

The logistic regression

Jean-Michel Marin

University of Montpellier
Faculty of Sciences

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Definition

We study the binary logistic model

$Y \in \{0, 1\}$ the response

X explanatory variables possibly containing factors associated with contrasts

We observe $(X_1, Y_1), \dots, (X_n, Y_n)$

We suppose that

- ▶ $Y_i | X_i = x_i \sim \mathcal{B} \left(\frac{\exp(x_i^T \beta)}{1 + \exp(x_i^T \beta)} \right)$
- ▶ knowing $X_1 = x_1, \dots, X_n = x_n$ the variables Y_i are independent

Definition

We have

$$\mathbb{P}(Y = 1|X = \mathbf{x}) = \frac{\exp(\mathbf{x}^T \boldsymbol{\beta})}{1 + \exp(\mathbf{x}^T \boldsymbol{\beta})} = \mathbb{E}(Y|X = \mathbf{x})$$

$$\mathbb{V}(Y|X = \mathbf{x}) = \mathbb{E}(Y|X = \mathbf{x})(1 - \mathbb{E}(Y|X = \mathbf{x})) = \frac{\exp(\mathbf{x}^T \boldsymbol{\beta})}{(1 + \exp(\mathbf{x}^T \boldsymbol{\beta}))^2}$$

Definition

We have directly modeled the distribution of $Y|X = x$ via its expectation

This is a generalized linear model associated with the logistic link function

$$\log \left(\frac{\mathbb{E}(Y|X = x)}{1 - \mathbb{E}(Y|X = x)} \right) = x^T \beta$$

Estimation de β

The likelihood function is

$$V(\beta) = \prod_{i=1}^n \left(\frac{\exp(x_i^T \beta)}{1 + \exp(x_i^T \beta)} \right)^{y_i} \left(\frac{1}{1 + \exp(x_i^T \beta)} \right)^{1-y_i}$$

The log-likelihood function is

$$L(\beta) = \sum_{i=1}^n \{y_i x_i^T \beta - \log(1 + \exp(x_i^T \beta))\}$$

Estimation de β

$$\frac{\delta L}{\delta \beta}(\beta^*) = 0_d$$
$$\iff \sum_{i=1}^n \left\{ y_i x_i - \frac{x_i \exp(x_i^T \beta^*)}{1 + \exp(x_i^T \beta^*)} \right\} = 0_d$$

There is no analytical solution to this system

We use an iterative algorithm derived from the Newton-Raphson procedure

Estimation de β

The hessian matrix of $L(\cdot)$ is given by

$$\frac{\delta^2 L}{(\delta \beta)^2}(\beta) = - \sum_{i=1}^n \frac{\exp(x_i^T \beta)}{(1 + \exp(x_i^T \beta))^2} x_i x_i^T$$

Estimation de β

The Newton-Raphson iterative procedure consists of calculating a sequence of $\beta^{(0)}, \dots, \beta^{(t)}, \dots$ telle que

$$\beta^{(t)} = \beta^{(t-1)} - \left[\frac{\delta L^2}{(\delta \beta)^2}(\beta^{(t-1)}) \right]^{-1} \frac{\delta L}{\delta \beta}(\beta^{(t-1)})$$

This sequence converges to a value β^* such that $\frac{\delta L}{\delta \beta}(\beta^*) = 0_d$

Variability of estimates

Since the maximum likelihood estimator is used, it is possible to construct asymptotic confidence intervals

$$\mathbb{V}(\hat{\beta}) \approx \left[\sum_{i=1}^n \frac{\exp(x_i^T \beta)}{(1 + \exp(x_i^T \beta))^2} x_i x_i^T \right]^{-1}$$