Master's degree in IEAP

Ingénierie et Ergonomie de l'Activité Physique

MATHEMATICS PROLEGOMENA

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by

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1

Complex Numbers

1.1. Set of complex numbers

One of the first uses of complex numbers was made by Scipione Dal Ferro (February 6, 1465 - November 5, 1526) in the early 16th century. He was interested in solving polynomial equations of order 3 (cubic equations), and in particular the equation $x^3 + px = q$. He showed that the solution can be written as:

$$x = \sqrt[3]{\frac{q - \sqrt{q^2 + 4p^3/27}}{2}} + \sqrt[3]{\frac{q + \sqrt{q^2 + 4p^3/27}}{2}}. \tag{1.1}$$

Depending on the signs of p and q and their ratio p/q, x can then take on non-real values [1]. At the end of the 16th century, Raphaël Bombelli applied Dal Ferro's method of solving the equation $x^3 - 15x = 4$ and demonstrated that the solution required the definition of the root of a complex number:

$$x = \sqrt[3]{2 - 11\sqrt{-1}} + \sqrt[3]{2 + 11\sqrt{-1}}. \tag{1.2}$$

Note that the notation $\sqrt{-1}$ has no clear mathematical meaning and should not be used. It is only used here to demonstrate the necessity of creating a new type of number.

1.1.1. Different sets of numbers

Over the centuries, as mathematical research has progressed, different families of numbers have been created. Most of these creations resulted from the fact that certain equations had no solution in the pre-defined sets. It then became necessary to imagine new sets encompassing the first ones and integrating, as basic building blocks, the solutions to these equations. The most classical sets of numbers today are the following:

- Set of integers $\mathbb{N}$:
  - $0, 1, 2, 3, \ldots$
  - The equation $x + 1 = 0$ has no solution.
- Set of relative integers $\mathbb{Z}$:
  - $\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$
  - The equation $2x + 1 = 0$ has no solution.
- Set of rational numbers $\mathbb{Q}$:
  - ex: $\frac{1}{2}$, $\frac{5}{3}$, $1 \ldots$
  - The equation $x^2 = 2$ has no solution.

[1] Note that the terms "real", "imaginary" and "complex" correspond to the beliefs of the time, which assumed that real numbers possessed a certain reality that imaginary and complex numbers could not claim.

- Set of real numbers $\mathbb{R}$:
  - ex: $\mathbb{Q}$ + irrationals: π, $\sqrt{2}\ldots$
  - The equation $x^2 = -1$ has no solution.
- Set of complex numbers $\mathbb{C}$:
  - ex: $\mathbb{R} + i\mathbb{R}$: 1, $1+i$, $\pi - 2i$, $\sqrt{2}\ldots$
  - All polynomial equations have a number of solutions equal to their degree.

1.1.2. The imaginary number i

To simplify things, the creation of complex numbers comes from the fact that the equation $x^2 = -1$ has no solution in $\mathbb{R}$. **Let's imagine** that it does, and let's call it i as ... **imaginary**. Therefore:

$$i^2 = -1. \tag{1.3}$$

The equation $x^2 = -1$ can then be solved:

$$x^2 + 1 = 0, \tag{1.4}$$
$$x^2 - i^2 = 0, \tag{1.5}$$
$$(x+i)(x-i) = 0. \tag{1.6}$$

It therefore has two solutions i and $-i$. **Warning**, as noted above, writing $\sqrt{-1}$ makes no sense since there are two roots of -1.

1.1.3. The set of complex numbers $\mathbb{C}$

The set of complex numbers, denoted $\mathbb{C}$, is then defined as the set of pairs $(a, b) \in \mathbb{R} \times \mathbb{R}$. It includes an element, denoted $(0, 1) = i$, such that $i^2 = -1$. In $\mathbb{C}$, the usual addition and multiplication rules are preserved. Any element z of $\mathbb{C}$ is also uniquely written $z = a + ib$ where a and b are real.

1.1.4. Real and imaginary parts

Let $z = a + ib$ be a complex number, a is the real part of z and is denoted $a = Re(z)$. Similarly, b is the imaginary part of z and is denoted $b = Im(z)$. **Warning** , the imaginary part of z is a real number! Any complex number of the form $z = ib$ is a pure imaginary. The set of pure imaginary numbers is denoted $i\mathbb{R}$. The notion of order no longer exists in $\mathbb{C}$. This means that it is not possible to say that one complex number is larger, or smaller, than another.

1.1.5. Equality between two complex numbers

Let $z = a + ib$ and $z' = a' + ib'$ be two complex numbers. Their equality is then equivalent to the simultaneous equality of their real and imaginary parts:

$$z = z' \quad \Leftrightarrow \quad a = a' \quad \text{et} \quad b = b'. \tag{1.7}$$

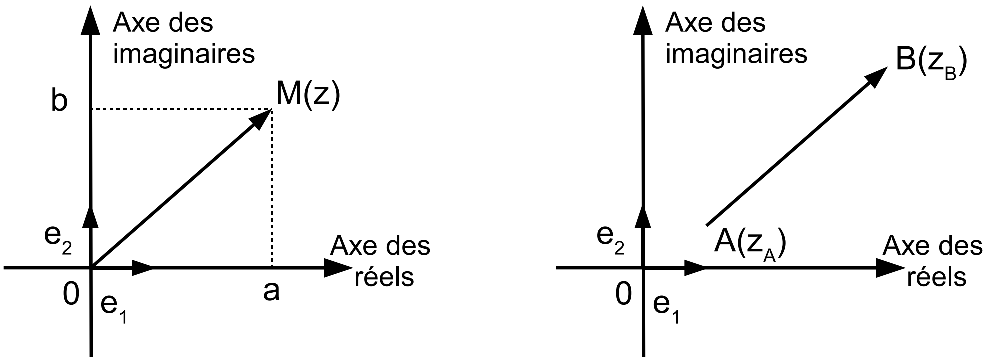
Application:
Let be the complex numbers $z_1 = 2 - i$ and $z_2 = 1 + 2i$.

1. What are the real and imaginary parts of z_1 and z_2?
2. Calculate $z_1 + z_2$, $z_1 - z_2$, $z_1 * z_2$ and $\frac{z_1}{z_2}$.

1.2. Graphic and geometric representation

Every complex number $z = a + ib$ has an associated point $M(a; b)$ in the plane. The point $M(a; b)$ is called the image of the number z. z is the affix of the point $M(a; b)$ ("affix" is feminine). It is denoted $z = \text{aff}(M)$.

As shown in figure 1.1, any complex number z can also be associated with a vector $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$.



(a) (b)

Figure 1.1: (a) Complex plane and (b) affix of the vector $\vec{AB}$.

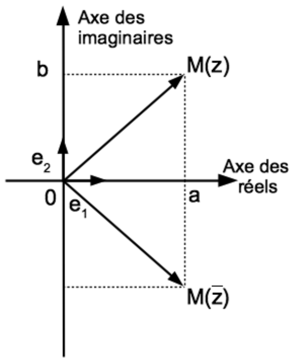


Figure 1.2: Conjugate numbers.

1.2.1. Application

Let z_A and z_B be the affixes of points A and B, then, the affix of the vector is ((see figure 1.1(b)):

$$aff(\vec{AB}) = z_B - z_A. \tag{1.8}$$

This translates geometric problems into algebraic ones.

1.3. Conjugate, modulus and argument of a complex number

The conjugate of a complex number $z = a + ib$ is the complex number $\overline{z} = a - ib$. Certain relationships follow from the definition of the conjugate of a complex number. These include:

- $Re(z) = Re(\overline{z})$ and $Im(z) = -Im(\overline{z})$,
- $z + \overline{z} = 2Re(z)$,
- $z - \overline{z} = 2i\,Im(z)$,
- z is real $\Leftrightarrow z = \overline{z}$,
- z is pure imaginary $\Leftrightarrow z = -\overline{z}$.

This also implies the geometric relationship shown in figure 1.2.

1.3.1. Properties of conjugation:

There are several properties of conjugation to be aware of:

$$\overline{z+z'} = \overline{z}+\overline{z'}, \quad (1.9)$$
$$\overline{-z} = -\overline{z}, \quad (1.10)$$
$$\overline{zz'} = \overline{z}\,\overline{z'}, \quad (1.11)$$
$$\overline{z^n} = \overline{z}^n, \quad (1.12)$$
$$\overline{\left(\frac{z}{z'}\right)} = \frac{\overline{z}}{\overline{z'}}. \quad (1.13)$$

Application:
Let be the complex numbers $z_1 = 2 - i$ and $z_2 = 1 + 2i$.

1. What are the conjugates $\overline{z_1}$ and $\overline{z_2}$ of z_1 and z_2?
2. Check that the above relationships hold.

1.3.2. Theorem:

For any complex number $z = a + ib$, $\overline{z}$ is a complex number such that $z\overline{z}$ is a real number equal to:

$$z\overline{z} = a^2 + b^2 \in \mathbb{R}. \quad (1.14)$$

Application:
Let be the complex number $z = a + ib$.

1. Calculate $z\overline{z}$.

1.3.3. Definition: Modulus

The modulus of a complex number $z = a + ib$ is the number **real positive**:

$$|z| = \sqrt{z\overline{z}} = \sqrt{a^2+b^2}. \quad (1.15)$$

Question:
Could the previous definition have been predicted by geometric reasoning?

1.3.4. Definition: Argument

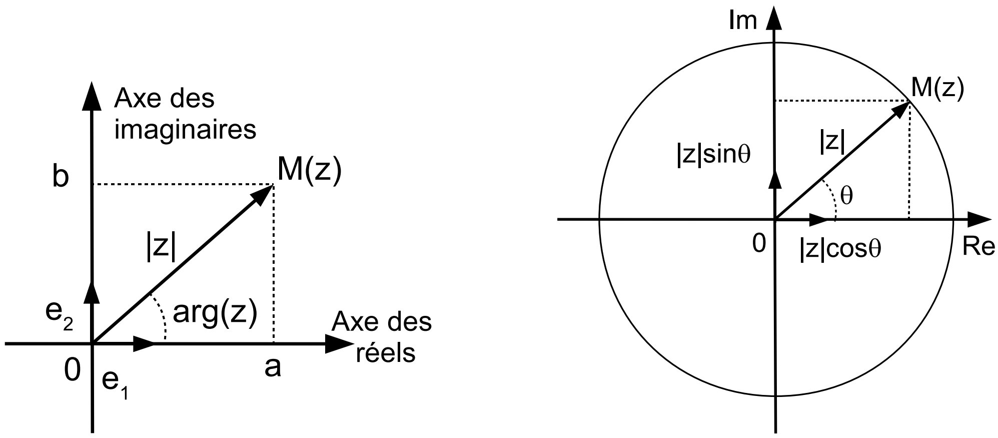
The argument of a **non-zero** complex number $z = a+ib$ is a measure of the oriented angle $\left(\vec{e}_1, \vec{OM}\right)$ (see figure 1.3(a)). It is denoted by $\theta = arg(z)$. A complex number actually has infinitely many arguments of the form $\theta \pm 2k\pi$. This is denoted $\theta\,[2\pi]$, read θ **modulo** 2π. The argument contained in $]-\pi, \pi]$ is called the main argument. Three specific cases are worth noting: the complex number 0 has no argument; a pure real number has an argument of zero modulo π; a pure imaginary number has an argument equal to $\frac{\pi}{2}\,[\pi]$.

