QFT, solutions to problem sheet 13

Problem 1: Schwinger-Dyson equations and Ward-Takahashi identities

1. *As warm-up in finitely many dimensions, show that*

$$\int \mathrm{d}^d x \, e^{-s(x)-j\cdot x} \left(\nabla s(x) + j\right) = 0$$

where $j \in \mathbb{R}^d$ is a constant vector and $s(x)$ is any real function (assumed to tend to infinity at $|x| \to \infty$ sufficiently quickly for the integral to converge).

A constant shift of the integration variable $x \to x' = x + z$ gives a trivial Jacobian, so the integration measure is invariant:

$$\begin{aligned}\int \mathrm{d}^d x \, e^{-s(x)-j\cdot x} &= \int \mathrm{d}^d x' \, e^{-s(x'-z)-(x'-z)\cdot j} \\ &= \int \mathrm{d}^d x' \, e^{-s(x')-x'\cdot j+\nabla s(x')\cdot z + j\cdot z} + \mathcal{O}(|z|^2) \\ &= \int \mathrm{d}^d x' \, e^{-s(x')-j\cdot x'}(1 + (\nabla s(x') + j)\cdot z + \mathcal{O}(|z|^2))\end{aligned}$$

Subtracting $\int \mathrm{d}^d x \, e^{-s(x)-j\cdot x}$ on both sides, one obtains the desired relation.

2. *Derive the corresponding formal functional identity by replacing $\phi \to \phi'$ in the generating functional $Z[J]$:*

$$\int \mathcal{D}\phi \int \mathrm{d}^4 y \left(\frac{\delta S}{\delta \phi(y)} + J(y)\right) \zeta(y) \, e^{i(S[\phi]+\int J\phi)} = 0 \, .$$

We have

$$\begin{aligned}Z[J] &= \int \mathcal{D}\phi' \, e^{i(S[\phi']+\int J\phi')} = \int \mathcal{D}\phi \, e^{i(S[\phi]+\int J\phi + \alpha\int(\frac{\delta S}{\delta\phi}+J)\zeta)+\mathcal{O}(|\alpha|^2)} \\ &= \int \mathcal{D}\phi \, e^{i(S[\phi]+\int J\phi)} \left(1 + i\alpha \int \mathrm{d}^4 y \left(\frac{\delta S}{\delta\phi(y)} + J(y)\right)\zeta(y) + \mathcal{O}(|\alpha|^2)\right)\end{aligned}$$

and subtracting $Z[J]$ on both sides gives the result.

3. *By setting $\zeta(y) = \delta^{(4)}(y-x)$ for some fixed x, conclude that*

$$\begin{aligned}&\Box_x \langle 0|\, \mathrm{T}\, \phi(x)\phi(x_1)\ldots\phi(x_n)|0\rangle + \langle 0|\, \mathrm{T}\, \mathcal{V}'(\phi(x))\, \phi(x_1)\ldots\phi(x_n)|0\rangle \\ &\quad = -i\sum_{k=1}^{n} \langle 0|\, \mathrm{T}\, \phi(x_1)\ldots\phi(x_{k-1})\delta^{(4)}(x-x_k)\phi(x_{k+1})\ldots\phi(x_n)|0\rangle \, .\end{aligned}$$

Starting from

$$\int \mathcal{D}\phi \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) e^{i(S[\phi]+\int J\phi)} = 0$$

and taking n functional derivatives $\frac{1}{i}\frac{\delta}{\delta J(x_k)}$ ($k = 1 \ldots n$), the derivatives can either all act on the exponential (which brings down n factors of ϕ) or one of them can act on the $J(x)$ in front of the exponential (which gives $\frac{1}{i}\delta^{(4)}(x-x_k)$, and only $n-1$ factors of ϕ from the others acting on the exponential). Setting $J = 0$ after differentiation, we get

$$\begin{aligned}
0 &= \int \mathcal{D}\phi \, \frac{\delta S}{\delta\phi(x)} \left(\prod_{j=1}^{n} \phi(x_j)\right) e^{iS[\phi]} + \frac{1}{i}\sum_{k=1}^{n} \int \mathcal{D}\phi \, \delta^{(4)}(x-x_k) \left(\prod_{j\neq k} \phi(x_j)\right) e^{iS[\phi]} \\
&= \int \mathcal{D}\phi \, \left(-\Box\phi(x) - \mathcal{V}'(\phi(x))\right) \left(\prod_{j=1}^{n} \phi(x_j)\right) e^{iS[\phi]} \\
&\qquad - i\sum_{k=1}^{n} \int \mathcal{D}\phi \, \delta^{(4)}(x-x_k) \left(\prod_{j\neq k} \phi(x_j)\right) e^{iS[\phi]} \\
&= -\Box_x \int \mathcal{D}\phi \, \phi(x) \left(\prod_{j=1}^{n} \phi(x_j)\right) e^{iS[\phi]} - \int \mathcal{D}\phi \, \mathcal{V}'(\phi(x)) \left(\prod_{j=1}^{n} \phi(x_j)\right) e^{iS[\phi]} \\
&\qquad - i\sum_{k=1}^{n} \int \mathcal{D}\phi \, \delta^{(4)}(x-x_k) \left(\prod_{j\neq k} \phi(x_j)\right) e^{iS[\phi]}
\end{aligned}$$

Dividing by $Z[0]$ gives the desired result.

