dx$$

$$= \int_{-1}^{1} \frac{2}{L}\,{}^t[B]\,[D]\,\frac{2}{L}\,[B]\,\frac{L}{2}dr$$

$$= \frac{2}{L}\,[K]_{ref}$$

$$\frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx}$$

$$dx = \frac{dx}{dr}dr = \frac{L}{2}dr$$

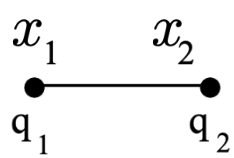
$$[K]_e = \frac{ES}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

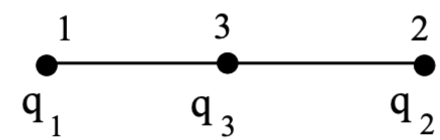
Modes propre de déformation ?

Lagrange - (les plus simples)

Les fonctions de bases locales de l'élément fini de Lagrange (K, Σ, P) à N points sont les N fonctions de P telles que $p_i(a_i) = \delta_{ij} pour 1 \leq i, j \leq N$

Éléments finis unidimensionnels

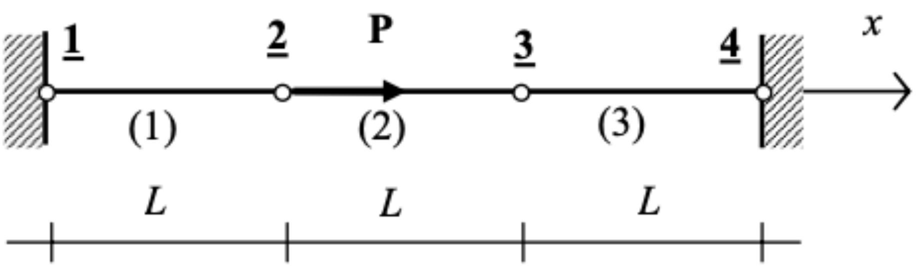




Elément réel

Élément	P_1	P_2	...	P_m
K	$[a\,;b]$	$[a\,;b]$	...	$[a\,;b]$
Σ	$\{a, b\}$	$\{a, \frac{a+b}{2}, b\}$	...	$\{a + i\,\frac{b-a}{m}, i = 0, \ldots, m\}$
P	P_1	P_2	...	P_m

Cas d'un système à trois barres

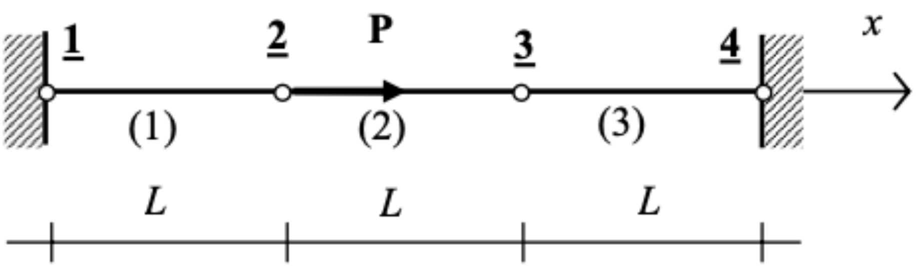


$$[K]_{e(i)} = \frac{ES}{L_{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K] = \frac{ES}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad [Q] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad [F] = \begin{bmatrix} 0 \\ P \\ 0 \\ 0 \end{bmatrix}$$

$$u_1 = u_4 = 0 \qquad \boxed{\frac{ES}{L} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}}$$

Cas d'un système à trois barres



$$\begin{cases} N_1(x) = 1 - \dfrac{x - x_1}{L} \\ N_2(x_1) = \dfrac{x - x_1}{L} \end{cases}$$

sur l'élément i

$$u(x) = N_i(x) q_i$$

$$N(x) = ESu'(x) = N_i'(x) q_i$$

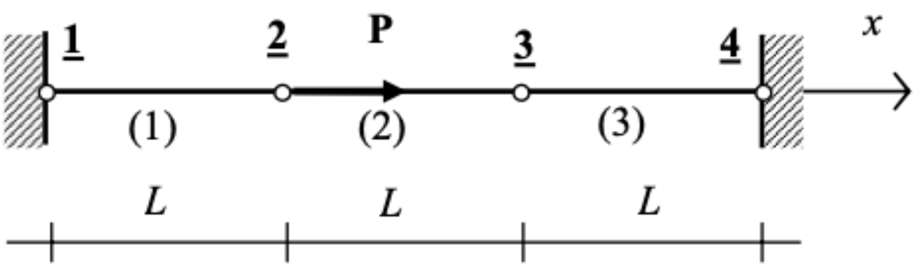
$$\boxed{\begin{bmatrix} u_2 \\ \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{2L}{3ES} P \\ \\ \frac{L}{3ES} P \end{bmatrix}}$$

$$R_0 = -N(0) = -ES\frac{1}{L}\frac{2L}{3ES}P \;\; = -\frac{2}{3}P$$

$$R_L = N(3L) = ES\frac{-1}{L}\frac{L}{3ES}P \;\; = -\frac{1}{3}P$$

$$\boxed{\sum \vec{F} = \vec{P} + \vec{R}_0 + \vec{R}_L = \vec{0}}$$

Cas d'un système à trois barres



$$\begin{cases} N_1(x) = 1 - \dfrac{x - x_1}{L} \\ N_2(x_1) = \dfrac{x - x_1}{L} \end{cases}$$

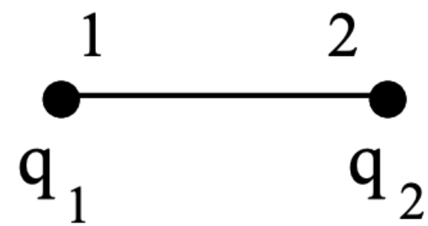
sur l'élément i

$$u(x) = N_i(x) q_i$$

$$\boxed{\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{2L}{3ES}P \\ \frac{L}{3ES}P \end{bmatrix}}$$

$$\begin{aligned} u(\tfrac{3L}{2}) &= N_1^{(2)}(\tfrac{3L}{2}).u_2 + N_2^{(2)}(\tfrac{3L}{2}).u_3 \\ &= \frac{1}{2}.\frac{2L}{3ES}P + \frac{1}{2}.\frac{L}{3ES}P \\ &= \frac{1}{2}\frac{PL}{ES} \end{aligned}$$

Calcul de la matrice de rigidité élémentaire sur l'Elément réel



Deux remarques :

Les matrices de rigidités élémentaires se déduisent de la matrice de rigidité élémentaire de référence

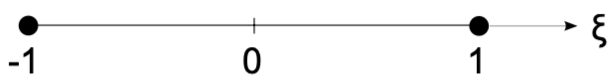
Comme l'interpolation des déplacements est linéaire, les contraintes et les déformations sont constantes sur l'élément.