1.3.5. Argument calculation

Figures 1.3(a) and 1.3(b) show that whatever θ:

$$\cos\theta = \frac{a}{|z|}, \quad (1.16)$$
$$\sin\theta = \frac{b}{|z|}. \quad (1.17)$$

Warning , a and b can be negative. Furthermore, the function $acos\frac{a}{|z|}$ on the calculator only provides the absolute value of θ. The sign of the principal value of θ is then equal to the sign of $\sin\theta$.



(a) (b)

Figure 1.3: (a) Modulus and argument and (b) Trigonometric circle.

Application:
Let be the complex numbers $z_1 = 2 - i$ and $z_2 = 1 + 2i$.

1. Calculate their moduli and arguments.

1.3.6. Properties of modules:

For all complex numbers z and z':

- Modulus of a product:

$$|zz'| = |z||z'|. \tag{1.18}$$

- Consequence: if $\lambda \in \mathbb{R}$, $|\lambda z| = |\lambda||z|$,
- Modulus of a quotient:

$$|\frac{z}{z'}| = \frac{|z|}{|z'|}. \tag{1.19}$$

- Consequence: for all $z \neq 0$, $|\frac{1}{z}| = \frac{1}{|z|}$,
- Triangular inequality:

$$|z + z'| leq |z| + |z'|. \tag{1.20}$$

1.3.7. Properties of arguments:

For any complex number $z \in \mathbb{C}^*$ (see figure 1.4):

- Conjugate argument:

$$arg(\overline{z}) = -arg(z)\,[2\pi]\,. \tag{1.21}$$

- Argument of the opposite:

$$arg(-z) = arg(z) + \pi\,[2\pi]\,. \tag{1.22}$$

- Argument of the opposite of the conjugate:

$$arg(-\overline{z}) = \pi - arg(z)\,[2\pi]\,. \tag{1.23}$$

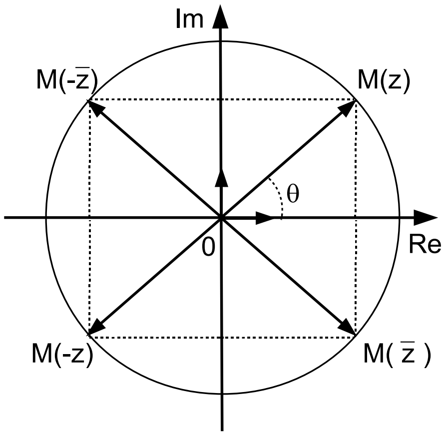


Figure 1.4: Conjugate and opposite arguments.

1.3.8. Note:

- If $\lambda \in \mathbb{R}_+^*$:

$$arg(\lambda z) = arg(z)\,[2\pi]\,, \tag{1.24}$$

- If $\lambda \in \mathbb{R}_-^*$:

$$arg(\lambda z) = arg(z) + \pi\,[2\pi]\,, \tag{1.25}$$

1.3.9. Properties of the arguments:

For all complex numbers $z, z' \in \mathbb{C}^*$:

- Argument of a product:

$$arg(zz') = arg(z) + arg(z')\,[2\pi]\,. \tag{1.26}$$

- Inverse argument:

$$arg(\frac{1}{z}) = -arg(z)\,[2\pi]\,. \tag{1.27}$$

- Argument of a quotient:

$$arg(\frac{z}{z'}) = arg(z) - arg(z')\,[2\pi]\,. \tag{1.28}$$

- Argument of a power:

$$arg(z^n) = narg(z)\,[2\pi]\,\forall n \in \mathbb{Z}. \tag{1.29}$$

1.4. Polar shape

1.4.1. Writing a complex number:

As a result of the properties described above, a complex number can be written in several different ways, and it's up to the user to choose the most appropriate form for the situation in hand:

- Cartesian or algebraic form:

$$z = a + ib. \tag{1.30}$$

- Polar form:

$$\begin{aligned} z &= |z|(\cos(\theta) + i\sin(\theta)), && (1.31)\\ z &= |z|e^{i\theta}. && (1.32) \end{aligned}$$

It may be worth noting here that equations 1.16 and 1.17 are in agreement with the above relationships and that in particular:

$$\begin{aligned} a &= Re(z) = |z|\cos(\theta), && (1.33)\\ b &= Im(z) = |z|\sin(\theta). && (1.34) \end{aligned}$$

1.4.2. Complex exponential:

For any real θ, the complex number $\cos(\theta) + i\sin(\theta)$ is denoted $e^{i\theta}$. The properties of the exponential function are then preserved. In particular:

- Exponential of a sum:

$$e^{i(\theta+\theta')} = e^{i\theta}e^{i\theta'}. \quad (1.35)$$

- Exponential of a subtraction:

$$e^{i(\theta-\theta')} = \frac{e^{i\theta}}{e^{i\theta'}}. \quad (1.36)$$

- Power of an exponential:

$$(e^{i\theta})^n = e^{ni\theta} \quad \text{for any } n \in \mathbb{Z}. \quad (1.37)$$

1.5. Moivre and Euler formulas

These formulas greatly simplify the solution of certain problems. The downside is that you need to be familiar with them. However, it can be useful to use the trigonometric circle (figure 1.5) to find them.

1.5.1. Moivre formulas:

For any real θ and any $n \in \mathbb{Z}$,

$$\begin{aligned} (\cos\theta + i\sin\theta)^n &= \cos(n\theta) + i\sin(n\theta), && (1.38)\\ (\cos\theta - i\sin\theta)^n &= \cos(n\theta) - i\sin(n\theta). && (1.39) \end{aligned}$$

1.5.2. Euleur formulas:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad (1.40)$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \quad (1.41)$$

1.6. Square roots of a complex number

1.6.1. Square root of a complex number:

For any complex number $z \in \mathbb{C}$, any complex number z', such that $z'^2 = z$, is called the complex square root of z. There are exactly two square roots of any complex number $z \in \mathbb{C}$. Writing $\sqrt{a + ib}$ therefore makes no sense, and we prefer $(a + ib)^{1/2}$ (which corresponds to two complex numbers!).

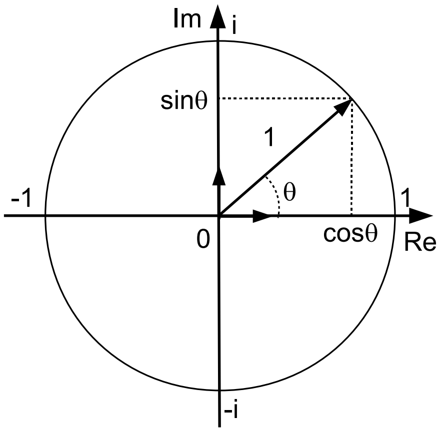


Figure 1.5: Trigonometric circle.

1.6.2. Calculating a square root:

According to the equation (1.37), for any complex number $z = |z|e^{i(\theta+2k\pi)} \in \mathbb{C}$, with $k \in \mathbb{Z}$:

$$z^{1/2} = \sqrt{|z|}e^{i(\frac{\theta}{2}+k\pi)}. \tag{1.42}$$

This clearly demonstrates the existence of two complex numbers $\sqrt{|z|}e^{i(\frac{\theta}{2}+2k\pi)}$ and $\sqrt{|z|}e^{i(\frac{\theta}{2}+\pi+2k\pi)}$.

Application:

1. Determine, in the complex plane, the positions of the two square roots of a complex number z. Use the trigonometric circle to do this.

1.7. n^{th}-Roots of a complex number

For any complex number $z \in \mathbb{C}$, is called complex n^{th} root of z any complex number z', such that $z'^n = z$. There are exactly n n^{th} -roots of any complex number $z \in \mathbb{C}$. Writing $\sqrt[n]{a+ib}$ therefore makes no sense, and we prefer $(a+ib)^{1/n}$ (which corresponds to n complex numbers!).

1.7.1. Calculating a n^{th}-root:

According to equation (1.37), for any complex number $z = |z|e^{i(\theta+2k\pi)} \in \mathbb{C}$, with $k \in \mathbb{Z}$:

$$z^{1/n} = \sqrt[n]{|z|}e^{i(\frac{\theta+2k\pi}{n})}. \tag{1.43}$$

Application:

1. Determine, in the complex plane, the positions of the n^{th}-roots of a complex number z. Use the trigonometric circle to do this.

1.8. Elementary functions of a complex number

1.8.1. Solving a second-degree equation:

Let be an equation of the second degree of the form $ax^2 + bx + c = 0$. Its discriminant Δ is of the form $\Delta = b^2 - 4ac$. If $\Delta > 0$, the equation has **exactly** two real solutions:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}. \tag{1.44}$$

If $\Delta = 0$, the equation has a double root: $x = \frac{-b}{2a}$. If $\Delta < 0$, the equation has **exactly** two complex conjugate solutions:

$$x_{1,2} = \frac{-b \pm i\sqrt{-\Delta}}{2a}. \tag{1.45}$$

Consequences: In $\mathbb{C}$, a polynomial equation of order n always has n (complex) roots.

1.8.2. Exponential and sine functions:

From the above relationships, we can deduce the properties of functions of complex numbers, in particular exponential and sine functions.

- Exponential function: Let $z = x + iy$ be any complex number.

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)). \tag{1.46}$$

- Sine functions: These are reconstructed from Euler's formulas. Example:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}. \tag{1.47}$$

Application:

1. Recall the expression of $\sin(z)$ as a function of e^{iz} and e^{-iz}.

2

Linear algebra

2.1. Linear systems and matrices

Let be a linear system of two equations with two unknowns:

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad (2.1)$$
$$a_{21}x_1 + a_{22}x_2 = b_2. \quad (2.2)$$

It can be rewritten in condensed form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \quad (2.3)$$

In condensed form, this expression becomes:

$$\mathbf{AX} = \mathbf{B}. \quad (2.4)$$

$\mathbf{A}$ is then a matrix of dimension 2×2:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \quad (2.5)$$

Its elements are defined by:

$$\mathbf{A} = [a_{ij}]_{(i,j)\in\{1,2\}}.. \quad (2.6)$$

In the element a_{ij}, i represents the row at which a_{ij} lies, and j represents the column at which a_{ij} lies. A mnemonic way of remembering this is to recall that a matrix is written in "row-column" form. By generalization, a matrix $\mathbf{A}$ of dimension $m \times n$ is defined by:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(m-1)1} & a_{(m-1)2} & \dots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \dots & a_{m(n-1)} & a_{mn} \end{bmatrix}. \quad (2.7)$$

In general:

- A matrix of dimension $(1, n)$ is called a row matrix,
- A matrix of dimension $(m, 1)$ is called a column matrix,
- A matrix of dimension (m, m) is called a square matrix,
- A matrix of dimension (m, n), $m \neq n$, is called a rectangular matrix,
- A matrix is said to be real or complex depending on whether its elements are contained in $\mathbb{R}$ or $\mathbb{C}$.