4. *Write down the Schwinger-Dyson equation explicitly for the case of $n = 1$ and a free field. Does this look familiar?*

   It should: Using $\frac{\delta S}{\delta\phi} = -(\Box + m^2)\phi$, one recovers the familiar statement that the Feynman propagator is a Green function for the Klein-Gordon operator, $(\Box_x + m^2)D_F(x - x_1) = -i\delta^{(4)}(x - x_1)$.

5. *Using the LSZ reduction formula, convince yourself that contact terms can never contribute to the invariant matrix elements $\mathcal{M}_{\text{fi}}$.*

   According to LSZ, the connected transition amplitudes $\langle f|i\rangle$ are given by the residues at the poles of the on-shell correlation functions in momentum space, i.e. for an n-point function

$$\begin{aligned}
\langle f|i\rangle &= (-i)^n (p_1^2 - m^2)\ldots(p_n^2 - m^2) G(p_1, \ldots, p_n) \qquad \text{at } p_i^2 = m^2 \\
G(p_1, \ldots, p_n) &= \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n e^{\pm ip_1x_1 \ldots \pm ip_nx_n} \langle 0|\mathrm{T}\,\phi(x_1)\ldots\phi(x_n)|0\rangle \,.
\end{aligned}$$

   Any term in G which is missing a pole at $p_k^2 = m^2$ for some k will not contribute to $\langle f|i\rangle$, since it comes with a zero prefactor $p_k^2 - m^2$ without a corresponding $p_k^2 - m^2$ in the denominator. For a contact term,

$$\begin{aligned}
&\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n e^{\pm ip_1x_1 \ldots \pm ip_nx_n} \langle 0|\mathrm{T}\,\phi(x_1)\ldots\phi(x_{k-1})\phi(x_{k+1})\ldots\phi(x_n)|0\rangle \delta^{(4)}(x_k - x) \\
&\quad = e^{ip_kx} \times (\text{some function independent of } p_k)\,.
\end{aligned}$$

   This obviously doesn't have the right pole structure to contribute.

6. By setting $\zeta(y) = \delta\phi(x)\,\delta^{(4)}(x - y)$ in the result of 2., show that

$$\int \mathcal{D}\phi \, \left(\partial_\mu j^\mu(x) - J(x)\,\delta\phi(x)\right) e^{i(S[\phi]+\int J\phi)} = 0$$

and deduce that

$$\frac{\partial}{\partial x^\mu}\langle 0|\,\mathrm{T}\; j^\mu(x)\phi(x_1)\dots\phi(x_n)|0\rangle$$
$$= i\sum_{k=1}^{n}\langle 0|\,\mathrm{T}\;\phi(x_1)\dots\phi(x_{k-1})\;\delta\phi(x_k)\;\delta^{(4)}(x-x_k)\,\phi(x_{k+1})\dots\phi(x_n)|0\rangle\,.$$

This is easily shown by following the same steps as in 3.

7. *Use the Schwinger-Dyson equation for $A_\mu(x)$ and the result of part 5. to show that*

$$\langle f|i\rangle = -i\varepsilon^\mu(k)\int \mathrm{d}^4x\; e^{-ikx}\dots\langle 0|\mathrm{T}\; j_\mu(x)\dots|0\rangle\,.$$

The equation of motion for A_μ in Lorenz gauge is

$$\frac{\delta S}{\delta A^\mu} = \Box A_\mu + j_\mu$$

hence the Schwinger-Dyson equation reads

$$i\varepsilon^\mu(k)\int \mathrm{d}^4x\; e^{-ikx}\Box_x\dots\langle 0|\mathrm{T}\; A_\mu(x)\dots|0\rangle$$
$$= -i\varepsilon^\mu(k)\int \mathrm{d}^4x\; e^{-ikx}\dots\langle 0|\mathrm{T}\; j_\mu(x)\dots|0\rangle + \text{ contact terms.}$$

8. *Use the Ward-Takahashi identities to prove the Ward identity of QED: If $\mathcal{M}_{\mathrm{fi}}$ is the corresponding invariant matrix element and $\mathcal{M}_\mu(k)$ is defined by $\mathcal{M}_{\mathrm{fi}} = \varepsilon^\mu(k)\mathcal{M}_\mu(k)$, then*

$$k^\mu\mathcal{M}_\mu(k) = 0\,.$$

We have

$$k^\mu\int \mathrm{d}^4x\; e^{-ikx}\Box_x\dots\langle 0|\mathrm{T}\; A_\mu(x)\dots|0\rangle$$
$$= \int \mathrm{d}^4x\; e^{-ikx}\dots k^\mu\left(\langle 0|\mathrm{T}\; j_\mu(x)\dots|0\rangle + \text{ contact terms}\right)$$
$$= -i\int \mathrm{d}^4x\; e^{-ikx}\dots\left(\partial^\mu\langle 0|\mathrm{T}\; j_\mu(x)\dots|0\rangle + \text{ contact terms}\right)$$
$$= -i\int \mathrm{d}^4x\; e^{-ikx}\dots(\text{ contact terms only})$$
$$= 0\,.$$