Elément Barre (Hermite)

Elément sub-paramétrique :
Géométrie Linéaire - continuité de déplacement et de sa dérivée

$$u = {}^{\mathrm{T}}(\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3, \mathrm{N}_4)\,\mathbf{u_n}$$

$${}^{\mathrm{T}}\mathbf{u_n} = {}^{\mathrm{T}}\left(u_1, u_{,x_1}, u_2, u_{,x_2}\right)$$



(a) Élément de référence

(b) Segment réel (2 nœuds, 4ddl)

Fonction d'interpolation en déplacement

$$N_1 = \frac{1}{4}(1-\xi)^2(2+\xi)$$

$$N_2 = \frac{1}{8}(\xi^2-1)(1-\xi)$$

$$N_3 = \frac{1}{4}(1+\xi)^2(2-\xi)$$

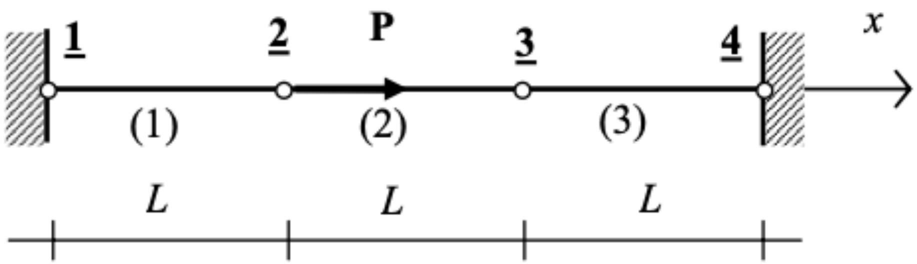
$$N_4 = \frac{1}{8}(\xi^2-1)(1+\xi)$$

Fonction d'interpolation géométrique

$$x = x_1\frac{1-\xi}{2} + x_2\frac{1+\xi}{2}$$

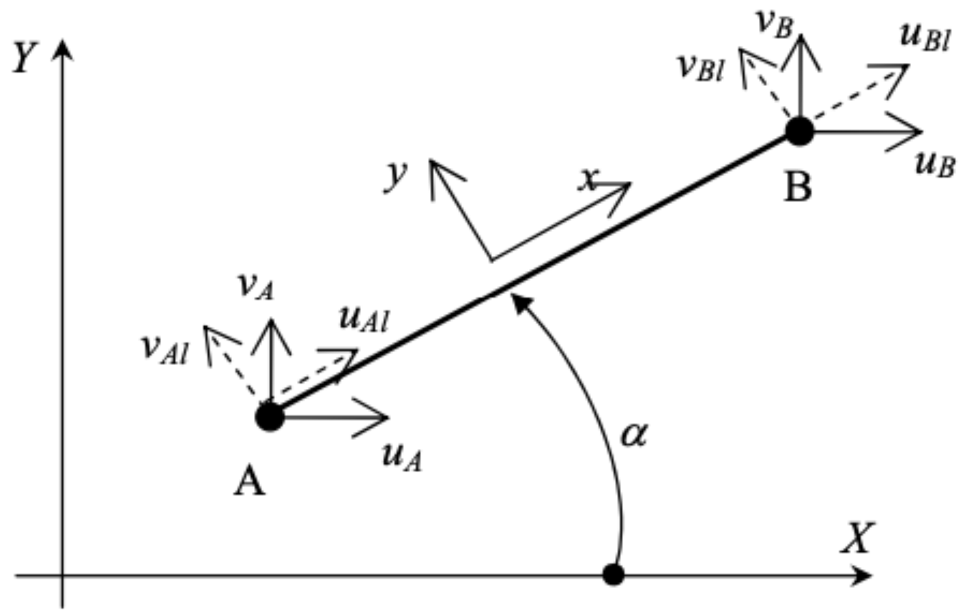
$$x = x_1 G_1(\xi) + x_2 G_2(\xi)$$

Déterminez K et résoudre



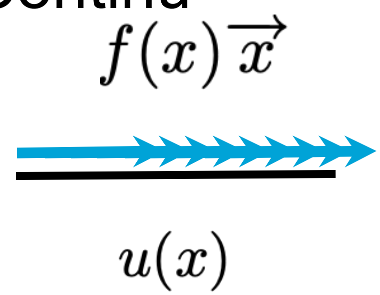
Comparer avec Lagrange

Elément Barre (2ddl) dans la plan treillis

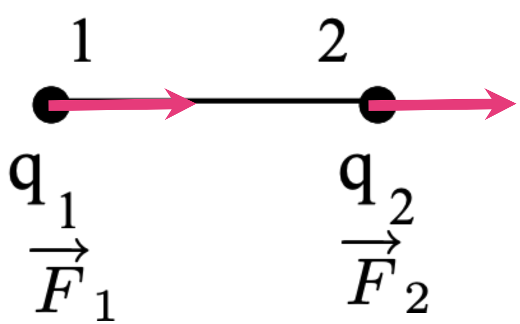


$$[K]_e = \frac{ES}{L_e} \begin{bmatrix} cos^2(\alpha) & cos(\alpha)sin(\alpha) & -cos^2(\alpha) & -cos(\alpha)sin(\alpha) \\ cos(\alpha)sin(\alpha) & sin^2(\alpha) & -cos(\alpha)sin(\alpha) & -sin^2(\alpha) \\ -cos^2(\alpha) & -cos(\alpha)sin(\alpha) & cos^2(\alpha) & cos(\alpha)sin(\alpha) \\ -cos(\alpha)sin(\alpha) & -sin^2(\alpha) & cos(\alpha)sin(\alpha) & sin^2(\alpha) \end{bmatrix}$$

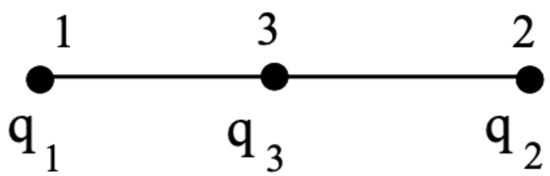
Problème Continu



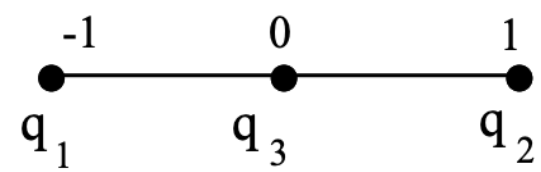
Problème Discret



Elément Barre (3ddl)



Elément réel



Elément de référence

Déplacement

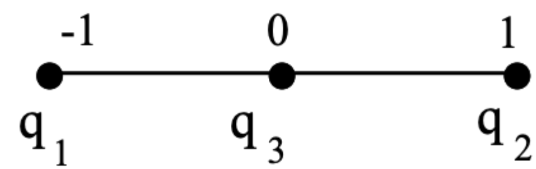
$$u(x) = a_1 + a_2 x + a_3 x^2$$

$$u(x) = N_1(x) q_1 + N_2(x) q_2 + N_3(x) q_3 = N_i(x) q_i$$

Fonction d'interpolation

$$\begin{cases} u(-1) = q_1 \\ u(0) = q_3 \\ u(1) = q_2 \end{cases} \Leftrightarrow \begin{cases} N_1(-1) = 1 \\ N_2(-1) = 0; \\ N_3(-1) = 0 \end{cases} \begin{cases} N_1(0) = 0 \\ N_2(0) = 0 et \\ N_3(0) = 1 \end{cases} \begin{cases} N_1(1) = 0 \\ N_2(1) = 1 \\ N_3(1) = 0 \end{cases}$$

Fonction d'interpolation du déplacement



Elément de référence

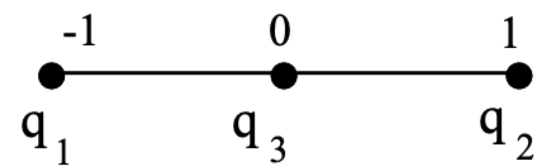
$$\begin{cases} N_1(r) = \dfrac{r(r-1)}{2} \\ N_2(r) = \dfrac{r(r+1)}{2} \\ N_3(r) = (1-r^2) \end{cases}$$

