M2 Cosmos, Champs et Particules — Faculté des sciences de Montpellier

Quantum Field Theory, problem sheet 11

Problem 1: Hamiltonian of the Dirac field

1. Let $p = (p^0, \vec{p})$ be a momentum 4-vector, $p^2 = m^2$, and let $\bar{p} = (p^0, -\vec{p})$ be the same 4-vector with the sign of the spatial components reversed. Prove that

$$\gamma^0 \not{p} + \not{p}\gamma^0 = 2\,p^0\,, \qquad \gamma^0 \not{p} - \not{\bar{p}}\gamma^0 = 0\,.$$

2. Prove that

$$\begin{aligned} \bar{u}_s(\vec{p})\gamma^0 u_r(\vec{p}) &= 2\,p^0\,\delta_{sr}\,, & \bar{v}_s(\vec{p})\gamma^0 v_r(\vec{p}) &= 2\,p^0\,\delta_{sr}\,, \\ \bar{u}_s(-\vec{p})\gamma^0 v_r(\vec{p}) &= 0\,, & \bar{v}_s(-\vec{p})\gamma^0 u_r(\vec{p}) &= 0\,. \end{aligned}$$

   Hint: Use the result of 1., remembering that $u_s(\vec{p})$ and $v_s(\vec{p})$ satisfy

$$\begin{aligned} (\not{p} - m)u_s(\vec{p}) &= 0\,, & (\not{p} + m)v_s(\vec{p}) &= 0\,, \\ \bar{u}_s(\vec{p})(\not{p} - m) &= 0\,, & \bar{v}_s(\vec{p})(\not{p} + m) &= 0\,, \\ \bar{u}_s(\vec{p})u_r(\vec{p}) &= 2m\,\delta_{rs}\,, & \bar{v}_s(\vec{p})v_r(\vec{p}) &= -2m\,\delta_{rs}\,. \end{aligned}$$

3. Starting from the Fourier mode expansion of a free Dirac field

$$\psi(x) = \sum_{s=+,-} \int \widetilde{\mathrm{d}p}\,\left(a_s(\vec{p})u_s(\vec{p})e^{-ipx} + b_s^\dagger(\vec{p})v_s(\vec{p})e^{ipx}\right)\,,$$

$$\overline{\psi}(x) = \sum_{s=+,-} \int \widetilde{\mathrm{d}p}\,\left(b_s(\vec{p})\bar{v}_s(\vec{p})e^{-ipx} + a_s^\dagger(\vec{p})\bar{u}_s(\vec{p})e^{ipx}\right)\,,$$

   and using the relations derived in 2., show that the Dirac Hamiltonian $H = \int \mathrm{d}^3x\,((\partial\mathcal{L}/\partial\dot{\psi})\dot{\psi} - \mathcal{L})$ can be written as

$$H = \sum_s \int \widetilde{\mathrm{d}p}\;\omega_{\vec{p}}\left(a_s^\dagger(\vec{p})a_s(\vec{p}) - b_s(\vec{p})b_s^\dagger(\vec{p})\right)\,.$$

It follows that the Hamiltonian can be made positive definite by imposing canonical anticommutation relations (rather than commutation relations).