

## QUANTUM FIELD THEORY, SOLUTIONS TO PROBLEM SHEET 13

## Problem 1: Schwinger-Dyson equations and Ward-Takahashi identities

1. *As warm-up in finitely many dimensions, show that*

$$\int d^d x e^{-s(x)-j \cdot x} (\nabla s(x) + j) = 0$$

where  $j \in \mathbb{R}^d$  is a constant vector and  $s(x)$  is any real function (assumed to tend to infinity at  $|x| \rightarrow \infty$  sufficiently quickly for the integral to converge).

A constant shift of the integration variable  $x \rightarrow x' = x + z$  gives a trivial Jacobian, so the integration measure is invariant:

$$\begin{aligned} \int d^d x e^{-s(x)-j \cdot x} &= \int d^d x' e^{-s(x'-z)-(x'-z) \cdot j} \\ &= \int d^d x' e^{-s(x')-x' \cdot j + \nabla s(x') \cdot z + j \cdot z} + \mathcal{O}(|z|^2) \\ &= \int d^d x' e^{-s(x')-j \cdot x'} (1 + (\nabla s(x') + j) \cdot z + \mathcal{O}(|z|^2)) \end{aligned}$$

Subtracting  $\int d^d x e^{-s(x)-j \cdot x}$  on both sides, one obtains the desired relation.

2. *Derive the corresponding formal functional identity by replacing  $\phi \rightarrow \phi'$  in the generating functional  $Z[J]$ :*

$$\int \mathcal{D}\phi \int d^4 y \left( \frac{\delta S}{\delta \phi(y)} + J(y) \right) \zeta(y) e^{i(S[\phi] + \int J\phi)} = 0.$$

We have

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi' e^{i(S[\phi'] + \int J\phi')} = \int \mathcal{D}\phi e^{i(S[\phi] + \int J\phi + \alpha \int (\frac{\delta S}{\delta \phi} + J)\zeta) + \mathcal{O}(|\alpha|^2)} \\ &= \int \mathcal{D}\phi e^{i(S[\phi] + \int J\phi)} \left( 1 + i\alpha \int d^4 y \left( \frac{\delta S}{\delta \phi(y)} + J(y) \right) \zeta(y) + \mathcal{O}(|\alpha|^2) \right) \end{aligned}$$

and subtracting  $Z[J]$  on both sides gives the result.

3. *By setting  $\zeta(y) = \delta^{(4)}(y - x)$  for some fixed  $x$ , conclude that*

$$\begin{aligned} \square_x \langle 0 | \mathbb{T} \phi(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle + \langle 0 | \mathbb{T} \mathcal{V}'(\phi(x)) \phi(x_1) \dots \phi(x_n) | 0 \rangle \\ = -i \sum_{k=1}^n \langle 0 | \mathbb{T} \phi(x_1) \dots \phi(x_{k-1}) \delta^{(4)}(x - x_k) \phi(x_{k+1}) \dots \phi(x_n) | 0 \rangle. \end{aligned}$$

Starting from

$$\int \mathcal{D}\phi \left( \frac{\delta S}{\delta \phi(x)} + J(x) \right) e^{i(S[\phi] + \int J\phi)} = 0$$

and taking  $n$  functional derivatives  $\frac{1}{i} \frac{\delta}{\delta J(x_k)}$  ( $k = 1 \dots n$ ), the derivatives can either all act on the exponential (which brings down  $n$  factors of  $\phi$ ) or one of them can act on the  $J(x)$  in front of the exponential (which gives  $\frac{1}{i} \delta^{(4)}(x - x_k)$ , and only  $n - 1$  factors of  $\phi$  from the others acting on the exponential). Setting  $J = 0$  after differentiation, we get

$$\begin{aligned}
0 &= \int \mathcal{D}\phi \frac{\delta S}{\delta \phi(x)} \left( \prod_{j=1}^n \phi(x_j) \right) e^{iS[\phi]} + \frac{1}{i} \sum_{k=1}^n \int \mathcal{D}\phi \delta^{(4)}(x - x_k) \left( \prod_{j \neq k} \phi(x_j) \right) e^{iS[\phi]} \\
&= \int \mathcal{D}\phi \left( -\square \phi(x) - \mathcal{V}'(\phi(x)) \right) \left( \prod_{j=1}^n \phi(x_j) \right) e^{iS[\phi]} \\
&\quad - i \sum_{k=1}^n \int \mathcal{D}\phi \delta^{(4)}(x - x_k) \left( \prod_{j \neq k} \phi(x_j) \right) e^{iS[\phi]} \\
&= -\square_x \int \mathcal{D}\phi \phi(x) \left( \prod_{j=1}^n \phi(x_j) \right) e^{iS[\phi]} - \int \mathcal{D}\phi \mathcal{V}'(\phi(x)) \left( \prod_{j=1}^n \phi(x_j) \right) e^{iS[\phi]} \\
&\quad - i \sum_{k=1}^n \int \mathcal{D}\phi \delta^{(4)}(x - x_k) \left( \prod_{j \neq k} \phi(x_j) \right) e^{iS[\phi]}
\end{aligned}$$

Dividing by  $Z[0]$  gives the desired result.