Fonction d'interpolation géométrique

$$x = x_1 \frac{1-r}{2} + x_2 \frac{1+r}{2}$$
$$= x_1 G_1(r) + x_2 G_2(r)$$

Elément Barre (3ddl)

$$[K]_{ref} = \int_{-1}^{1} {}^t[B]\,[D]\,[B]\,dr$$

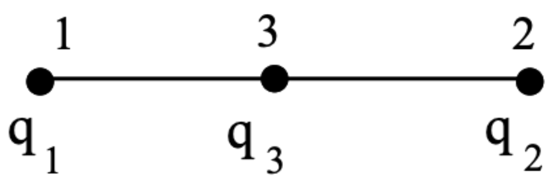


Elément de référence

$$\text{où} \qquad {}^t[B] = \begin{bmatrix} N_{1,r} \\ N_{2,r} \\ N_{3,r} \end{bmatrix} = \begin{bmatrix} \frac{2r-1}{2} \\ \frac{2r+1}{2} \\ -2r \end{bmatrix}$$

$$[K]_{ref} = \int_{-1}^{1} ES \begin{bmatrix} (N_{1,r})^2 & N_{1,r}N_{2,r} & N_{1,r}N_{3,r} \\ & (N_{2,r})^2 & N_{2,r}N_{3,r} \\ Symm & & (N_{3,r})^2 \end{bmatrix} dr$$

Elément Barre (3ddl)



Elément réel

$$[K]_e = \int_{x_1}^{x_2} {}^t[B]_e [D][B]_e \, dx$$

$${}^t[B]_e = \begin{bmatrix} N_{1,x} \\ N_{2,x} \\ N_{3,x} \end{bmatrix} = \begin{bmatrix} N_{1,r} \dfrac{dr}{dx} \\ N_{2,r} \dfrac{dr}{dx} \\ N_{3,r} \dfrac{dr}{dx} \end{bmatrix} = \frac{2}{L} {}^t[B]$$

$$[K]_e = \int_{x_1}^{x_2} {}^t\,[B]_e\,[D]\,[B]_e\,dx$$

$$= \int_{-1}^{1} \frac{2}{L} {}^t\,[B]\,[D]\,\frac{2}{L}\,[B]\,\frac{L}{2} dr$$

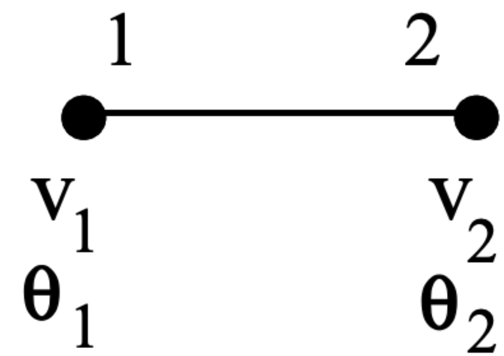
$$= \frac{2}{L}\,[K]_{ref}$$

$$[K]_e = \frac{2}{L}\int_{-1}^{1} ES \begin{bmatrix} (N_{1,r})^2 & N_{1,r}N_{2,r} & N_{1,r}N_{3,r} \\ & (N_{2,r})^2 & N_{2,r}N_{3,r} \\ Symm & & (N_{3,r})^2 \end{bmatrix} dr$$

Méthode de Newton-Cotes

Méthode de Gauss

Problèmes de flexion



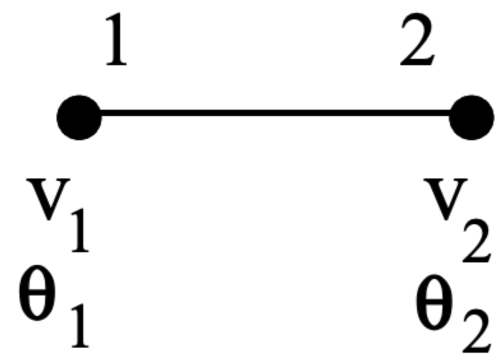
Energie de déformation de flexion :

$$W_e = \frac{1}{2} \int_{x_1}^{x_2} \left(EIv"(x)^2 \right) dx$$

Inconnues nodales réelles

$$[Q] = [v_1, \theta_1, v_2, \theta_2] \text{ où } \theta_i = \frac{\partial v}{\partial x} (x_i)$$

Problèmes de flexion



Energie de déformation de flexion

Écriture matricielle :

$$W_e = \frac{1}{2} \int_{x_1}^{x_2} {}^t[Q]\, {}^t[B]_e\, [D]\, [B]_e\, [Q]\, dx$$

$$= {}^t[Q]\, [K_e]\, [Q]$$

où

$$[Q] = [v_1, \theta_1, v_2, \theta_2]$$

Problèmes de flexion

$$W_{ref} = \frac{1}{2} \int_{-1}^{1} \left(EIv"(r)^2 \right) dr$$

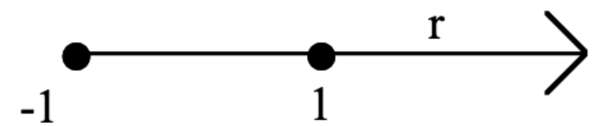
Inconnues nodales sur l'élément de réfence

$$[Q]^r = [v_1, \theta_1^r, v_2, \theta_2^r]$$

où

$$\theta_i^r = \frac{\partial v}{\partial r} (-1 \text{ ou } 1)$$

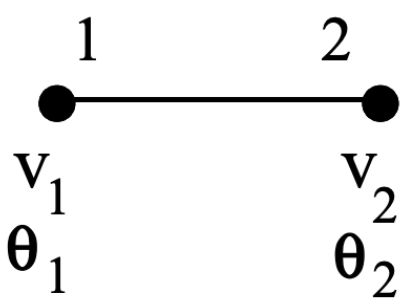
Elément de référence



Fonction d'interpolation géométrique

$$\begin{aligned} x &= x_1 \frac{1-r}{2} + x_2 \frac{1+r}{2} \\ &= x_1 G_1(r) + x_2 G_2(r) \end{aligned}$$

Elément réel

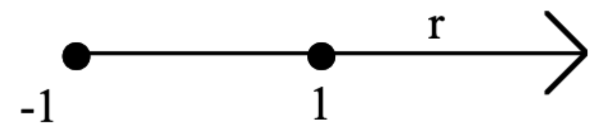


$$[Q] = [v_1, \theta_1, v_2, \theta_2]$$

où

$$\theta_i = \frac{\partial v}{\partial x}(x_i)$$

Elément de référence



$$[Q]^r = [v_1, \theta_1^r, v_2, \theta_2^r]$$

où

$$\theta_i^r = \frac{\partial v}{\partial r}(-1 \text{ ou } 1)$$

Elément réel

$$v\left(x\right)=N_1\left(x\right)v_1+N_2\left(x\right)\theta_1+N_3\left(x\right)v_2+N_4\left(x\right)\theta_2$$

Elément de référence

$$v\left(r\right)=N_1\left(r\right)v_1+N_2\left(r\right)\theta_1^r+N_3\left(r\right)v_2+N_4\left(r\right)\theta_2^r$$

