Quantum Field Theory, problem sheet 12

Solutions to be discussed on 15/01/2025.

Problem 1: The Grassmann algebra

1. Find the vector space dimension of the Grassmann algebra generated by $\mathbb{1}$ and $\{\theta_i\}_{i=1\ldots n}$.
2. Let $U = (U_{ij})$ be a unitary matrix of commuting numbers ($i, j = 1 \ldots n$). Show that the Grassmann integral
$$\int \mathrm{d}^n\theta^* \, \mathrm{d}^n\theta \, f(\theta_i, \theta_i^*)$$
is invariant under the change of variables $\theta_i \to U_{ij}\theta_j$, $\theta_i^* \to U_{ij}^*\theta_j^*$.
3. Let A be a hermitian $n \times n$ matrix of commuting numbers. Show that
$$\int \mathrm{d}^n\theta^* \, \mathrm{d}^n\theta \, e^{-\theta^\dagger A\, \theta} = \det A\,.$$

Problem 2: Yukawa theory

Consider the Lagrangian
$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - M)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - y\,\phi\overline{\psi}\psi,$$
where ψ is a Dirac field, ϕ is a real scalar field, and y is a coupling.

1. Compute the matrix element $\mathcal{M}_{\mathrm{fi}}$ for $\overline{\psi}\psi \to \phi\phi$ scattering to lowest order in perturbation theory.

   (If you are motivated: Use this result to further calculate the total cross section as a function of the Mandelstam variables s, t, and u, assuming that the incoming particles are unpolarized — but be warned, this calculation is long and tedious, and best done with the help of a computer algebra system.)
2. Show that the 1-point function $\langle 0|\phi(x)|0\rangle$ is divergent, and that the divergence can be subtracted by introducing a counterterm $\delta_t\phi$ in $\mathcal{L}$ such that $\langle 0|\phi(x)|0\rangle = 0$. Compute δ_t at the one-loop level in dimensional regularization.
3. Draw the Feynman diagrams contributing to all the 2-point and 3-point correlation functions at the one-loop level, including the above counterterm with a suitable Feynman rule. Which of them do you expect to diverge? (There is no need to evaluate them in detail.)
4. Draw the *amputated* Feynman diagrams contributing to the $\phi\phi \to \phi\phi$ and $\psi\psi \to \psi\psi$ matrix elements at the one-loop level. Which of them do you expect to diverge?
5. In view of the divergent diagrams, what other terms should we have included in the Lagrangian and why?