2.2. Operations on matrices

2.2.1. Sum of two matrices:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{kl}]$ be two matrices of dimension $m \times n$, The matrix $\mathbf{C} = [c_{ij}]$, sum of $\mathbf{A}$ and $\mathbf{B}$ is also of dimension $m \times n$ and is defined by:

$$c_{ij} = a_{ij} + b_{ij} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.8}$$

Two matrices of different dimensions cannot be summed. The neutral element of addition is the null matrix $[0]_{ij} = \mathbf{0}, \quad \forall i, j$.

Application:

Calculate $\begin{bmatrix} 2 & 1 & 8 \\ 4 & 4 & 6 \\ 1 & 3 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 7 & -6 \\ 1 & -4 & 3 \\ 11 & -13 & 2 \end{bmatrix}$.

2.2.2. Multiplication by a scalar:

To multiply a matrix by a scalar, simply multiply each of its elements by that scalar:

$$\lambda \mathbf{A} = \lambda [a_{ij}]_{i,j} = [\lambda a_{ij}]_{i,j} = \mathbf{A}\lambda. \tag{2.9}$$

Application:

Calculate $3 \begin{bmatrix} 5 & 7 & -6 \\ 1 & -4 & 3 \\ 11 & -13 & 2 \end{bmatrix}$.

2.2.3. Transpose matrix:

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m \times n}$ (of dimension $m \times n$), the transpose matrix of $\mathbf{A}$, denoted $\mathbf{A}^T$, then belongs to $\mathbb{R}^{n \times m}$ (is of dimension $n \times m$) and is written:

$$\mathbf{A}^T_{ij} = a_{ji} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.10}$$

It follows that:

$$(\mathbf{A}^T)^T = \mathbf{A}. \tag{2.11}$$

Application:

What is the transpose of $\begin{bmatrix} 6 & 0 & 9 \\ -3 & 4 & -5 \end{bmatrix}$?

2.2.4. Conjugate transpose:

Let $\mathbf{A} = [a_{ij}] \in \mathbb{C}^{m \times n}$ (of dimension $m \times n$), the conjugate transpose matrix of $\mathbf{A}$, denoted $\mathbf{A}^*$, then belongs to $\mathbb{C}^{n \times m}$ (is of dimension $n \times m$) and is written:

$$\mathbf{A}^*_{ij} = \overline{a_{ji}} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.12}$$

Application:

What is the conjugate transpose of $\begin{bmatrix} 6+i & 0 & 9-i \\ -3i & 4 & -5 \end{bmatrix}$?

2.2.5. Product of two matrices:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be two matrices of dimensions $k \times l$ and $l \times m$ respectively (note that dimension l is present in both matrices). The matrix $\mathbf{C} = [c_{ij}]$, product of $\mathbf{A}$ and $\mathbf{B}$ is of dimension $k \times m$ (dimension l has disappeared) and is defined by:

$$c_{ij} = \sum_{p=1}^{l} a_{ip} b_{pj} \quad \forall 1 \leq i \leq k,\ 1 \leq j \leq m. \tag{2.13}$$

Because of their dimensions, it is possible to define the matrix product $\mathbf{A}\mathbf{A}^T$.

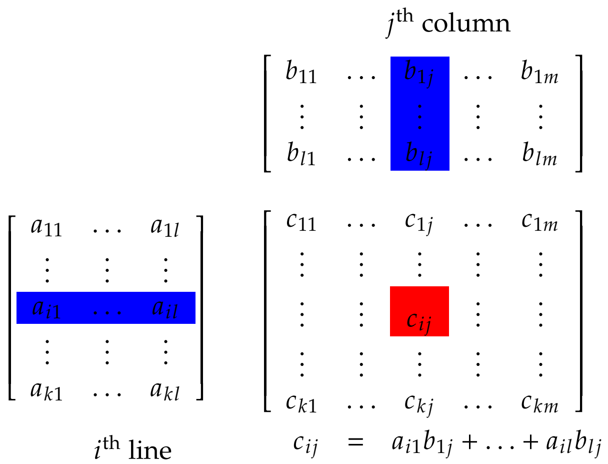
Application:
Let be the following matrix product:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \tag{2.14}$$

1. Determine the expressions of c_{11}, c_{12}, c_{21} and c_{22}.
2. Perform the application with the following matrices:

$$\mathbf{A} = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 7 \\ 6 & 0 \end{bmatrix}. \tag{2.15}$$

It can be useful to visualize the multiplication of two matrices. To do this, the matrices are placed as follows and each element ij of the product is written, in the center, by "projecting" the column j of $\mathbf{B}$ onto the row i of $\mathbf{A}$.



CAUTION! The product of two matrices is not commutative. In general, $\mathbf{AB} \neq \mathbf{BA}$ (if only in terms of dimensions). Subject to compatible dimensions, the product of several matrices is associative. Example:

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}. \tag{2.16}$$

The neutral element of the multiplication is the unit matrix $\mathbf{I}$:

$$\mathbf{I}_{ii} = 1, \quad \forall i = j, \tag{2.17}$$
$$\mathbf{I}_{ij} = 0, \quad \forall i \neq j. \tag{2.18}$$

The product of two matrices is right- and left-hand distributive with respect to the addition of the matrices:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}, \tag{2.19}$$
$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}. \tag{2.20}$$

Application:
A certain species of insect behaves as follows: on average, half the population dies at the end of the first year of life. Of the survivors, the $\frac{2}{3}$ die at the end of the second year of life. The remainder die at the end of the third year, but each female has already given birth to an average of 6 female individuals. The population census gives, for the year 2012: x_n individuals in the first year of life, y_n individuals in the second year of life and z_n individuals in the last year of life.

1. Determine the matrix $\mathbf{A}$ such that:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}. \tag{2.21}$$

2. Calculate $\mathbf{A}^2$ and $\mathbf{A}^3$.
3. What conclusions can be drawn about the evolution of the insect population?

2.2.6. Transpose of the sum of two matrices:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{kl}]$ be two matrices of the same dimension, then:

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T. \tag{2.22}$$

2.2.7. Transpose of the product of two matrices:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{kl}]$ be two matrices of dimensions $k \times l$ and $l \times m$ respectively, then:

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T. \tag{2.23}$$

2.3. Special Matrices

2.3.1. Squared Matrix:

A matrix is said to be square if it has the same number of rows as columns.

Example:

$$\mathbf{A} = \begin{bmatrix} 15 & 21 & -18 \\ 6 & -12 & 9 \\ 33 & -39 & 6 \end{bmatrix}. \tag{2.24}$$

2.3.2. Symmetric matrix:

A matrix is symmetric if it has elements that are symmetric with respect to the diagonal:

$$a_{ij} = a_{ji}. \tag{2.25}$$

It is necessarily a square matrix and $\mathbf{A} = \mathbf{A}^T$.

Example:

$$\mathbf{A} = \begin{bmatrix} 15 & 6 & 33 \\ 6 & -12 & 9 \\ 33 & 9 & 6 \end{bmatrix}. \tag{2.26}$$

2.3.3. Antisymmetric matrix:

A matrix is said to be antisymmetric if its diagonal-symmetric elements are opposite:

$$a_{ij} = -a_{ji}. \tag{2.27}$$

This is necessarily a square matrix.

Example:

$$\mathbf{A} = \begin{bmatrix} 15 & -6 & 33 \\ 6 & 0 & 9 \\ -33 & -9 & -6 \end{bmatrix}. \quad (2.28)$$

2.3.4. Diagonal matrix:

A matrix is said to be diagonal if its off-diagonal elements are zero:

$$a_{ij} = 0 \quad \text{si } i \neq j. \quad (2.29)$$

This is necessarily a square matrix.

Example:

$$\mathbf{A} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix} \; ; \; \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.30)$$

2.3.5. Triangular matrix:

A matrix is said to be upper (resp. lower) triangular if its elements below (resp. on) the diagonal are zero:

$$a_{ij} = 0 \quad \text{si } i > j \text{ (resp. } i < j). \quad (2.31)$$

This is necessarily a square matrix.

Example:

$$\mathbf{A} = \begin{bmatrix} 15 & 8 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & -6 \end{bmatrix} ; \mathbf{B} = \begin{bmatrix} 15 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 8 & -6 \end{bmatrix}. \quad (2.32)$$

2.3.6. Idempotent matrix:

A matrix is said to be idempotent if:

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}. \quad (2.33)$$

This is necessarily a square matrix.

Exemple:

$$\mathbf{I}. \quad (2.34)$$

2.3.7. Translation matrix:

In a space of dimension 3×3, a translation matrix of dimension 4×4 is defined by:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2.35)$$

Il vient alors:

$$\mathbf{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + t_1 \\ x_2 + t_2 \\ x_3 + t_3 \\ 1 \end{bmatrix}. \quad (2.36)$$

2.3.8. Rotation matrix:

Rotation matrices (of dimension 3×3) group together the three projections of an orthonormal basis $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ into another $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$. The 'canonical' rotations about $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are written:

$$\mathbf{Rot}(\mathbf{x}, \alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \tag{2.37}$$

$$\mathbf{Rot}(\mathbf{y}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \tag{2.38}$$

$$\mathbf{Rot}(\mathbf{z}, \gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.39}$$

The orthonormal basis $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ is thus transformed into another $(\mathbf{x}', \mathbf{y}', \mathbf{z}')$ according to (for example for $\mathbf{Rot}(\mathbf{x}, \alpha)$):

$$\mathbf{Rot}(\mathbf{x}, \alpha) = \begin{matrix} \\ \mathbf{x}' \ldots \\ \mathbf{y}' \ldots \\ \mathbf{z}' \ldots \end{matrix} \overset{\begin{matrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \vdots & \vdots & \vdots \end{matrix}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}}. \tag{2.40}$$

Application:
Using the rotation matrices defined above:

1. Determine the expressions of the three rotation matrices of 90°, about the principal axes.
2. By applying them successively to the vector $\mathbf{T} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, what are the consequences for its components?
3. Determine the expressions of the three rotation matrices of 60°, about the principal axes.
4. By applying them successively to the vector $\mathbf{T} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, what are the consequences for its components?
5. Do the application with vectors (make a diagram):
   (a) $\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$,
   (b) $\mathbf{T} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.
6. How do you model a rotation around a non-principal axis?