Fonction d'interpolation du déplacement

$$\begin{cases} N_1\left(r\right)=\frac{1}{4}\left(2+r\right)\left(r-1\right)^2, & N_2\left(r\right)=\frac{1}{4}\left(1-r^2\right)\left(1-r\right) \\ N_3\left(r\right)=\frac{1}{4}\left(2-r\right)\left(r+1\right)^2, & N_4\left(r\right)=\frac{1}{4}\left(r^2-1\right)\left(1+r\right) \end{cases}$$

$$x=x_1\frac{1-r}{2}+x_2\frac{1+r}{2}=x_1G_1\left(r\right)+x_2G_2\left(r\right)$$

$$W_{ref} = \frac{1}{2}\int_{-1}^{1}\left(EIv"(r)^2\right)dr$$

Mise en place de la matrice de rigidité élémentaire

$$\begin{aligned}
\frac{d^2}{dr^2}v(r) &= N_{1,rr}v_1 + N_{2,rr}\theta_1^r + N_{3,rr}v_{2r} + N_{4,rr}\theta_2^r \\
&= {}^t[Q]^r\,{}^t[B]^r \\
&= N_{1,rr}v_1 + N_{2,rr}\theta_1\frac{\mathrm{dx}}{\mathrm{dr}} + N_{3,rr}v_{2r} + N_{4,rr}\theta_2\frac{\mathrm{dx}}{\mathrm{dr}} \\
&= {}^t[Q]\,{}^t\left[N_{1,rr}\left(N_{2,rr}\frac{\mathrm{dx}}{\mathrm{dr}}\right)N_{3,rr}\left(N_{4,rr}\frac{\mathrm{dx}}{\mathrm{dr}}\right)\right] \\
&= {}^t[Q]\,{}^t[B]
\end{aligned}$$

$$W_{ref} = \frac{1}{2}\int_{-1}^{1}\left(EIv"(r)^2\right)dr$$

Mise en place de la matrice de rigidité élémentaire

$$\begin{aligned}
\frac{d^2}{dx^2}v\left(x\right) &= {}^t\left[Q\right]{}^t\left[B\right]_e \\
\left[B\right]_e &= \left[N_{1,xx}N_{2,xx}N_{3,xx}N_{4,xx}\right] \\
\left[B\right]_e &= \left(\frac{dr}{dx}\right)^2\left[B\right]^r = \frac{4}{L^2}\left[B\right]^r \\
&= \frac{4}{L^2}\left[N_{1,rr}\left(\frac{dx}{dr}N_{2,rr}\right)N_{3,rr}\left(\frac{dx}{dr}N_{4,rr}\right)\right]
\end{aligned}$$

Mise en place de la matrice de rigidité élémentaire

$$W_{el} = \frac{1}{2}\int_{x_1}^{x_2} EIv"(x)^2 dx$$

$$= \frac{1}{2}{}^t[Q]\,[K]_{el}\,[Q]$$

$$[K]_{el} = \int_{x_1}^{x_2} {}^t[B]_e\,[D]\,[B]_e\,dx = \int_{-1}^{1} \frac{4}{L^2}{}^t[B]^r\,[D]\,\frac{4}{L^2}[B]^r\,\frac{L}{2}dr$$

Mise en place de la matrice de rigidité élémentaire - poutre fle

$$[K]_{el} = \frac{8}{L^3} \int_{-1}^{1} EI \begin{bmatrix} (N_{1,rr})^2 & \frac{L}{2} N_{1,rr} N_{2,rr} & N_{1,rr} N_{3,rr} & \frac{L}{2} N_{1,rr} N_{4,rr} \\ & \frac{L^2}{4} (N_{2,rr})^2 & \frac{L}{2} N_{2,rr} N_{3,rr} & \frac{L^2}{4} N_{2,rr} N_{4,rr} \\ & & (N_{3,rr})^2 & \frac{L}{2} N_{3,rr} N_{4,rr} \\ sym & & & \frac{L^2}{4} (N_{4,rr})^2 \end{bmatrix} dr$$

Élément triangulaire - 2D

Nous allons illustrer ce qui précède dans un cas simple : élasticité plane, éléments triangulaires à trois noeuds.

En élasticité plane

$$\begin{aligned}\sigma_{ij}\epsilon_{ij} &= \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{12}\epsilon_{12} + \sigma_{21}\epsilon_{21} \\ &= \sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{12}\gamma_{12}\end{aligned}$$

$$\underline{\sigma}.\underline{\epsilon} = \{\epsilon\}^T \underline{E} \{\epsilon\} \quad avec \quad \{\epsilon\} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix}$$

avec, en contraintes planes,

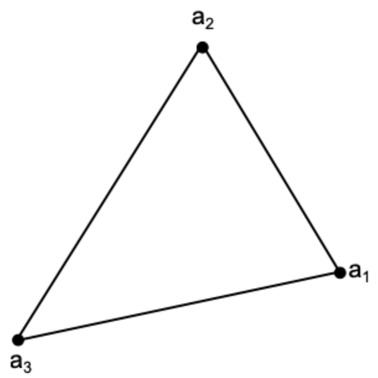
$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

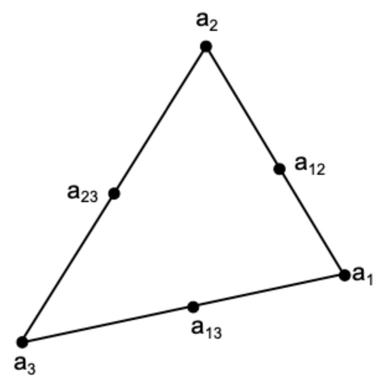
et, en déformations planes,

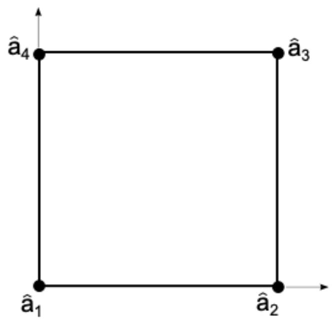
$$\underline{D} = \begin{bmatrix} \lambda+2\mu & \lambda & 0 \\ \lambda & \lambda+2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

Lagrange - (les plus simples)

Éléments finis bidimensionnels





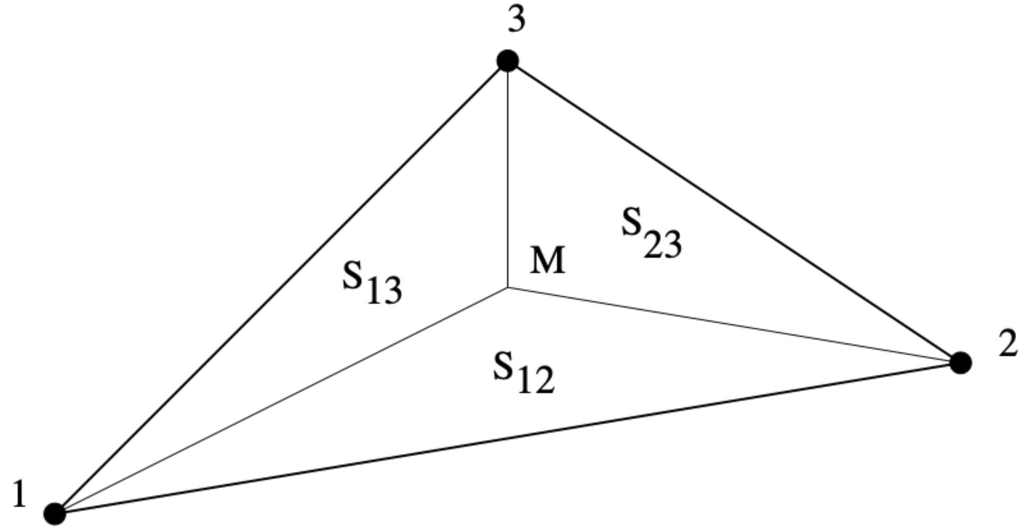


Élément	P_1	P_2
K	triangle de sommets $\{a_1, a_2, a_3\}$	triangle de sommets $\{a_1, a_2, a_3\}$
Σ	$\{a_1, a_2, a_3\}$	$\{a_{ij} = \frac{a_i + a_j}{2}, 1 \leqslant i, j \leqslant 3\}$
P	P_1	P_2

