9. À l'aide des formules de dérivées usuelles, déterminer les fonctions dérivées des fonctions suivantes :

1) $f(x) = \dfrac{\sin x}{1-\cos x}$ **2)** $f(x) = \dfrac{3x^2-4x+1}{x^2+1}$ **3)** $f(x) = \exp\left[\tan\left(x^2+1\right)\right]$

4) $f(x) = \sin(x)\cdot\cos(x)$ **5)** $f(x) = \sin^2(x)\cdot\cos^3(x)$ **6)** $f(x) = \sqrt{2x^2-3x+1}$

7) $f(x) = \arctan(3x)$ **8)** $f(x) = \arctan(3x)$ **9)** $f(x) = \arccos(2x+1)$

10) $f(x) = \arcsin\left(x^2\right)$ **11)** $f(x) = \arctan\left(\sqrt{\dfrac{1+x}{1-x}}\right)$

$\rightarrow \dfrac{u}{v} = \dfrac{u'v - uv'}{v^2}$

$\rightarrow uv = u'v + uv'$

$\rightarrow u\circ v\circ w = w' \times v'\circ w \times u'\circ v\circ w$

$u(v(w(x))) = w'(x) \times v'(w(x)) \times u'(v(w(x)))$

$\arctan' x = \dfrac{1}{1+x^2}$

1) $f(x) = \dfrac{\sin x}{1-\cos x} \Rightarrow f'(x) = \dfrac{\cos x(1-\cos x) - \sin x \sin x}{(1-\cos x)^2} = \dfrac{\cos x - \overbrace{(\cos^2 x + \sin^2 x)}^{-1}}{(1-\cos x)^2} = \dfrac{\cos x - 1}{(\cos x - 1)^2} = \dfrac{1}{\cos x - 1}$

2) $f(x) = \dfrac{3x^2-4x+1}{x^2+1} \Rightarrow f'(x) = \dfrac{(6x-4)(x^2+1) - (3x^2-4x+1)(2x)}{(x^2+1)^2} = \dfrac{\cancel{6x^3} + 6x - 4x^2 - 4 - \cancel{6x^3} + 8x^2 - 2x}{(x^2+1)^2} = \dfrac{4x^2+4x-4}{(x^2+1)^2} = 4\,\dfrac{x^2+x-1}{(x^2+1)^2}$

3) $f(x) = \exp\left[\tan(x^2+1)\right] = u\circ v\circ w(x)$ avec $u(x) = e^x \Rightarrow u'(x) = e^x$

$v(x) = \tan(x) \Rightarrow v'(x) = 1+\tan^2(x) = \dfrac{1}{\cos^2(x)}$

$w(x) = x^2+1 \Rightarrow w'(x) = 2x$

$f'(x) = 2x \times \left(1+\tan^2(x^2+1)\right) \times e^{\tan(x^2+1)} = \dfrac{2x}{\cos^2(x^2+1)}\, e^{\tan(x^2+1)}.$

$f(x) = e^u \quad f'(x) = u'e^u$

$u(x) = \tan(x^2+1)$

$\Rightarrow u'(x) = \dfrac{2x}{\cos^2(x^2+1)}$

$\Rightarrow f'(x) = \dfrac{2x}{\cos^2(x^2+1)}\, e^{\tan(x^2+1)}$

$(u^n)' = n u^{n-1} \times u'$

4) $f(x) = \sin(x) \cdot \cos(x)$ **5)** $f(x) = \sin^2(x) \cdot \cos^3(x)$ **6)** $f(x) = \sqrt{2x^2 - 3x + 1}$

7) $f(x) = \arctan(3x)$ ~~**8)** $f(x) = \arctan(3x)$~~ **9)** $f(x) = \arccos(2x+1)$

10) $f(x) = \arcsin\left(x^2\right)$ **11)** $f(x) = \arctan\left(\sqrt{\frac{1+x}{1-x}}\right)$

4) $f(x) = \sin(x)\cos(x) \Rightarrow f'(x) = \cos x \cdot \cos x + \sin(x) \times (-\sin x) = \cos^2 x - \sin^2 x = \cos(2x).$