2.4. Partitioned matrices:

It can sometimes be interesting to partition a matrix into sub-blocks. So let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{(m+n)\times(p+q)}$. It is possible to write:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tag{2.41}$$

with $\mathbf{A}_{11} \in \mathbb{R}^{m\times p}$, $\mathbf{A}_{12} \in \mathbb{R}^{m\times q}$, $\mathbf{A}_{21} \in \mathbb{R}^{n\times p}$, $\mathbf{A}_{22} \in \mathbb{R}^{n\times q}$. It is of course possible to partition $\mathbf{A}$ into more sub-blocks.

2.4.1. Operations:

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{(m+n)\times(p+q)}$ and $\mathbf{B} \in \mathbb{R}^{(p+q)\times 1}$. It is possible to write:

$$\mathbf{AB} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_1 + \mathbf{A}_{12}\mathbf{B}_2 \\ \mathbf{A}_{21}\mathbf{B}_1 + \mathbf{A}_{22}\mathbf{B}_2 \end{bmatrix}. \tag{2.42}$$

2.4.2. Transposed matrix:

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{(m+n)\times(p+q)}$. $\mathbf{A}^T$ is written:

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T \end{bmatrix}. \tag{2.43}$$

2.5. Determinant of a matrix:

The determinant of a square matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a number, denoted $det(\mathbf{A})$ or $|\mathbf{A}|$, defined by:

$$|\mathbf{A}| = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} |\mathbf{A}_i|. \tag{2.44}$$

Application:

1. Calculate $|\mathbf{A}|$ for a square matrix of dimension $n = 2$.
2. Do the application with $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.
3. Calculate $|\mathbf{A}|$ for a square matrix of dimension $n = 3$.
4. Apply with $\mathbf{A} = \begin{bmatrix} 5 & -5 & 1 \\ 0,1 & 3 & 2 \\ 2 & -1 & -0,5 \end{bmatrix}$.

2.5.1. Properties:

- When the determinant of a matrix is zero, the matrix is said to be singular.
- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$,
- $|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}$,
- Si $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, then:

$$|\mathbf{A}| = |\mathbf{A}_{22}||\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}| \tag{2.45}$$

$$= |\mathbf{A}_{11}||\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}|. \tag{2.46}$$

2.5.2. Theorems:

- If all the elements of a row (column) of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ are zero, then $|\mathbf{A}| = 0$,
- If all the elements of a row (column) of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ are multiplied by a scalar λ, then $|\mathbf{B}| = \lambda|\mathbf{A}|$.
- If $\mathbf{B} \in \mathbb{R}^{n\times n}$ is obtained from $\mathbf{A}$ by exchanging two of its rows (columns), then $|\mathbf{B}| = -|\mathbf{A}|$.
- If $\mathbf{B} \in \mathbb{R}^{n\times n}$ is obtained from $\mathbf{A}$ by passing the i^{th} row (column) over the p^{th} row (column), then $|\mathbf{B}| = (-1)^{(i-p)}|\mathbf{A}|$.
- If two rows (columns) of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ are identical, then $|\mathbf{A}| = 0$,
- If the elements of a row (column) of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ are added k times the elements of another row (column), the value of the determinant remains unchanged.

2.6. Minors and cofactors of a matrix:

The minor m_{ij} of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the determinant of the remaining part of $\mathbf{A}$ when row i and column j are "deleted". The cofactor of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, denoted c_{ij} or Δ_{ij}, is defined by:

$$c_{ij} = (-1)^{i+j} m_{ij}. \tag{2.47}$$

The comatrix (adjoint matrix) of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, denoted $\tilde{\mathbf{A}}$, is defined by:

$$\tilde{\mathbf{A}} = [c_{ij}]^T. \tag{2.48}$$

2.7. Cofactors and determinant of a matrix:

2.7.1. Rules for calculating the determinant from cofactors:

The value of the determinant of a square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the sum of the n products obtained by multiplying each element of a given row (column) of the matrix by its cofactor.

Application:

1. Check the validity of the above rule by verifying the following calculation:

$$\left\| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right\| =$$

$$a_{11} \left\| \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right\| + a_{21} \left(- \left\| \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \right\| \right) + a_{31} \left\| \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \right\|$$

$$= -a_{12} \left\| \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \right\| + a_{22} \left\| \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \right\| - a_{32} \left\| \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} \right\|.$$

Consequence:
By judiciously choosing the row (or column) through which to expand, you can greatly simplify calculations. Rows (or columns) with zero elements are particularly sought-after.

Application:

1. Calculate $|\mathbf{A}|$ for a square matrix of dimension $n = 2$.
2. Make the application with:

   (a) $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$,

   (b) $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$.

3. Calculate $|\mathbf{A}|$ for a square matrix of dimension $n = 3$.
4. Make the application with:

   (a) $\mathbf{A} = \begin{bmatrix} 15 & -6 & 33 \\ 6 & 0 & 9 \\ -33 & -9 & -6 \end{bmatrix}$,

   (b) $\mathbf{A} = \begin{bmatrix} 5 & -5 & 1 \\ 0,1 & 3 & 2 \\ 2 & -1 & -0,5 \end{bmatrix}$.

2.8. Solving a system of linear equations:

2.8.1. Kramer's rule:

Let be a linear system of three equations with three unknowns:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

then:

$$x_1 = \frac{\left\|\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}\right\|}{|\mathbf{A}|}. \tag{2.49}$$

Likewise:

$$x_2 = \frac{\left\|\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}\right\|}{|\mathbf{A}|}; \; x_3 = \frac{\left\|\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}\right\|}{|\mathbf{A}|}. \tag{2.50}$$

This rule can be applied to any system of n equations with n unknowns **provided that the determinant of A is non-zero**.

Application:

A mouse population contains 100 individuals. It is made up of gray males and white females. After one month of uncontrolled development, the number of individuals in the population is 284. During the course of the experiment, 25% of the reproductions resulted in a white mouse. Given that no mice died during the experiment, and that the male and female populations were multiplied by 2 and 3 respectively:

1. Determine the number of males and females at the start of the experiment.
2. Determine the number of gray mice and the number of white mice at the end of the experiment.

Application:

A farmer delivered 30 tonnes of wheat, 45 tonnes of sunflower and 75 tonnes of sorghum to his cooperative. In exchange, he received 234 M€. The price per tonne of wheat is the average of the price per tonne of sunflower and the price per tonne of sorghum. On the other hand, if he had delivered one tonne of each of these products, he would have received 5.1 million.

1. What is the price paid per tonne for each of the products delivered?

2.9. Inverse matrix:

2.9.1. Definition:

Two square matrices **A** and **B** are inverses if their product is equal to the identity matrix:

$$\mathbf{AB} = \mathbf{I}. \tag{2.51}$$

B is then denoted $\mathbf{A}^{-1}$ and $\mathbf{BA} = \mathbf{I}$. The inverse matrix is used to solve systems of linear equations of the form $\mathbf{Ax} = \mathbf{y}$. In fact, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$ (attention, multiplication on the left).

2.10. Pseudo-inverse matrix:

This is the extension of the inverse matrix concept to non-square systems. Let $\mathbf{A} \in \mathbb{R}^{mtimesn}$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$, the pseudo-inverse of **A**, denoted $\mathbf{A}^+$ satisfies the following conditions:

$$\mathbf{AA}^+\mathbf{A} = \mathbf{A}, \tag{2.52}$$

$$\mathbf{A}^+\mathbf{AA}^+ = \mathbf{A}^+, \tag{2.53}$$

$$(\mathbf{AA}^+)^T = \mathbf{AA}^+ \in \mathbb{R}^{m\times m}, \tag{2.54}$$

$$(\mathbf{A}^+\mathbf{A})^T = \mathbf{A}^+\mathbf{A} \in \mathbb{R}^{n\times n}. \tag{2.55}$$

There is a pseudo-inverse matrix on the left ($\in \mathbb{R}^{n\times m}$) and a pseudo-inverse matrix on the right ($\in \mathbb{R}^{n\times m}$). The resolution of the system is then $\mathbf{x} = \mathbf{A}^{+}\mathbf{y}$ (note that the solution does not necessarily exist). The pseudo-inverse matrix on the left can be used to solve oversized problems (more equations than unknowns). It minimizes the mean square error (solution closest to $\mathbf{y}$ in the least squares sense). The right-hand pseudo-inverse matrix is used to solve under-dimensioned problems (fewer equations than unknowns). It minimizes the mean square norm.

2.10.1. Properties of the inverse matrix:

If the system is square and $\mathbf{A} \in \mathbb{R}^{n\times n}$ (and $\mathbf{B} \in \mathbb{R}^{n\times n}$) non-singular(s) (non-zero determinant):

$$\mathbf{A}^{-1} = \mathbf{A}^{+}, \tag{2.56}$$
$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}, \tag{2.57}$$
$$(\mathbf{A}^{-1})^{T} = (\mathbf{A}^{T})^{-1}, \tag{2.58}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}, \tag{2.59}$$
$$|\mathbf{A}^{-1}| = |\mathbf{A}|^{-1}. \tag{2.60}$$

The system can then be solved thanks to $\mathbf{x} = \mathbf{A}^{+}\mathbf{y}$.

2.10.2. Inverse matrix calculation:

The inverse matrix $\mathbf{A}^{-1}$ of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is given by the relation (cofactor or adjoint matrix divided by the determinant of $\mathbf{A}$):

$$\mathbf{A}^{-1} = \frac{\mathbf{A}}{|\mathbf{A}|}. \tag{2.61}$$

The determinant of $\mathbf{A}$ must therefore be non-zero! Otherwise, the system has no solution. The matrix is then said to be regular or non-singular. There are other methods for calculating the inverse of a matrix, such as the Gauss pivot.

Application:
Returning to the previous exercise on the delivery of grain by a farmer to his cooperative:

1. Determine the inverse matrix of the matrix of the solved system.
2. Check the validity of your calculation by solving the problem again using the inverse matrix.

2.11. Rank of a matrix:

The rank of a matrix is the maximum number of linearly independent columns or rows. It is also the order of the largest non-zero determinant. A matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ is said to have full rank if $rang(\mathbf{A}) = min(m, n)$. A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible if its rank is equal to n.