Élément	Q_1
K	rectangle de sommets $\{a_1, a_2, a_3, a_4\}$ de côtés parallèles aux axes.
Σ	$\{a_1, a_2, a_3, a_4\}$
P	Q_1

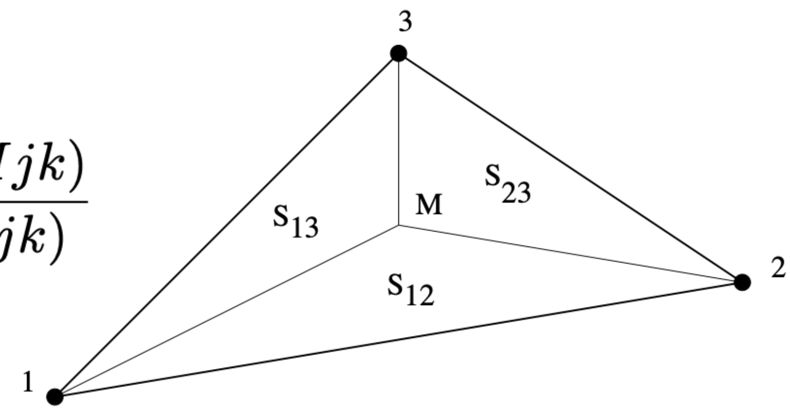
Élement triangulaire

Nous utilisons des éléments triangles à trois noeuds, encore appelés triangles à déformation constante (TDC).



On utilise les fonctions d'interpolation

$$L_i(M) = \frac{S(Mjk)}{S(ijk)}$$



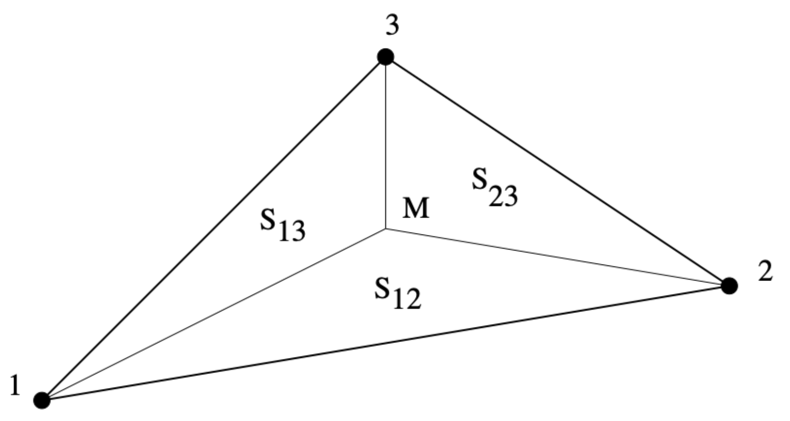
qui vérifient à l'évidence

$$L_i(M) + L_j(M) + L_k(M) = 1 \quad , \quad L_i(j) = \delta_{ij} \quad , \quad M \in (jk) \Rightarrow L_i(M) = 0$$

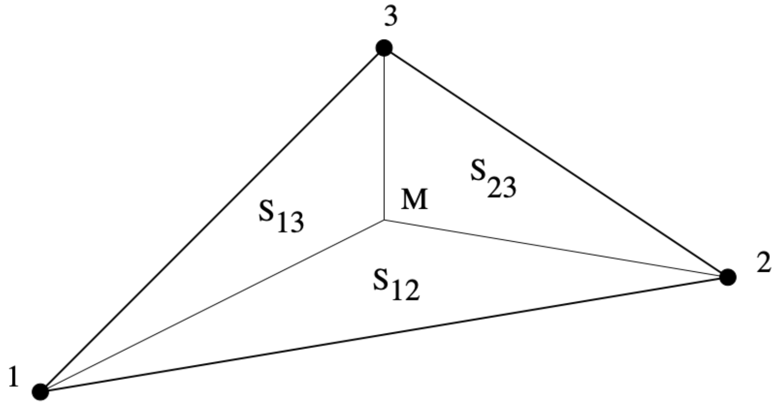
$$\int_{\Delta} L_1^p L_2^q L_3^r \, dS = 2\Delta \, \frac{p!\, q!\, r!}{(p+q+r+2)!}$$

Le champ de déplacements est donné par

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



le champ de déformations est donné par



$$\{\epsilon\} = [\underline{B}] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

avec

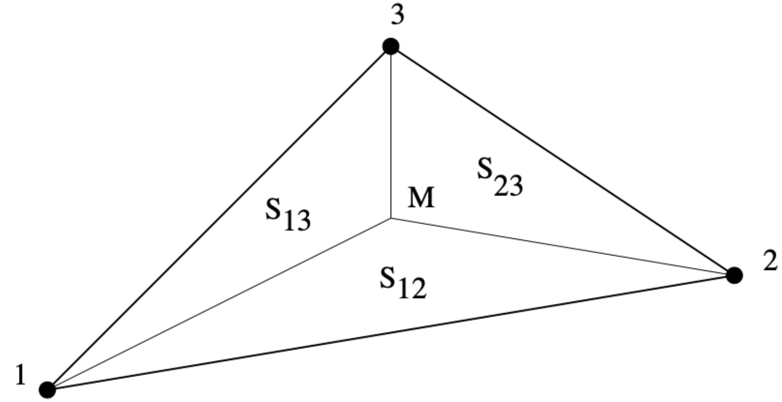
$$[\underline{B}] = \begin{bmatrix} L_{1,x} & L_{2,x} & L_{3,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{1,y} & L_{2,y} & L_{3,y} \\ L_{1,y} & L_{2,y} & L_{3,y} & L_{1,x} & L_{2,x} & L_{3,x} \end{bmatrix}$$

La matrice de rigidité élémentaire est donnée par

$$[\underline{k}] = e \int_{(ijk)} [\underline{B}]^T \; \underline{E} \; [\underline{B}] \; dS$$

avec e l'épaisseur de la plaque

$$S(ABC) = \frac{1}{2} \begin{vmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{vmatrix}$$



La matrice de rigidité élémentaire est donnée par

$$[\underline{k}] = e \int_{(ijk)} [\underline{B}]^T \; \underline{E} \; [\underline{B}] \; dS$$

avec e l'épaisseur de la plaque

on obtient, pour le triangle $(1, 2, 3)$ de surface Δ

$$[\underline{B}] = \frac{1}{2\Delta} \begin{bmatrix} y_3 - y_2 & y_1 - y_3 & y_2 - y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 - x_3 & x_3 - x_1 & x_1 - x_2 \\ x_2 - x_3 & x_3 - x_1 & x_1 - x_2 & y_3 - y_2 & y_1 - y_3 & y_2 - y_1 \end{bmatrix}$$

