QFT, SOLUTIONS TO PROBLEM SHEET 9

Problem 1: Decay of a scalar particle

Consider the following Lagrangian, involving two real scalar fields ϕ and χ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\chi^2 - \frac{1}{2}\mu\,\chi\phi^2.$$

Suppose that $M > 2m$, so that the decay $\chi \to \phi\phi$ is kinematically possible. Calculate the lifetime of χ to leading order in the coupling μ.

The only Feynman diagram contributing to the 3-point function $\langle 0|\mathrm{T}\chi(x_1)\phi(x_2)\phi(x_3)|0\rangle$ at leading order in μ is the one given in the exercise. According to the LSZ formalism, we should amputate the external propagators and Fourier transform. This gives the transition amplitude, for a χ particle of momentum p_1 to decay into two ϕ particles of momenta $p_{2,3}$:

$$\langle f|i\rangle = -i\mu\,(2\pi)^4\,\delta^{(4)}(p_1 - p_2 - p_3) \quad\Leftrightarrow\quad \mathcal{M}_{\mathrm{fi}} = -\mu\,.$$

In the decay width, a factor $\frac{1}{2}$ must be included to account for the two identical particles in the final state. Hence, in the χ rest frame (where $p_1 = (M, 0, 0, 0)$)

$$\mathrm{d}\Gamma = \frac{\mu^2}{4M}\widetilde{\mathrm{d}p_2}\widetilde{\mathrm{d}p_3}\,(2\pi)^4\;\delta^{(4)}(p_1 - p_2 - p_3)$$

The total decay width is

$$\begin{aligned}
\Gamma &= \frac{\mu^2}{4M}\int\frac{\mathrm{d}^3p_2}{(2\pi)^3\,2E_2}\frac{\mathrm{d}^3p_3}{(2\pi)^3\,2E_3}(2\pi)^4\;\delta(M - E_2 - E_3)\;\delta^{(3)}(\vec{p}_2 + \vec{p}_3)\\
&= \frac{\mu^2}{64\pi^2\,M}\int\mathrm{d}^3p_2\frac{1}{E_2^2}\delta(M - 2\,E_2)\\
&= \frac{\mu^2}{16\pi\,M}\int_0^\infty\mathrm{d}p\frac{p^2}{p^2 + m^2}\delta(M - 2\sqrt{p^2 + m^2})
\end{aligned}$$

Remembering the δ function transformation rule

$$\delta(f(x)) = \sum_{x_i\,:\,f(x_i)=0}\frac{1}{|f'(x_i)|}\delta(x - x_i)$$

here we have, for $f(p) = M - 2\sqrt{p^2 + m^2}$,

$$f'(p) = -\frac{2\,p}{\sqrt{p^2 + m^2}}\,, \qquad f(p) = 0 \Leftrightarrow p^2 + m^2 = \frac{M^2}{4}$$

and therefore

$$\Gamma = \frac{\mu^2}{16\pi\,M}\frac{\frac{M^2}{4} - m^2}{\frac{M^2}{4}}\frac{M}{2\sqrt{M^2 - 4\,m^2}} = \frac{\mu^2}{32\,\pi\,M}\sqrt{1 - 4\frac{m^2}{M^2}}$$

or

$$\tau = \frac{1}{\Gamma} = \frac{32\,\pi\,M}{\mu^2}\left(1 - 4\frac{m^2}{M^2}\right)^{-1/2}.$$

This expression satisfies a number of obvious consistency checks, e.g. it tends to infinity if the coupling approaches zero, or if the decay becomes kinematically forbidden ($m \to M/2$).

Problem 2: Compton scattering

Consider an $e\gamma \to e\gamma$ scattering process. The four-momenta in the initial state are p_1 for the electron and p_2 for the photon, while in the final state they are p_2' for the photon and $p_1' = p_1 + p_2 - p_2'$ for the electron. A tree-level calculation in quantum electrodynamics gives the squared matrix element

$$|\overline{\mathcal{M}}|^2 = 32\pi^2\,\alpha^2 \left(\frac{p_1 p_2'}{p_1 p_2} + \frac{p_1 p_2}{p_1 p_2'} + 2\,m^2 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right) + m^4 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right)^2 \right) .$$

Here α is the fine-structure constant, m is the electron mass, and the bar in $\overline{\mathcal{M}}$ indicates that we have averaged over initial spin and polarization states and summed over final ones.

Starting from this expression, derive the *Klein-Nishina formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2} \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right) ,$$

where ω and ω' are the initial and final photon energies, and θ is the scattering angle between the two photons, in a frame where the initial electron is at rest.