2.11.1. Properties:

Whatever the matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$:

$$rang(\mathbf{A}) \quad = \quad rang(\mathbf{A}^{T}) = rang(\mathbf{A}\mathbf{A}^{T}) = rang(\mathbf{A}^{T}\mathbf{A}). \tag{2.62}$$

2.12. Trace of a matrix:

The trace of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is equal to the sum of the elements of its diagonal:

$$trace(\mathbf{A}) \quad = \quad \sum_{i=1}^{n} a_{ii}. \tag{2.63}$$

Application:

1. Prove the following property:
   - $trace(\mathbf{AB}) = trace(\mathbf{BA})$:

2.12.1. Eigenvalues and eigenvectors of a matrix:

2.12.2. Definition:

Let a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. The eigenvalues λ_i of this matrix are the roots (solutions) of the polynomial equation:

$$|\mathbf{A} - \lambda_i \mathbf{I}| = 0. \tag{2.64}$$

The set of eigenvalues of a matrix is its spectrum. The $\mathbf{v}_i$ eigenvectors of the matrix can be deduced from:

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i. \tag{2.65}$$

They are defined to within one constant.

2.12.3. Properties:

Let a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$:

$$|\mathbf{A}| = \prod_{i=1}^{n} \lambda_i, \tag{2.66}$$

$$trace(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i. \tag{2.67}$$

2.13. Diagonalization of a matrix:

If the eigenvalues of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are real and simple (all different, no multiple root), it is possible to transform $\mathbf{A}$ and “make” it diagonal in the form:

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots \lambda_i & \\ 0 & & \ddots \lambda_n \end{bmatrix}. \tag{2.68}$$

Let $\mathbf{P}$ be the matrix formed by the eigenvectors:

$$\mathbf{P} = [\mathbf{v_1} \ldots \mathbf{v_i} \ldots \mathbf{v_n}]. \tag{2.69}$$

This is called the $\mathbf{P}$ model matrix. The $\mathbf{D}$ matrix is then defined by:

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}. \tag{2.70}$$

$\mathbf{A}$ is then **conjugate** by $\mathbf{P}$. Diagonalization is an elegant way of solving problems that may seem complex at first glance. Numerical solutions are often used to calculate the eigenvalues and eigenvectors of a matrix.

3

Analysis

3.1. Real number sequences

A numerical sequence is an application u of $\mathbb{N}$, possibly deprived of a finite number of elements, in $\mathbb{R}$. The notation u_n is often preferred to $u(n)$ for the general term of the sequence (u_n). A sequence (u_n) is often defined by the expression of its general term or by a recurrence relation.

Examples:

- $u_n = \frac{1}{n}$,
- $u_n = \frac{n^2+2}{-3n^3}$,
- $u_n = \ln(\cos^2(n) + 2\frac{1}{n^2}e^{\sqrt{n}})$.

Application:

1. A pair of rabbits produces a new pair every month, which only becomes productive in the second month of its existence. Assuming pair fidelity and no mortality, how many rabbits will have been produced after one year?
2. What is the expression of the sequence that defines the number of rabbits as a function of the number of months?

3.1.1. Monotonous sequences:

- A sequence (u_n) is stationary if $u_n = u_{n+1}$, $\forall n \in \mathbb{N}$,
- A sequence (u_n) is increasing if $u_n \leq u_{n+1}$, $\forall n \in \mathbb{N}$,
- A sequence (u_n) is decreasing if $u_n \geq u_{n+1}$, $\forall n \in \mathbb{N}$,

3.1.2. Bounded sequences:

- A sequence (u_n) is increased if $\exists M \in \mathbb{R}$, such that $u_n \leq M$, $\forall n \in \mathbb{N}$,
- A sequence (u_n) is minorized if $\exists m \in \mathbb{R}$, such that $u_n \geq m$, $\forall n \in \mathbb{N}$,
- A sequence (u_n) is bounded if it is major and minor:

$$\exists M \in \mathbb{R}, \qquad |u_n| \leq M, \forall n. \tag{3.1}$$

3.1.3. Definition of $\lim\limits_{n \to +\infty} u_n = l$:

If any open interval containing a given number $l \in \mathbb{R}$ contains all the terms of the sequence from a certain rank, the sequence is said to converge to l.

3.1.4. Definition of $\lim_{n\to+\infty} u_n = +\infty$:

If any interval of the type $]A, +\infty[$ contains all the terms of the sequence from a certain rank, the sequence is said to diverge towards $+\infty$.

3.1.5. Definition of $\lim_{n\to+\infty} u_n = -\infty$:

If any interval of the type $]-\infty, -A]$ contains all the terms of the sequence from a certain rank, (u_n) is said to diverge towards $-\infty$.

3.1.6. Operations on convergent sequences:

3.1.7. Linear combination:

If (u_n) converges to l_1, and if (v_n) converges to l_2, then the sequence $(\lambda u_n + \mu v_n)$ converges to $\lambda l_1 + \mu l_2$.

Product:

If (u_n) converges to l_1, and if (v_n) converges to l_2, then the sequence $(u_n v_n)$ converges to $l_1 l_2$.

Quotient:

If (u_n) converges to l_1, and if (v_n) converges to $l_2 \neq 0$, then the sequence $\left(\frac{u_n}{v_n}\right)$ converges to $\frac{l_1}{l_2}$.

3.1.8. Image of a convergent sequence:

Let f be a function defined on an interval I, and a a point of I. f has limit l at point a if, and only if, for any sequence (x_n) converging to a, the sequence $(f(x_n))$ converges to l, finite or not. To show that a function has no limit when x tends to a, it is sufficient to provide a single example of a sequence (x_n) that tends to a and such that $(f(x_n))$ is divergent or does not tend to l.

3.1.9. Relation of order:

If (u_n) converges to l_1, and if (v_n) converges to l_2, and if $u_n \leq v_n$, $\forall n$, then $l_1 \leq l_2$.

3.1.10. Framing theorem:

If (u_n) and (v_n) are sequences converging to the same limit l, and if $u_n \leq w_n \leq v_n$, $\forall n$, then (w_n) has limit l.

3.1.11. Existence of limits:

Convergence of monotonic sequences:

- Any increasing sequence majoring by a given M is convergent to a limit $l \leq M$,
- Any non-major increasing sequence tends to $+\infty$,
- Any decreasing sequence minorized by a given m converges to a limit $l \geq m$,
- Any nonminorized decreasing sequence tends to $-\infty$.

Extracted sequences:

- A sequence (v_n) is said to be extracted from a sequence (u_n) if it is defined by $v_n = u_{h(n)}$ where h is a strictly increasing application of $\mathbb{N}$ in $\mathbb{N}$,
- (v_n) is also called sub-suite (u_n).
- If (u_n) is a convergent sequence of limit l, then any sub-sequence extracted from (u_n) is also convergent and converges to l,
- If two sequences extracted from (u_n) have distinct limits, then (u_n) is divergent,
- If two sequences extracted from (u_n) have the same limit, it is only possible to conclude if they cover all of (u_n).

Adjacent suites:

- Two sequences, (u_n) and (v_n), are adjacent if (u_n) is increasing, (v_n) is decreasing and $\lim_{n\to+\infty}(v_n - u_n) = 0$,
- If two sequences are adjacent, they converge and have the same limit.

Application:
Let u_n be the size of a population at generation n. This population follows a logistic growth if the passage from generation n to the next generation obeys the following formula:

$$\forall n \in \mathbb{N}, \quad u_{n+1} - u_n = a u_n \left(1 - \frac{u_n}{K}\right). \tag{3.2}$$

1. Interpret the constants a and K when the headcount is much less than K or, on the contrary, very close to K.
2. What happens if u_n becomes twice K?
3. What happens if $u_n = 1.5K$?
4. Starting with $u_1 = 100$ and choosing $K = 1000$, visualize and comment on the evolution of the population according to various values of a:

$$a = 1,2 \quad ; \quad a = 1,8 \quad ; \quad a = 2,2 \quad ; \quad a = 2,8.$$

5. With $K = 100$ and $u_1 = 10$, represent the generation of the first values of u_n based on the graphical representation of the f function of the model:
   (a) with $a = 1,2$,
   (b) with $a = 2,8$.

3.1.12. Suites Particuliers

3.1.13. Arithmetic sequences:

A sequence (u_n) is arithmetic of reason r if:

$$\forall n \in \mathbb{N}, \quad u_{n+1} = u_n + r. \tag{3.3}$$

To show that a sequence is arithmetic, we need to show that the difference between two consecutive terms is constant (does not depend on n). The general form of an arithmetic sequence is $u_n = u_0 + nr$. The sum of the first n terms is given by:

$$\sum_{k=0}^{n-1} u_k = n\frac{u_0 + u_{n-1}}{2}. \tag{3.4}$$

Application:
Five people are in a room. One of them notices that their ages are in arithmetical progression. Knowing that the sum of the squares of their ages is equal to the year in which this story takes place (i.e. 1980), and that between them they total 90 years, what is the age of each of them?

3.1.14. Geometric sequences:

A sequence (u_n) is geometric of reason $q \neq 0$ if:

$$\forall n \in \mathbb{N}, \quad u_{n+1} = q u_n. \tag{3.5}$$

To show that a sequence is geometric, we need to show that the quotient of two consecutive terms is constant (does not depend on n). The general form of a geometric sequence is $u_n = u_0 q^n$. The sum of the first n terms is given by:

$$\sum_{k=0}^{n-1} u_k = u_0 \frac{1 - q^n}{1 - q} \quad \text{si } q \neq 1, \tag{3.6}$$

$$= n u_0, \quad \text{si } q = 1. \tag{3.7}$$

The sequence (u_n) converges to 0 if $|q| < 1$. It is stationary if $q = 1$. It diverges in all other cases.

Application:

1. A water lily doubles in size every day. After 40 days, it has covered the entire pond. In how many days has it covered half the pond?
2. At a book fair, a textbook loses 12% of its value each year. A book bought new in 2005 cost 150 €. What is its price at the 2010 book fair?
3. What will its price be in 2015?

3.1.15. Linear recurrent sequences of order 2:

Linear recurrent sequences of order 2 are determined by knowing their first two terms u_0 and u_1, and a relation of the type:

$$\forall n \in \mathbb{N}, \quad au_{n+2} + bu_{n+1} + cu_n \quad = \quad 0, \tag{3.8}$$

where a, b, and c are real constants. They can be decomposed into sums of geometric sequences to simplify their study.

3.2. Functions of a real variable

f is a function defined on a set $D \subset \mathbb{R}$ with values in $\mathbb{R}$.

3.2.1. Direction of variation:

f is increasing on an interval I if $I \subset D$ and :

$$\forall x_1 \in I, \forall x_2 \quad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \leq f(x_2). \tag{3.9}$$

f is decreasing on an interval I if $I \subset D$ and :

$$\forall x_1 \in I, \forall x_2 \quad x_1 < x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2). \tag{3.10}$$

With strict inequalities, f is said to be strictly increasing or strictly decreasing. f is monotonic on an interval I if it is only increasing or only decreasing on I.