4. *Write down the Schwinger-Dyson equation explicitly for the case of  $n = 1$  and a free field. Does this look familiar?*

It should: Using  $\frac{\delta S}{\delta \phi} = -(\square + m^2)\phi$ , one recovers the familiar statement that the Feynman propagator is a Green function for the Klein-Gordon operator,  $(\square_x + m^2)D_F(x - x_1) = -i\delta^{(4)}(x - x_1)$ .

5. *Using the LSZ reduction formula, convince yourself that contact terms can never contribute to the invariant matrix elements  $\mathcal{M}_{\text{fi}}$ .*

According to LSZ, the connected transition amplitudes  $\langle f|i \rangle$  are given by the residues at the poles of the on-shell correlation functions in momentum space, i.e. for an  $n$ -point function

$$\begin{aligned}
\langle f|i \rangle &= (-i)^n (p_1^2 - m^2) \dots (p_n^2 - m^2) G(p_1, \dots, p_n) \quad \text{at } p_i^2 = m^2 \\
G(p_1, \dots, p_n) &= \int d^4x_1 \dots d^4x_n e^{\pm ip_1 x_1 \dots \pm ip_n x_n} \langle 0|T \phi(x_1) \dots \phi(x_n)|0 \rangle.
\end{aligned}$$

Any term in  $G$  which is missing a pole at  $p_k^2 = m^2$  for some  $k$  will not contribute to  $\langle f|i \rangle$ , since it comes with a zero prefactor  $p_k^2 - m^2$  without a corresponding  $p_k^2 - m^2$  in the denominator. For a contact term,

$$\begin{aligned}
&\int d^4x_1 \dots d^4x_n e^{\pm ip_1 x_1 \dots \pm ip_n x_n} \langle 0|T \phi(x_1) \dots \phi(x_{k-1}) \phi(x_{k+1}) \dots \phi(x_n)|0 \rangle \delta^{(4)}(x_k - x) \\
&= e^{ip_k x} \times (\text{some function independent of } p_k).
\end{aligned}$$

This obviously doesn't have the right pole structure to contribute.

6. *By setting  $\zeta(y) = \delta\phi(x) \delta^{(4)}(x - y)$  in the result of 2., show that*

$$\int \mathcal{D}\phi \left( \partial_\mu j^\mu(x) - J(x) \delta\phi(x) \right) e^{i(S[\phi] + \int J\phi)} = 0$$

and deduce that

$$\begin{aligned} & \frac{\partial}{\partial x^\mu} \langle 0 | T j^\mu(x) \phi(x_1) \dots \phi(x_n) | 0 \rangle \\ &= i \sum_{k=1}^n \langle 0 | T \phi(x_1) \dots \phi(x_{k-1}) \delta\phi(x_k) \delta^{(4)}(x - x_k) \phi(x_{k+1}) \dots \phi(x_n) | 0 \rangle. \end{aligned}$$

This is easily shown by following the same steps as in 3.

7. Use the Schwinger-Dyson equation for  $A_\mu(x)$  and the result of part 5. to show that

$$\langle f | i \rangle = -i\varepsilon^\mu(k) \int d^4x e^{-ikx} \dots \langle 0 | T j_\mu(x) \dots | 0 \rangle.$$

The equation of motion for  $A_\mu$  in Lorenz gauge is

$$\frac{\delta S}{\delta A^\mu} = \square A_\mu + j_\mu$$

hence the Schwinger-Dyson equation reads

$$\begin{aligned} & i\varepsilon^\mu(k) \int d^4x e^{-ikx} \square_x \dots \langle 0 | T A_\mu(x) \dots | 0 \rangle \\ &= -i\varepsilon^\mu(k) \int d^4x e^{-ikx} \dots \langle 0 | T j_\mu(x) \dots | 0 \rangle + \text{contact terms.} \end{aligned}$$

8. Use the Ward-Takahashi identities to prove the Ward identity of QED: If  $\mathcal{M}_{\text{fi}}$  is the corresponding invariant matrix element and  $\mathcal{M}_\mu(k)$  is defined by  $\mathcal{M}_{\text{fi}} = \varepsilon^\mu(k) \mathcal{M}_\mu(k)$ , then

$$k^\mu \mathcal{M}_\mu(k) = 0.$$

We have

$$\begin{aligned} & k^\mu \int d^4x e^{-ikx} \square_x \dots \langle 0 | T A_\mu(x) \dots | 0 \rangle \\ &= \int d^4x e^{-ikx} \dots k^\mu (\langle 0 | T j_\mu(x) \dots | 0 \rangle + \text{contact terms}) \\ &= -i \int d^4x e^{-ikx} \dots (\partial^\mu \langle 0 | T j_\mu(x) \dots | 0 \rangle + \text{contact terms}) \\ &= -i \int d^4x e^{-ikx} \dots (\text{contact terms only}) \\ &= 0. \end{aligned}$$