Quantum Field Theory, problem sheet 11

Solutions to be discussed on 17/12/2024.

Problem 1: Hamiltonian of the Dirac field

1. Let $p = (p^0, \vec{p})$ be a momentum 4-vector, $p^2 = m^2$, and let $\bar{p} = (p^0, -\vec{p})$ be the same 4-vector with the sign of the spatial components reversed. Prove that

$$\gamma^0 \not{p} + \not{p}\gamma^0 = 2\, p^0 \,, \qquad \gamma^0 \not{p} - \not{\bar{p}}\gamma^0 = 0 \,.$$

2. Prove that

$$\begin{aligned} \bar{u}_s(\vec{p})\gamma^0 u_r(\vec{p}) = 2\, p^0\, \delta_{sr} \,, \quad \bar{v}_s(\vec{p})\gamma^0 v_r(\vec{p}) = 2\, p^0\, \delta_{sr} \,, \\ \bar{u}_s(-\vec{p})\gamma^0 v_r(\vec{p}) = 0 \,, \quad \bar{v}_s(-\vec{p})\gamma^0 u_r(\vec{p}) = 0 \,. \end{aligned}$$

   Hint: Use the result of 1., remembering that $u_s(\vec{p})$ and $v_s(\vec{p})$ satisfy

$$\begin{aligned} (\not{p} - m) u_s(\vec{p}) = 0 \,, \qquad & (\not{p} + m) v_s(\vec{p}) = 0 \,, \\ \bar{u}_s(\vec{p})(\not{p} - m) = 0 \,, \qquad & \bar{v}_s(\vec{p})(\not{p} + m) = 0 \,, \\ \bar{u}_s(\vec{p}) u_r(\vec{p}) = 2m\, \delta_{rs} \,, \qquad & \bar{v}_s(\vec{p}) v_r(\vec{p}) = -2m\, \delta_{rs} \,. \end{aligned}$$

3. Starting from the Fourier mode expansion of a free Dirac field

$$\begin{aligned} \psi(x) &= \sum_{s=+,-} \int \widetilde{\mathrm{d}p} \left(a_s(\vec{p}) u_s(\vec{p}) e^{-ipx} + b_s^\dagger(\vec{p}) v_s(\vec{p}) e^{ipx} \right) \,, \\ \overline{\psi}(x) &= \sum_{s=+,-} \int \widetilde{\mathrm{d}p} \left(b_s(\vec{p}) \bar{v}_s(\vec{p}) e^{-ipx} + a_s^\dagger(\vec{p}) \bar{u}_s(\vec{p}) e^{ipx} \right) \,, \end{aligned}$$

   and using the relations derived in 2., show that the Dirac Hamiltonian $H = \int \mathrm{d}^3x \left((\partial\mathcal{L}/\partial\dot{\psi})\dot{\psi} - \mathcal{L} \right)$ can be written as

$$H = \sum_s \int \widetilde{\mathrm{d}p}\; \omega_{\vec{p}} \left(a_s^\dagger(\vec{p}) a_s(\vec{p}) - b_s(\vec{p}) b_s^\dagger(\vec{p}) \right) \,.$$

It follows that the Hamiltonian can be made positive definite by imposing canonical anticommutation relations (rather than commutation relations).