3.2.2. Parity, periodicity:

f is even if:

$$\forall x \in D, \quad -x \in D \text{ and } f(x) = f(-x), \tag{3.11}$$

Its graph is then symmetrical with respect to O_y. f is odd if:

$$\forall x \in D, \quad -x \in D \text{ et } f(x) = -f(-x), \tag{3.12}$$

Its graph is then symmetrical with respect to O. f is periodic with period T if:

$$\forall x \in D, \quad x + T \in D \text{ et } f(x+T) = f(x), \tag{3.13}$$

Its graph is then invariant through vectors $kT\vec{i}$ with $k \in \mathbb{Z}$.

3.2.3. Limits:

By definition:

- f admits a limit l when x tends towards a means that it's possible to get $f(x)$ as close to l as desired, provided x is sufficiently close to a. This is noted: $\lim_{x \to a} f(x) = l$.
- f admits as limit $\pm\infty$ when x tends towards a means that $f(x)$ can be as large (positive or negative) as desired provided x is sufficiently close to a. This is noted: $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$.

Application:
Do the following functions have limits at $x = 0$, when x tends to $\pm\infty$ or at any other point?

1. $f_1 = x^4 + x^2 - 1$,
2. $f_2 = \frac{x^2-1}{x+1}$,
3. $f_3 = \frac{\sqrt{x}}{2x-1}$.

3.2.4. Limit properties:

3.2.5. Framing theorem

Let f, g and h be three functions defined in the vicinity of x_0 and satisfying $f \leq g \leq h$. If f and h have the same limit l (finite or infinite) when x tends towards x_0 then g has limit l when $x \longrightarrow x_0$.

Algebraic operations

Let f and g be two functions defined in the vicinity of x_0 and admitting limits l and m there, and λ a real number. Then, the functions $f + g$, λf and fg have respective limits when $x \longrightarrow x_0$: $l + m$, λl and lm. Moreover, if $m \neq 0$, $1/g$ has a limit of $1/m$.

Compound function

Let f be a function defined in the vicinity of x_0 such that $\lim_{x \longrightarrow x_0} f(x) = u_0$. Let a second function g be defined in the neighborhood of u_0 and such that $\lim_{u \longrightarrow u_0} g(u) = v$. Then the function $g \circ f$ is defined in the vicinity of x_0 and:

$$\lim_{x \longrightarrow x_0} g \circ f(x) \quad = \quad \lim_{x \longrightarrow x_0} g(f(x)) = v. \tag{3.14}$$

3.2.6. Equivalent functions:

Let f and g be two functions defined on I and x_0 a point, finite or not, belonging to I or end of I. If $\frac{f}{g}$ is defined in the vicinity of x_0, except perhaps at x_0, f and g are said to be equivalent in the vicinity of x_0 if:

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 1. \tag{3.15}$$

This is noted: $f \sim g$ or $f \sim_{x_0} g$.

3.2.7. Properties of equivalent functions:

- If $f_1 \sim_{x_0} g_1$ and $f_2 \sim_{x_0} g_2$, then $f_1 f_2 \sim_{x_0} g_1 g_2$ $f_1/f_2 \sim_{x_0} g_1/g_2$.
- If $f \sim_{x_0} g$ and $\lim_{x \longrightarrow x_0} g(x) = l$, then $\lim_{x \longrightarrow x_0} f(x) = l$.
- To find the limit of a product or quotient, each function can be replaced by an equivalent function, chosen to simplify the calculation. Note that this must not be done for sums or compound functions.

3.2.8. Classic equivalent functions:

$$e^x - 1 \sim_0 x \quad ; \quad \sin x \sim_0 x \quad ; \quad 1 - \cos x \sim_0 \frac{x^2}{2} \quad ; \tag{3.16}$$

$$\ln(1 + x) \sim_0 x \quad ; \quad \tan x \sim_0 x \quad ; \quad (1 + x)^\alpha \sim_0 1 + \alpha x \tag{3.17}$$

A polynomial function is equivalent to:

- Its highest degree term when x tends to $\pm\infty$,
- Its lowest degree term when x tends to 0.

Question:

1. What's the point of this kind of relationship?
2. Give a practical example.

3.2.9. Continuity:

A function f is continuous at a point x_0 if, and only if,

$$\lim_{x \to x_0} f(x) = f(x_0) \tag{3.18}$$

(see figures 3.1(a) and 3.1(b)). A function f defined on an interval or union of intervals I is continuous on I if it is continuous at any point on I.

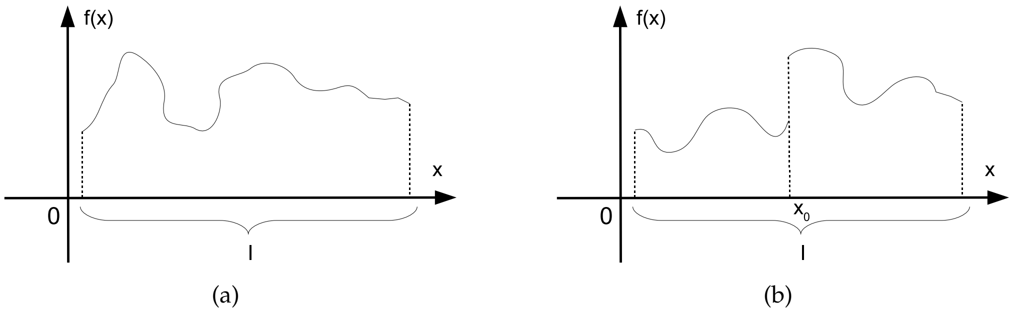


Figure 3.1: (a) Continuous and (b) discontinuous functions.

3.2.10. Continuity and operations:

Let f and g be two continuous functions in x_0, then:

- $f + g$ and fg are continuous at x_0,
- If $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous in x_0 and equals $\frac{f(x_0)}{g(x_0)}$.

If f is continuous in x_0 and g is continuous in $f(x_0)$, then $g \circ f$ is continuous in x_0.

3.2.11. Image of an interval by a continuous function:

If f is a continuous function on an interval I, then $f(I)$ is an interval.

Linear combination:

If f is a continuous function on a closed interval I, then $f(I)$ is a closed interval.

Consequence:

If a function f is continuous on $[a, b]$, and if $f(a)f(b) \leq 0$, then the equation $f(x) = 0$ has at least one solution in $[a, b]$.

3.2.12. Reciprocal function of a continuous strictly monotone function:

Let f be a continuous, strictly monotonic (increasing or decreasing) function on an interval I. Then:

- f is a bijection of I on $f(I)$, its reciprocal f^{-1} is continuous and strictly monotonic (resp. increasing or decreasing) on $f(I)$,
- In an orthonormal frame of reference, the graphs of f and f^{-1} are symmetrical about the first bisector of the axes (see figure 3.2).

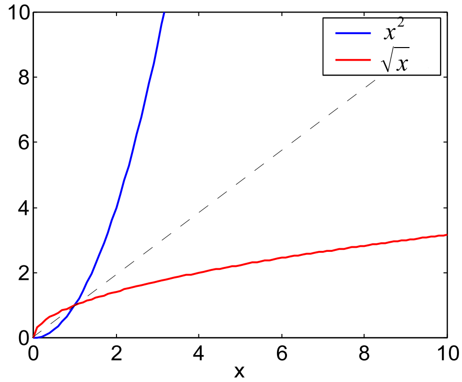


Figure 3.2: Reciprocal functions.

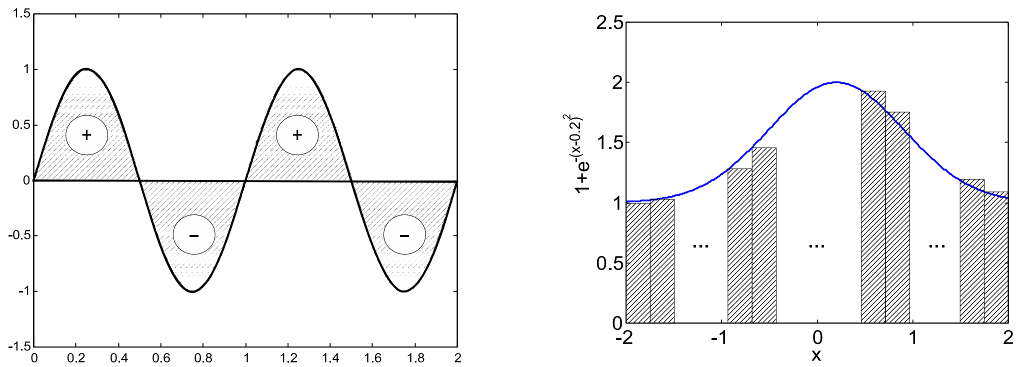
3.2.13. Integral calculation
3.2.14. Geometric interpretation:

Let f be a continuous function on an interval $[a, b]$. Then $\int_a^b f(x)dx$ reads "integral from a to b of $f(x)$, dx". It corresponds to the area of the plane domain below the graph of f, counted positively for the part above the x axis, negatively for the part below (see figure 3.3(a)).

3.2.15. Riemann integration:

From a numerical point of view, this involves calculating the limit of the sum of the areas of all the small rectangles contained under the curve, as the width of these rectangles tends towards 0 (see figure 3.3(b)):

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{\delta x \to 0} \sum_{k=0}^{n-1} f(x)\delta x \\ x &= a + k(b-a)/n, \\ \delta x &= (b-a)/n. \end{aligned}$$



(a) (b)

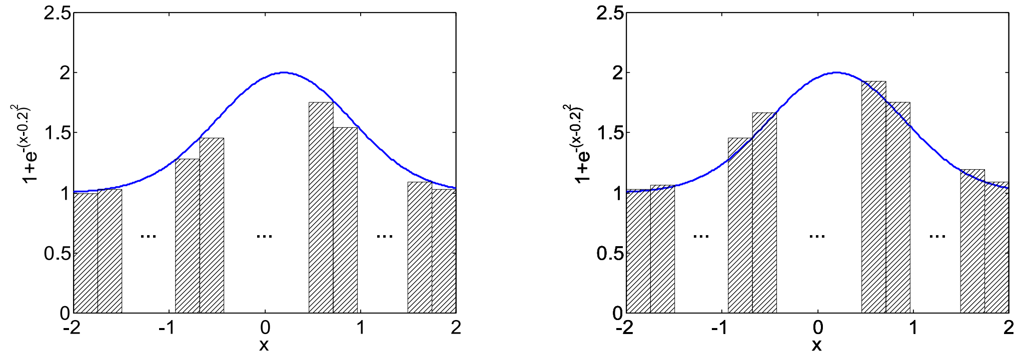
Figure 3.3: (a) Area and integral and (b) Riemann integration.