La matrice de rigidité élémentaire est donnée par

$$[\underline{k}] = e \int_{(ijk)} [\underline{B}]^T \; \underline{\underline{E}} \; [\underline{B}] \; dS$$

avec e l'épaisseur de la plaque

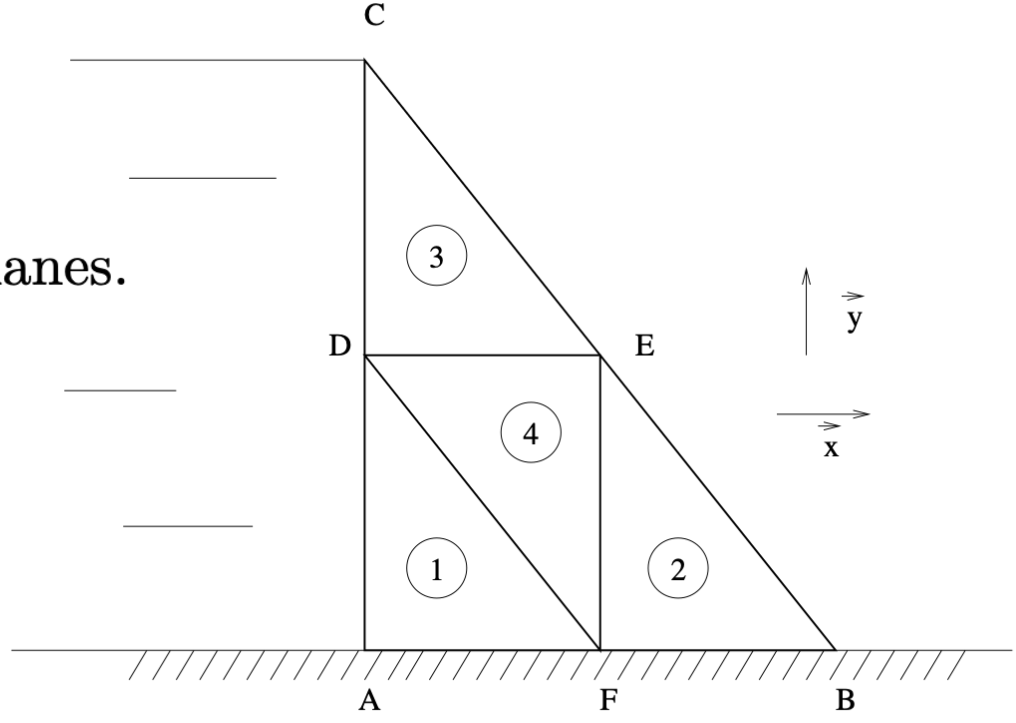
Pour $[\underline{k}]$, comme il n'y a pas de dépendance en x, y

$$[\underline{k}] = e \, \Delta \; [\underline{B}]^T \; \underline{\underline{E}} \; [\underline{B}]$$

La suite du calcul est aisée mais fastidieuse.

Exemple : barrage masse

Nous considérons un barrage dont la section est un triangle rectangle de base $2a$, de hauteur $2h$.



On travaille en déformations planes.

Exemple : barrage masse

Nous considérons un barrage dont la section est un triangle rectangle de base $2a$, de hauteur $2h$.

Le chargement comporte une pression hydrostatique sur la face AC de densité ,

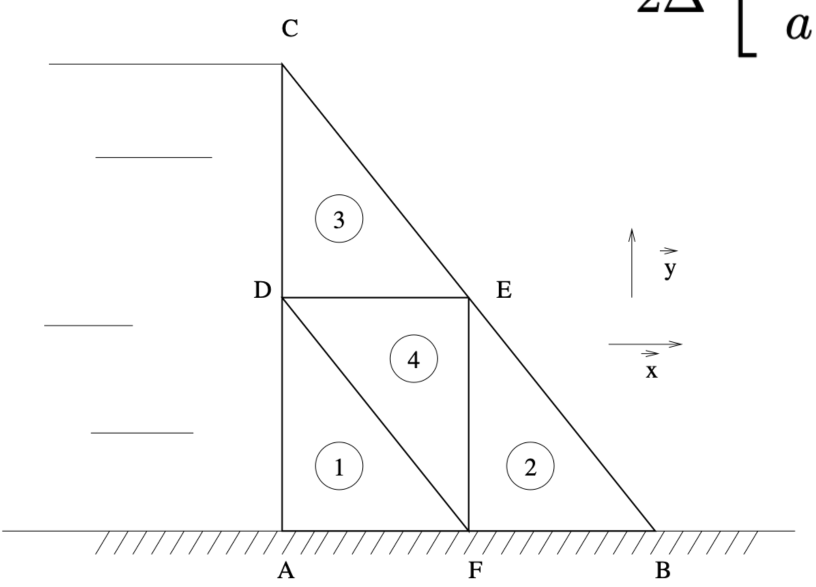
$$\vec{F} = \rho\, g\, (2h - y)\vec{x}$$

ainsi que le poids propre du barrage, force volumique de densité

$$-\rho_b g\, \vec{y}.$$

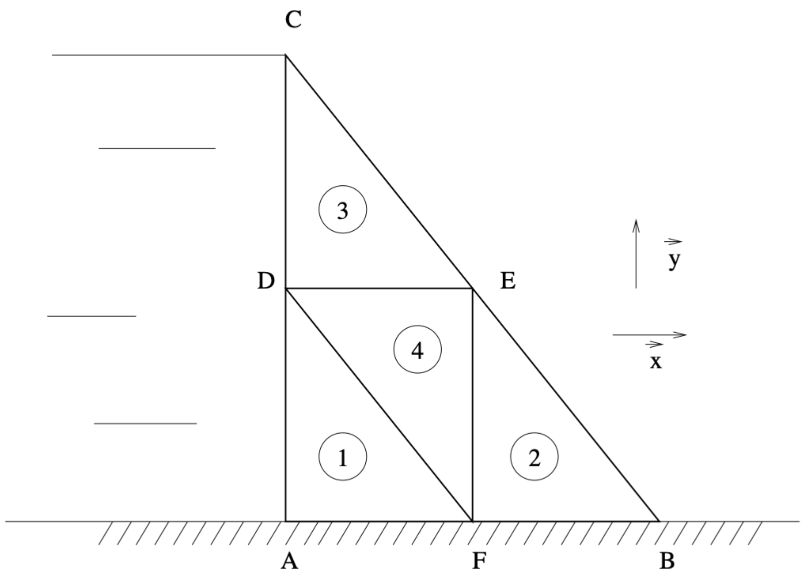
Il est évident que les éléments 1, 2, et 3 ont même matrice de rigidité élémentaire. Pour ces éléments, en numérotant les noeuds $1 = A$, $2 = F$, et $3 = D$, il vient

$$[\underline{B}] = \frac{1}{2\Delta}\begin{bmatrix} h & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & -a \\ a & 0 & -a & h & -h & 0 \end{bmatrix}$$



