

QUANTUM FIELD THEORY, PROBLEM SHEET 12

Problem 1: The Grassmann algebra

1. Find the vector space dimension of the Grassmann algebra generated by $\mathbb{1}$ and $\{\theta_i\}_{i=1\dots n}$.
2. Let $U = (U_{ij})$ be a unitary matrix of commuting numbers ($i, j = 1 \dots n$). Show that the Grassmann integral

$$\int d^n \theta^* d^n \theta f(\theta_i, \theta_i^*)$$

is invariant under the change of variables $\theta_i \rightarrow U_{ij} \theta_j$, $\theta_i^* \rightarrow U_{ij}^* \theta_j^*$.

3. Let A be a hermitian $n \times n$ matrix of commuting numbers. Show that

$$\int d^n \theta^* d^n \theta e^{-\theta^\dagger A \theta} = \det A.$$

Problem 2: Yukawa theory

Consider the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - M)\psi + \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2 \phi^2 - y \phi \bar{\psi} \psi,$$

where ψ is a Dirac field, ϕ is a real scalar field, and y is a coupling.

1. Compute the matrix element \mathcal{M}_{fi} for $\bar{\psi}\psi \rightarrow \phi\phi$ scattering to lowest order in perturbation theory.
(If you are motivated: Use this result to further calculate the total cross section as a function of the Mandelstam variables s , t , and u , assuming that the incoming particles are unpolarized — but be warned, this calculation is long and tedious, and best done with the help of a computer algebra system.)
2. Show that the 1-point function $\langle 0|\phi(x)|0\rangle$ is divergent, and that the divergence can be subtracted by introducing a counterterm $\delta_t \phi$ in \mathcal{L} such that $\langle 0|\phi(x)|0\rangle = 0$. Compute δ_t at the one-loop level in dimensional regularization.
3. Draw the Feynman diagrams contributing to all the 2-point and 3-point correlation functions at the one-loop level, including the above counterterm with a suitable Feynman rule. Which of them do you expect to diverge? (There is no need to evaluate them in detail.)
4. Draw the *amputated* Feynman diagrams contributing to the $\phi\phi \rightarrow \phi\phi$ and $\psi\psi \rightarrow \psi\psi$ matrix elements at the one-loop level. Which of them do you expect to diverge?
5. In view of the divergent diagrams, what other terms should we have included in the Lagrangian and why?