There are several ways to calculate this integral. By lower limit (see figure 3.4(a)):

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{n \to +\infty} \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x) \\ x_1 &= a, x_n = b. \end{aligned}$$

By upper limit (see figure 3.4(b)):

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to+\infty}\sum_{i=1}^{n}(x_i - x_{i-1})\sup_{x\in[x_{i-1},x_i]} f(x)\\ x_1 &= a, x_n = b.\end{aligned}$$



(a) (b)

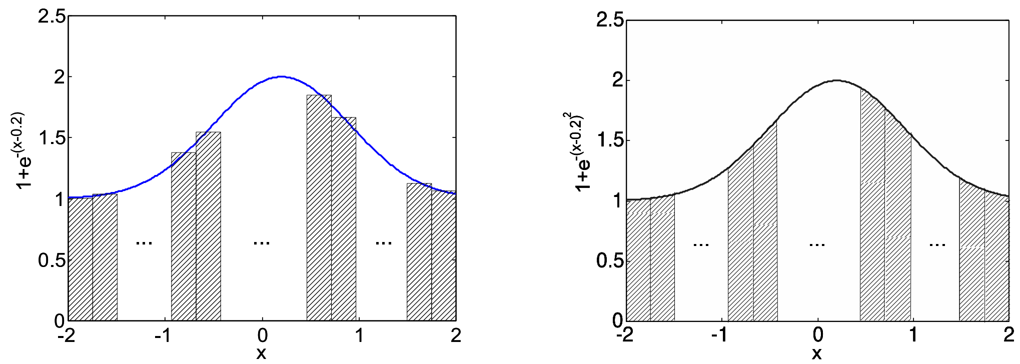
Figure 3.4: (a) lower and (b) upper limits

Using median and not necessarily mean values (see figure 3.5(a)):

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to+\infty}\sum_{i=1}^{n}(x_i - x_{i-1})f(\frac{x_i + x_{i-1}}{2})\\ x_1 &= a, x_n = b.\end{aligned}$$

Using average values:

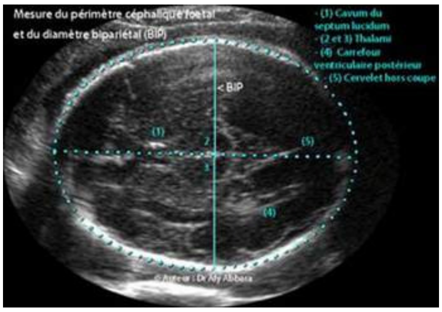
$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to+\infty}\sum_{i=1}^{n}(x_i - x_{i-1})\frac{f(x_{i-1}) + f(x_i)}{2}\\ x_1 &= a, x_n = b.\end{aligned}$$



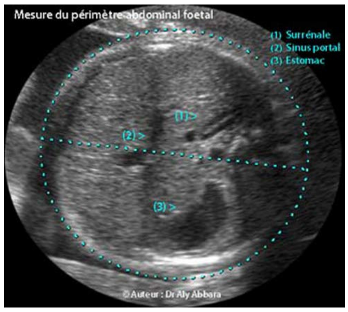
(a) (b)

Figure 3.5: Limits (a) by median values and (b) the trapezoid method

By the trapezoid method (see figure 3.5(b)). A practical interest of these calculations for medical purposes lies, for example, in the measurement of areas or volumes, as can be seen in figures 3.6(a) to 3.7(d).

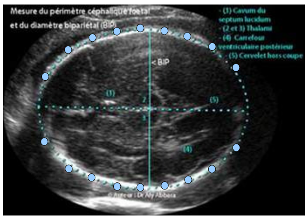


(a)

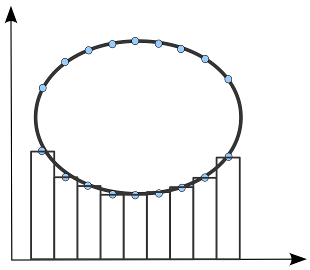


(b)

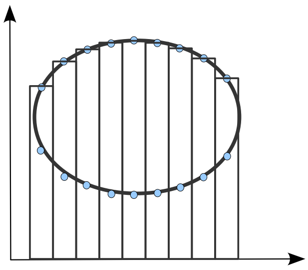
Figure 3.6: Fetal ultrasound and associated measurements



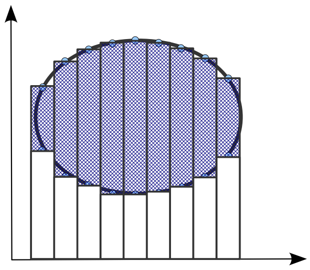
(a)



(b)



(c)



(d)

Figure 3.7: (a) Fetal ultrasound and associated surface measurement principles.

Furthermore, if $f(t)$ designates the temporal evolution of a flow rate, for example the flow of blood through an artery or of medication through an infusion, then $\int_a^b f(t)dt$ corresponds to the quantity of fluid flowed between instants a and b. As an example, the SERINGUE INFUSER - PHOENIX M-CP - CE 0459 (figure 3.8) allows:

- select infusion volume, infusion time and flow rate,
- select a drug library,
- monitor infusion pressure,
- select an adjustable occlusion alarm threshold.

For some of these functions, integral calculations are required.

Figure 3.8: SERINGUE INFUSER - PHOENIX M-CP - CE 0459.

Application:

1. For each of the methods presented above, write an algorithm for calculating the integral of any function f.
2. Apply this algorithm to the following calculations:
   (a) $\int_0^{2\pi} \sin(x)dx$,
   (b) $\int_0^1 x\cos(x)dx$.
3. What problems were encountered during the calculations?
4. Which is the most appropriate solution?

3.2.16. Properties of integral calculus:

3.2.17. Linearity:

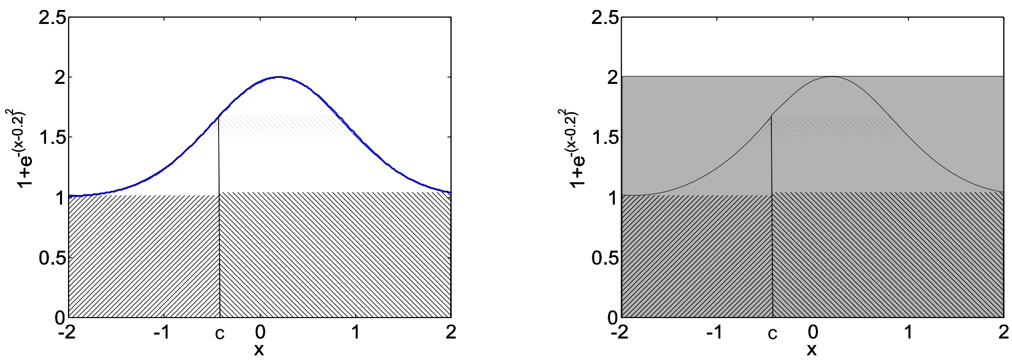
Integration is a linear function:

$$\int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x))dx. \quad (3.19)$$

Chasles relation:

The integral from a to c of a function $f(x)$ is equal to the sum of the integral from a to b and that from b to c (figure 3.9(a)):

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \quad (3.20)$$



(a) (b)

Figure 3.9: (a) Chasles relationship and (b) Inequality of the mean

3.2.18. Integral enhancement:

3.2.19. Absolute value:

$$\text{Si } a < b, \left|\int_a^b f(x)dx\right| \leq \int_a^b \left|f(x)\right| dx.$$

Framework:
If, for all $x \in [a, b]$, $m \leq f(x) \leq M$, then:

$$m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

Integral and average:
The number $\frac{1}{b-a}\int_a^b f(x)dx$ is the mean value of f on $[a, b]$.

Inequality of the mean:

$$\left|\int_a^b f(x)dx\right| \leq |b-a| \sup_{x\in[a,b]} \left|f(x)\right|.$$

This relationship is obvious from a geometric point of view (see figure 3.9(b)).

3.3. Calculating primitives

3.3.1. Primitives of a continuous function:

Let f be a function defined on an interval I. A function F, defined on I, is a primitive of f if it is derivable (of derivative function F') on I and if:

$$\forall x \in I,\ F'(x) = f(x). \tag{3.21}$$

Two primitives of a function f differ by only one constant. If f is continuous on an interval I containing a, the function F, defined on I by $F(x) = \int_a^x f(t)dt$ is a primitive of f. It is the only primitive of f that cancels at a. For any primitive h of f on an interval I:

$$\int_a^x f(t)dt = [h(t)]_a^x = h(x) - h(a). \tag{3.22}$$

Calculating integrals therefore boils down to finding primitive functions. For any function f with a continuous derivative on I:

$$f(x) - f(a) = \int_a^x f'(t)dt. \tag{3.23}$$

3.3.2. Linearity:

If F and G are respective primitives of the functions f and g on I, and λ and μ are two real numbers, then, on I, $\lambda F + \mu G$ is a primitive of $\lambda f + \mu g$.

3.3.3. Integration by parts:

Let u and v be two functions of class C^1 (derivable and continuously derived) on an interval I, and a and b be real numbers in I. Then:

$$\int_a^b u'(t)v(t)dt = [u(t)v(t)]_a^b - \int_a^b u(t)v'(t)dt. \tag{3.24}$$

Application:

1. To calculate $\int_a^b P(t)\sin(\alpha t+\beta)dt$, take $v(t) = P(t)$ and $u'(t) = sin(\alpha t + \beta)$,
2. To calculate $\int_a^b P(t)\cos(\alpha t+\beta)dt$, take $v(t) = P(t)$ and $u'(t) = \cos(\alpha t + \beta)$.

3.3.4. Integration by change of variable:

Let u be a function of class C^1 from $[\alpha,\beta]$ into $[a,b]$, and f a continuous function on $[a,b]$. Then:

$$\int_{u(\alpha)}^{u(\beta)} f'(u(t))u'(t)dt = \int_{u(\alpha)}^{u(\beta)} f(x)dx. \tag{3.25}$$

If moreover u is bijective:

$$\int_a^b f(x)dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f'(u(t))u'(t)dt. \tag{3.26}$$

Remember: if $x = u(t)$, then $dx = u'(t)dt$.

3.4. Generalized integrals

3.4.1. Function defined on an unbounded interval:

Let f be a function defined on $[a,+\infty[$ and integrable on any segment $[a,x]$. f is said to have a convergent integral on $[a,+\infty[$ or that the integral $\int_a^{+\infty} f(t)dt$ converges or exists if the function

$$x \mapsto \int_a^x f(t)dt \tag{3.27}$$

has a finite limit when x tends to $+\infty$. This is noted:

$$\lim_{x\to+\infty}\int_a^x f(t)dt = \int_a^{+\infty} f(t)dt. \tag{3.28}$$

Otherwise, the integral is said to be divergent.