On obtient:

$$[\underline{k}] = \frac{e}{4\Delta}\begin{bmatrix} (\lambda+2\mu)h^2+\mu a^2 & -(\lambda+2\mu)h^2 & -\mu a^2 & (\lambda+\mu)ah & -\mu ah & -\lambda ah \\ -(\lambda+2\mu)h^2 & (\lambda+2\mu)h^2 & 0 & -\lambda ah & 0 & \lambda ah \\ -\mu a^2 & 0 & \mu a^2 & -\mu ah & \mu ah & 0 \\ (\lambda+\mu)ah & -\lambda ah & -\mu ah & (\lambda+2\mu)a^2+\mu h^2 & -\mu h^2 & -(\lambda+2\mu)a^2 \\ -\mu ah & 0 & \mu ah & -\mu h^2 & \mu h^2 & 0 \\ -\lambda ah & \lambda ah & 0 & -(\lambda+2\mu)a^2 & 0 & (\lambda+2\mu)a^2 \end{bmatrix}$$



De même, pour DEF, en numérotant $1 = F$, $2 = E$, et $3 = D$,

$$[\underline{B}] = \frac{1}{2\Delta}\begin{bmatrix} 0 & -h & h & 0 & 0 & 0 \\ 0 & 0 & 0 & a & -a & 0 \\ a & -a & 0 & 0 & -h & h \end{bmatrix}$$

et

$$[\underline{k}] = \frac{e}{4\Delta}\begin{bmatrix} \mu a^2 & -\mu a^2 & 0 & 0 & -\mu ah & \mu ah \\ -\mu a^2 & (\lambda+2\mu)h^2+\mu a^2 & -(\lambda+2\mu)h^2 & -\lambda ah & (\lambda+\mu)ah & -\mu ah \\ 0 & -(\lambda+2\mu)h^2 & (\lambda+2\mu)h^2 & \lambda ah & -\lambda ah & 0 \\ 0 & -\lambda ah & \lambda ah & (\lambda+2\mu)a^2 & -(\lambda+2\mu)a^2 & 0 \\ -\mu ah & (\lambda+\mu)ah & -\lambda ah & -(\lambda+2\mu)a^2 & (\lambda+2\mu)a^2+\mu h^2 & -\mu h^2 \\ \mu ah & -\mu ah & 0 & 0 & -\mu h^2 & \mu h^2 \end{bmatrix}$$

En introduisant les notations

$$\alpha = (\lambda + 2\mu)h^2 \quad , \quad \beta = \mu a^2 \quad , \quad \gamma = (\lambda + \mu)ah \quad , \quad \delta = \mu ah$$

$$\epsilon = \lambda ah \quad , \quad \phi = (\lambda + 2\mu)a^2 \quad , \quad \theta = \mu h^2$$

et, pour la matrice de rigidité globale,

$$[\underline{K}] = \frac{e}{4\Delta}\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{uv} \\ \underline{K}_{vu} & \underline{K}_{vv} \end{bmatrix} \qquad (\underline{K}_{uv} = \underline{K}_{vu})$$

Exemple Barrage - Élément triangulaire

$$\underline{K}_{uu} = \frac{e}{4\Delta}\begin{bmatrix} \alpha+\beta & 0 & 0 & -\beta & 0 & -\alpha \\ 0 & \alpha & 0 & 0 & 0 & -\alpha \\ 0 & 0\beta & -\beta & 0 & 0 & \\ -\beta & 0 & -\beta & 2(\alpha+\beta) & -2\alpha & 0 \\ 0 & 0 & 0 & -2\alpha & 2(\alpha+\beta) & -2\beta \\ \alpha & -\alpha & 0 & 0 & -2\beta & 2(\alpha+\beta) \end{bmatrix}$$

$$\underline{K}_{vv} = \frac{e}{4\Delta}\begin{bmatrix} \phi+\theta & 0 & 0 & -\phi & 0 & -\theta \\ 0 & \theta & 0 & 0 & 0 & -\theta \\ 0 & 0 & \phi & -\phi & 0 & 0 \\ -\phi & 0 & -\phi & 2(\phi+\theta) & -2\theta & 0 \\ 0 & 0 & 0 & -2\theta & 2(\phi+\theta) & -2\phi \\ -\theta & -\theta & 0 & 0 & -2\phi & 2(\phi+\theta) \end{bmatrix}$$

$$\underline{K}_{uv} = \frac{e}{4\Delta}\begin{bmatrix} \gamma & 0 & 0-\epsilon & 0 & -\delta & \\ 0 & 0 & 0 & 0 & -\epsilon & \epsilon \\ 0 & 0 & 0 & -\delta & \delta & 0 \\ -\delta & 0 & -\epsilon & \gamma & -\delta-\epsilon & \delta+\epsilon \\ 0 & \delta & \epsilon & -\delta-\epsilon & \gamma & -\delta-\epsilon \\ -\epsilon & -\delta & 0 & \delta+\epsilon & -\delta-\epsilon & \gamma \end{bmatrix}$$

Ici, les déplacements imposés sont :

$$u_A = u_B = u_F = v_A = v_B = v_F = 0.$$

alors que les déplacements des noeuds non liés sont solution de

$$\begin{bmatrix} \beta & -\beta & 0 & 0 & -\delta & \delta \\ -\beta & 2(\alpha+\beta) & -2\alpha & -\epsilon & \gamma & -\delta-\epsilon \\ 0 & -2\alpha & 2(\alpha+\beta) & \epsilon & -\epsilon-\delta & \gamma \\ 0 & -\epsilon & \epsilon & \phi & -\phi & 0 \\ -\delta & \gamma & -\delta-\epsilon & -\phi & 2(\phi+\theta) & -2\theta \\ \delta & -\delta-\epsilon & \gamma & 0 & -2\theta & 2(\phi+\theta) \end{bmatrix} \begin{bmatrix} u_C \\ u_D \\ u_E \\ v_C \\ v_D \\ v_E \end{bmatrix} = \begin{bmatrix} F_{Cx} \\ F_{Dx} \\ F_{Ex} \\ F_{Cy} \\ F_{Dy} \\ F_{Ey} \end{bmatrix}$$

donc, pour les réactions d'appui, les relations

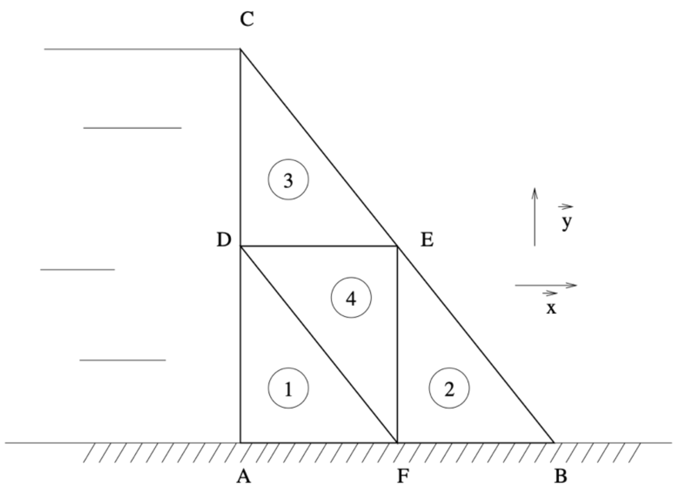
$$\begin{bmatrix} R_{Ax} \\ R_{Bx} \\ R_{Fx} \\ R_{Ay} \\ R_{By} \\ R_{Fy} \end{bmatrix} = \begin{bmatrix} 0 & -\beta & 0 & 0 & -\epsilon & 0 \\ 0 & 0 & 0 & 0 & \epsilon & \\ 0 & 0 & -2\beta & 0 & \delta+\epsilon & -\delta-\epsilon \\ 0 & -\delta & 0 & 0 & -\phi & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & \delta+\epsilon & -\delta-\epsilon & 0 & 0 & -2\phi \end{bmatrix} \begin{bmatrix} u_C \\ u_D \\ u_E \\ v_C \\ v_D \\ v_E \end{bmatrix}$$