5) $f(x) = \sin^2 x \cos^3 x \Rightarrow f'(x) = (\sin^2 x)' \underbrace{\cos^3 x}_{(\cos x)^3} + \underbrace{\sin^2 x}_{(\sin x)^2} (\cos^3 x)' = (2 \sin x \times \cos x) \times \cos^3 x + \sin^2 x \left(3\cos^2 x \times (-\sin x)\right)$

$= 2 \sin x \cos^4 x - 3 \sin^3 x \cos^2 x = (\sin x \cos^2 x)\left[2\cos^2 x - 3 \sin^2 x\right]$

6) $f(x) = \sqrt{2x^2 - 3x + 1} = (2x^2 - 3x + 1)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(2x^2 - 3x + 1)^{-1/2} \times (4x - 3) = \dfrac{4x-3}{2\sqrt{2x^2 - 3x + 1}}$

7) $f(x) = \arctan(3x)$ $\left[\arctan(u(x))\right]' = u'(x) \times \arctan' \circ u(x) = \dfrac{u'}{1 + (u(x))^2}$

$\Rightarrow f'(x) = 3 \times \dfrac{1}{1 + (3x)^2} = \dfrac{3}{1 + 9x^2}$

9) $f(x) = \arccos(2x+1)$ $f'(x) = (u \circ v(x))'$ avec $u(x) = \arccos(x) \to u'(x) = \dfrac{-1}{\sqrt{1-x^2}}$

$v(x) = 2x + 1 \to v'(x) = 2$

$\Rightarrow f'(x) = \dfrac{-2}{\underbrace{\sqrt{1-(2x+1)^2}}_{\text{positif si défini}}}$

$D_f = \{x \,/\, -1 \leq 2x + 1 \leq 1\}$

$-2 \leq 2x \leq 0 \Rightarrow -1 \leq x \leq 0 \Rightarrow x \in [-1;0].$

$D_f = [0;1]$

10) $f(x) = \arcsin\left(x^2\right)$ **11)** $f(x) = \arctan\left(\sqrt{\frac{1+x}{1-x}}\right)$

10) $D_f = \{x / \underbrace{-1 \leq x^2} \leq 1\}$ $\underbrace{x^2 \leq 1}_{x^2-1<0} \iff -1 \leq x \leq 1$ $D_f = [-1;1]$

$$f'(x) = 2x \frac{1}{\sqrt{1-(x^2)^2}} = \underbrace{\frac{2x}{\sqrt{1-x^4}}}$$

positif si défini c'est-à-dire $x \in]-1;1[$

11) $f(x) = \arctan\left(\sqrt{\frac{1+x}{1-x}}\right)$ $D_f = \left(\mathbb{R} \setminus \{1\}\right) \cap \left\{x / \frac{1+x}{1-x} \geq 0\right\}$

arctan x défini $\forall x \in \mathbb{R}$

Or $\sqrt{x}$ est définie $\forall x \geq 0$

$sg\left(\frac{1+x}{1-x}\right) = sg\left[\underbrace{(1+x)(1-x)}_{1-x^2}\right]$

WANTED: $x / 1-x^2 \geq 0 \iff x / \underbrace{-x^2}_{a<0}+1 \geq 0$

$\iff x \in [-1;1]$

Tableau de signe:

x	$-\infty$	-1		1	$+\infty$
$x+1$	$-$	0	$+$		$+$
$-x+1$	$+$		$+$	0	$-$
$\frac{x+1}{-x+1}$	$-$	0	$+$	$\Vert$	$-$

$D_f = [-1;1[$

$f(x) = u \circ v \circ w(x)$

$$\begin{cases} u(x) = \arctan(x) & \Rightarrow u'(x) = \frac{1}{1+x^2} \\ v(x) = \sqrt{x} & \Rightarrow v'(x) = \frac{1}{2\sqrt{x}} \\ w(x) = \frac{1+x}{1-x} & \Rightarrow w'(x) = \frac{1(1-x) + (1+x)\cancel{(-1)}}{(1-x)^2} = \frac{1-x+1+x}{(1-x)^2} = \frac{2}{(1-x)^2} \end{cases}$$

$$f'(x) = \frac{2}{(1-x)^2} \times \frac{1}{2\sqrt{\frac{1+x}{1-x}}} \times \frac{1}{1+\left(\sqrt{\frac{1+x}{1-x}}\right)^2}$$ >0 si définie