3.4.2. Function defined on an unbounded interval:

Analogously:

$$\lim_{x\to-\infty}\int_x^a f(t)dt = \int_{-\infty}^{a} f(t)dt. \tag{3.29}$$

Then, with any a:

$$\int_{-\infty}^{+\infty} f(t)dt = \int_{-\infty}^{a} f(t)dt + \int_a^{+\infty} f(t)dt. \tag{3.30}$$

3.4.3. Unbounded function on a bounded interval:

Let f be an integrable function on any segment $[x,b]$ of on $]a,b]$. If the limit

$$\lim_{x\to a}\int_x^b f(t)dt \tag{3.31}$$

exists, the integral $\int_a^b f(t)dt$ is said to be convergent. Otherwise, it is said to be divergent. The definition is similar for a function defined on an interval $[a,b[$.

3.4.4. Convergence rules:

The following rules are presented for positive functions. Similar rules can be deduced for negative functions.

3.4.5. Compare functions:

Let f and g be integrable on any segment and such that $0 \leq f \leq g$ on $[a, +\infty[$.

- If $\int_a^{+\infty} g(t)dt$ converges, then $\int_a^{+\infty} f(t)dt$ also converges.
- If $\int_a^{+\infty} f(t)dt$ diverges, then $\int_a^{+\infty} g(t)dt$ also diverges.

Equivalence of functions:

Let f and g be two positive functions.

- If $f(x) \sim_{+\infty} g(x)$ then the integrals $\int_a^{+\infty} f(t)dt$ and $\int_a^{+\infty} g(t)dt$ are of the same kind.
- If $f(x) \sim_a g(x)$ then the integrals $\int_a^b f(t)dt$ and $\int_a^b g(t)dt$ are of the same kind.
- It is important that the functions have the same sign in the vicinity of the problem under study. Otherwise, the functions may be equivalent, but their integrals of different natures.

Riemann integrals:

For $[a, +\infty[$, with $a > 0$:

$$\int_a^{+\infty} \frac{1}{t^\alpha} dt \text{ converges} \Longleftrightarrow \alpha > 1. \tag{3.32}$$

For $]0, a]$, with $a > 0$:

$$\int_0^{a} \frac{1}{t^\alpha} dt \text{ converges} \Longleftrightarrow \alpha < 1. \tag{3.33}$$

3.5. Differential equations

Many problems can only be solved in the form of a differential equation (DE), i.e. an equation combining certain terms and their derivatives. Specific methods have been developed to simplify the solution of such problems, depending on the form of the corresponding differential equation.

Example:

Consider a contagious disease in a population. Three subsets can be identified:

- $S(t)$: number of healthy individuals,
- $I(t)$: number of sick individuals,
- $R(t)$: number of individuals dead, or cured and immune to the disease.

Modeling data are as follows:

- Contamination proportional to the number of encounters between healthy and sick individuals,
- The sick have a certain probability of recovery per unit of time.

The equation then imposes:

$$\partial t S = -rSI, \tag{3.34}$$

$$\partial t I = rSI - aI, \tag{3.35}$$

$$\partial t R = aI. \tag{3.36}$$

This is clearly a system of coupled differential equations. Note: $S + I + R$ does not depend on time (conservation of the number of individuals).

3.5.1. Equations with separable variables

When the equation is of the form:

$$f(x(t))x'(t) = g(t), \quad (3.37)$$

where f and g are given functions whose primitives F and G are known, then

$$F(x(t)) = G(t) + C. \quad (3.38)$$

And if F has a reciprocal function F^{-1}, the ED solutions are given by:

$$x(t) = F^{-1}(G(t) + C). \quad (3.39)$$

3.5.2. Linear equations

They are of the form:

$$a(t)x'(t) + b(t)x(t) = c(t), \quad (3.40)$$

where a, b and c are given continuous functions on an interval I. For the solution, it will be assumed that the study takes place on an interval $J \subset I$ in which a does not cancel (otherwise, the equation is not differential and the solution is obvious).

Theorem due to linearity:

Any solution of (3.40) is of the form $x_P(t) + x_S(t)$ where $x_P(t)$ is a particular solution of (3.40) and $x_S(t)$ the general solution of the associated homogeneous equation (without second member):

$$a(t)x'(t) + b(t)x(t) = 0. \quad (3.41)$$

Resolution is therefore a two-stage process.

Solving the associated homogeneous equation:

This is an equation with separable variables. Its solutions are of the type:

$$x_S(t) = Ke^{-F(t)} \quad \text{ou} \quad F(t) = \int_{t_0}^{t} \frac{b(u)}{a(u)} du \quad (3.42)$$

with K an arbitrary constant and t_0 any element of I. They include the null function and functions that never cancel.

Search for a particular solution:

This method is called **constant variation method**. Since $x_1(t)$ is a non-zero solution of (3.41), an unknown auxiliary function $K(t)$ is introduced so that $x(t) = K(t)x_1(t)$ is a solution of (3.40). When $x'_1(t)$ is calculated and then $x'(t)$ and $x(t)$ are transferred to (3.40), $K(t)$ must disappear (self-checking), while $K'(t)$ must remain. It is then possible to deduce $K(t)$ and then $x(t)$. There are two equivalent variants:

- Find all $K(t)$ with an integration constant, then plot in $x(t)$,
- Find one $K(t)$, plot in $x(t)$ and add with $x_S(t)$.

Application (1):

1. "The relative increase in coronary risk r for the same increase in cholesterol C is constant at all cholesterol levels". Which of the following equalities, where k is constant, reflects this statement?

$$\mathbf{1.}\ \frac{dr}{r} = \frac{k}{C} \quad \mathbf{2.}\ \frac{dr}{r} = kC \quad \mathbf{3.}\ \frac{dr}{r} = kdC \quad (3.43)$$
$$\mathbf{4.}\ \frac{dr}{r} = kCdC \qquad \mathbf{5.}\ \frac{dr}{r} = k$$

2. How does coronary risk then vary as a function of cholesterol levels (bearing in mind that cholesterol levels cannot be lower than an s threshold)?

Application (2):

1. It is assumed that the equation describing the variation in blood concentration $C(t)$ of a drug, as a function of time t is:

$$C'(t) + C(t) = 3e^{-2t}.$$

2. The time origin is taken at the time of injection and the drug does not pre-exist in the blood ($C(0) = 0$).
3. Determine $C(t)$.

Application (3):

1. In the study of the preventive treatment of osteoporosis by transdermal administration of an estrogenic hormone, it is accepted that the variation in the blood concentration y of this hormone verifies:

$$\frac{dy}{dt} + ay = be^{-t},$$

where a and b are constants, and t is time.
2. Express blood concentration y as a function of time.

Application (4):

1. The chemical kinetics of the reaction $A + B \longrightarrow C$ is of the form:

$$x'(t) = k(a - x(t))(b - x(t)),$$

where a denotes the initial concentration of product A, b that of B, $x(t)$ the concentration at time t of product C and k a constant.
2. Integrate this differential equation and give the solution verifying $x(0) = 0$.
3. What happens after a very long time?

3.6. second-order linear DEs with constant coefficients

They are of the form:

$$ax''(t) + bx'(t) + cx(t) = f(t), \tag{3.44}$$

where a, b and c are real data constants.

3.6.1. Cauchy-Lipschitz theorem:

If initial conditions of the form $x(t_0) = x_0$ and $x'(t_0) = x_1$ are imposed, then the differential equation (3.44) has one and only one solution.

3.6.2. Theorems due to linearity:

- Any solution of (3.44) is of the form $x_P(t) + x_S(t)$ where $x_P(t)$ is a particular solution of (3.44) and $x_S(t)$ the general solution of the associated homogeneous equation (without second member):

$$ax''(t) + bx'(t) + cx(t) = 0, \tag{3.45}$$

- If x_1 (resp. x_2) is a particular solution of

$$ax''(t) + bx'(t) + cx(t) = f_1(t) \text{ resp. } f_2(t), \tag{3.46}$$

then $x_1 + x_2$ is a particular solution of

$$ax''(t) + bx'(t) + cx(t) = f_1(t) + f_2(t). \tag{3.47}$$

3.6.3. Solution of the homogeneous equation:

The function $t \mapsto e^{rt}$ is a solution of (3.45) if, and only if, r satisfies the characteristic equation:

$$ar^2 + br + c = 0. \tag{3.48}$$

Let $\Delta = b^2 - 4ac$. If $\Delta \neq 0$, (3.48) has two distinct (complex) roots r_1 and r_2, and:

$$x_S(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t}, \tag{3.49}$$

where K_1 and K_2 are any constants. If $\Delta < 0$, (3.48) has two conjugate complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, and:

$$x_S(t) = e^{\alpha t}(K_1 \cos(\beta t) + K_2 \sin(\beta t)), \tag{3.50}$$

where K_1 and K_2 are any constants. This is a special case of the previous resolution. Finally, if $\Delta = 0$, (3.48) has a double root r_0, and:

$$x_S(t) = (K_1 t + K_2) e^{r_0 t}, \tag{3.51}$$

where K_1 and K_2 are any constants.

3.6.4. Searching for a particular solution in certain cases:

If $f(t)$ is a polynomial $P(t)$ of degree n, there exists a particular solution of (3.44) in the form of a polynomial of degree

- n if $c \neq 0$,
- $n + 1$ if $c = 0$ and $b \neq 0$,
- $n + 2$ if $b = c = 0$ and $a \neq 0$.

The solution is determined by identification. If $f(t) = e^{kt}P(t)$ with P a polynomial and k a constant, through the change of variable

$$x(t) = e^{kt} z(t), \tag{3.52}$$

where z is a new unknown function, the equation is transformed into another of the previous type. If $f(t) = e^{\alpha t}\cos(\beta t)P(t)$ or $f(t) = e^{\alpha t}\sin(\beta t)P(t)$ with α and β real and P a polynomial with constant coefficients, the particular solution is the real part, or the imaginary part, of the particular solution obtained for the second member equation $e^{(\alpha+i\beta)t}$.

3.7. Examples of modeling

Example (1):
A drug is administered in the form of a homogeneous spherical implant. It is assumed that the implant remains spherical and that the drug is released at a rate proportional to the surface area of the implant. Express the quantity $q(t)$ of drug delivered as a function of time. Reminder: the surface area of a sphere of radius R is $S = 4\pi R^2$ and its volume $\frac{4}{3}\pi R^3$.

Example (2):

On a clear, calm night in an elevated desert location, the rate at which the temperature falls is proportional to the fourth power of the temperature (Stefan's law). The following measurements were taken:

- at $t_0 = 22$ h: $T_0 = 290$°K, or 17°C,
- at $t_1 = 24$ h: $T_1 = 280$°K, or 7°C.

What will the temperature be at 5 h in the morning?