Le calcul des forces nodales équivalentes est immédiat. Pour chaque élément e qui contient le noeud n, on a, en se limitant aux forces surfaciques

$$F^e_{ni} = \int_{\partial\Omega \bigcap e} F_i(\vec{x})\, \phi^e_n(\vec{x})\, dS$$

On obtient finalement

	DEC	AFD	Total
F_{Cx}	$\rho g e\, h^2/6$	0	$\rho g e\, h^2/6$
F_{Dx}	$\rho g e\, 2h^2/3$	$\rho g e\, 2h^2/3$	$\rho g e\, 4h^2/3$
F_{Ax}	0	$\rho g e\, 5h^2/6$	$\rho g e\, 5h^2/6$



Les forces nodales équivalentes pour le chargement volumique sont obtenues sans difficulté :

pour chaque noeud n, la contribution de l'élément e comportant le noeud n est

$$
\begin{aligned}
F^{vol}_{n,e,x} &= 0 \\
F^{vol}_{n,e,y} &= -\rho_b g \, e \, \frac{\Delta_e}{3}
\end{aligned}
$$

où Δ_e est la surface du triangle concerné.

Exemple Barrage - Élément triangulaire

alors y a plus qu'à résoudre ce système

$$\begin{bmatrix} \beta & -\beta & 0 & 0 & -\delta & \delta \\ -\beta & 2(\alpha+\beta) & -2\alpha & -\epsilon & \gamma & -\delta-\epsilon \\ 0 & -2\alpha & 2(\alpha+\beta) & \epsilon & -\epsilon-\delta & \gamma \\ 0 & -\epsilon & \epsilon & \phi & -\phi & 0 \\ -\delta & \gamma & -\delta-\epsilon & -\phi & 2(\phi+\theta) & -2\theta \\ \delta & -\delta-\epsilon & \gamma & 0 & -2\theta & 2(\phi+\theta) \end{bmatrix} \begin{bmatrix} u_C \\ u_D \\ u_E \\ v_C \\ v_D \\ v_E \end{bmatrix} = \begin{bmatrix} \frac{\rho g e\, h^2}{6} \\ \frac{\rho g e\, 4h^2}{3} \\ 0 \\ -\rho_b g\, e\, \frac{\Delta_e}{3} \\ -\rho_b g\, e\, \Delta_e \\ -\rho_b g\, e\, \Delta_e \end{bmatrix}$$

Merci à M.E. Haglund

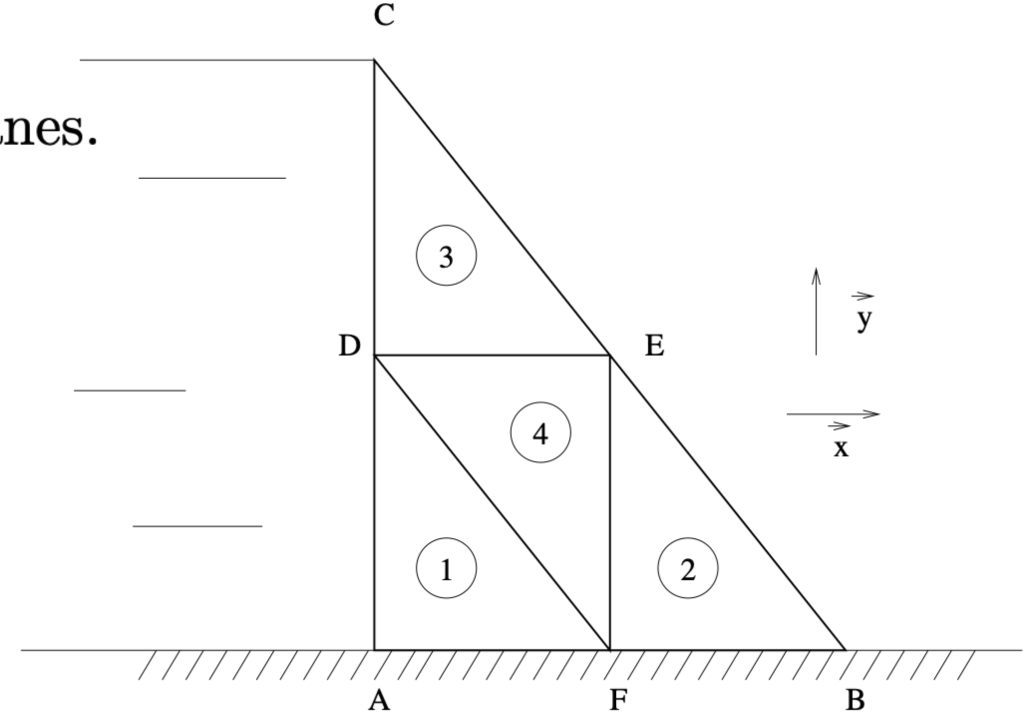
Exemple : barrage masse

A faire avec Ansys

Nous considérons un barrage dont la section est un triangle rectangle de base $2a$, de hauteur $2h$.

On travaille en déformations planes.

h=10 m
a=10 m



3 méthodes :

Méthode r : Pour un maillage et un type d'élément donné, il s'agit de déplacer les nœuds, en fonctions des indicateurs d'erreur.

Méthode h : En conservant le même type d'élément, on les subdivise dans les zones les plus sollicitées selon le ou les indicateurs d'erreur choisis.

Méthode p : À nombre d'éléments constant, dans les zones les plus sollicitées, on va modifier les éléments en introduisant des fonctions de formes polynomiales d'ordre plus élevé, dites hiérarchiques.

Généralités

Français	Anglais
Résistance des matériaux	Strength of Materials / Mechanics of Materials
Analyse numérique	Numerical Analysis
Méthode des éléments finis	Finite Element Method (FEM)
Maillage	Mesh
Élément fini	Finite element
Nœud	Node

Fonctions et champs

Français	Anglais
Fonction de base	Basis function
Fonction de forme	Shape function
Déplacement	Displacement
Champ de déplacement	Displacement field
Champ de contraintes	Stress field
Champ de déformations	Strain field
Contraintes	Stresses
Déformations	Strains

Énergie et principes

Français	Anglais
Énergie potentielle	Potential energy
Variation de l'énergie potentielle	Variation of potential energy
Principe des travaux virtuels	Principle of virtual work
Formule de Green	Green's theorem
Intégration par parties	Integration by parts

Forces et conditions aux limites

Français	Anglais
Force volumique	Body force
Traction (au bord)	Traction (on the boundary)
Conditions aux limites	Boundary conditions
Conditions aux limites de Dirichlet	Dirichlet boundary conditions
Conditions aux limites de Neumann	Neumann boundary conditions

Formulations mathématiques

Français	Anglais
Équation aux dérivées partielles (EDP)	Partial Differential Equation (PDE)
Système linéaire	Linear system
Matrice de rigidité	Stiffness matrix
Vecteur de charges	Load vector
Approximation	Approximation
Convergence	Convergence
Fonction test	Test function
Formulation faible	Weak formulation