According to the lecture, the differential cross-section for $2 \to 2$ scattering is given by

$$\mathrm{d}\sigma = \frac{1}{4\,E_1\,E_2}\frac{1}{|\vec{v}_1 - \vec{v}_2|}\frac{\mathrm{d}^3 p_1'}{2\,E_1'\,(2\pi)^3}\frac{\mathrm{d}^3 p_2'}{2\,E_2'\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}\left(p_1 + p_2 - p_1' - p_2'\right)|\overline{\mathcal{M}}|^2 .$$

We begin by choosing a coordinate system: Initially the electron is at rest at the origin, the incoming photon is aligned with the z-direction, and the two photons lie in the (y,z)-plane. This gives

$$p_1 = (m,\,\vec{0})\,, \quad p_2 = (\omega,\,\omega\vec{e}_z)\,, \quad p_1' = (E_1',\,\vec{p}_1{}')\,, \quad p_2' = (\omega',\,\omega'\sin\theta\vec{e}_y + \omega'\cos\theta\vec{e}_z)$$

where $E_1' = \sqrt{\vec{p}_1{}'^2 + m^2}$; here we have used that all particles are on shell and that the photon is massless. In this frame, $|\vec{v}_1 - \vec{v}_2| = 1$, and therefore

$$\mathrm{d}\sigma = \frac{1}{4\,\omega m}\frac{\mathrm{d}^3 p_1'}{2\,E_1'\,(2\pi)^3}\frac{\mathrm{d}^3 p_2'}{2\,\omega'\,(2\pi)^3}\,(2\pi)^4\,\delta^{(4)}\left(p_1 + p_2 - p_1' - p_2'\right)|\overline{\mathcal{M}}|^2 .$$

Splitting the four-dimensional delta function into an energy-conserving part and a 3-momentum conserving part,

$$\begin{aligned}&(2\pi)^4\,\delta^{(4)}(p_1 + p_2 - p_1' - p_2')\\ &\quad = (2\pi)\,\delta\left(m + \omega - E_1' - \omega'\right)\,(2\pi)^3\delta^{(3)}\left(\omega\vec{e}_z - \vec{p}_1{}' - \omega'\sin\theta\vec{e}_y - \omega'\cos\theta\vec{e}_z\right)\end{aligned}$$

we notice that the latter enforces

$$\vec{p}_1{}' = (\omega - \omega'\cos\theta)\vec{e}_z - \omega'\sin\theta\vec{e}_y\,, \quad E_1' = \sqrt{\omega^2 + \omega'^2 + m^2 - 2\omega\omega'\cos\theta} \qquad (1)$$

and therefore

$$\mathrm{d}\sigma = \frac{1}{8\,m\omega E_1'}\frac{\mathrm{d}^3p_2'}{2\,\omega'\,(2\pi)^3}\,(2\pi)\,\delta\left(m+\omega-E_1'-\omega'\right)\,|\overline{\mathcal{M}}|^2$$

where E_1' is now a function of ω, ω' and θ given in (1). Transforming to polar coordinates and integrating over the angle ϕ gives

$$\mathrm{d}^3p_2' = 2\pi\,\omega'^2\mathrm{d}\omega'\,\mathrm{d}\cos\theta$$

and hence

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'}{m\omega E_1'}\,\mathrm{d}\omega'\,\mathrm{d}\cos\theta\;\delta\left(m+\omega-E_1'-\omega'\right)\,|\overline{\mathcal{M}}|^2\,.$$

We should now transform the energy-conserving delta function, because its argument is a nontrivial function of the integration variable ω'. In general,

$$\delta(f(\omega')) = \sum_{\{\omega_0'\,:\,f(\omega_0')=0\}}\frac{1}{|\frac{\partial f}{\partial\omega'}(\omega_0')|}\,\delta(\omega'-\omega_0')\,.$$

Here, with $f(\omega') = m+\omega-E_1'(\omega')-\omega'$, we have

$$\left|\frac{\partial}{\partial\omega'}\left(m+\omega-E_1'-\omega'\right)\right| = \left|-1-\frac{\omega'-\omega\cos\theta}{E_1'}\right| = \left|\frac{E_1'+\omega'-\omega\cos\theta}{E_1'}\right| = \frac{m+\omega(1-\cos\theta)}{E_1'}$$

where the last equality holds only under the delta function. Therefore

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'}{m\omega(m+\omega(1-\cos\theta))}\mathrm{d}\cos\theta\,|\overline{\mathcal{M}}|^2\,.$$

In this expression, ω' is constrained by energy conservation to be a function of ω and θ. In fact,

$$\begin{aligned}
&E_1'^{\,2} = (m+\omega-\omega')^2\\
&\quad\Leftrightarrow\quad \omega^2+\omega'^2+m^2-2\omega\omega'\cos\theta = m^2+\omega^2+\omega'^2+2\,m\omega-2\,\omega\omega'-2\,m\omega'\\
&\quad\Leftrightarrow\quad \omega'(m+\omega(1-\cos\theta)) = m\omega
\end{aligned}$$

which gives

$$\mathrm{d}\sigma = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\mathrm{d}\cos\theta\,|\overline{\mathcal{M}}|^2\,.$$

Finally using the expression for $|\overline{\mathcal{M}}|^2$ gives

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} &= \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\,32\pi^2\alpha^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2\,m\,\frac{\omega'-\omega}{\omega\omega'}+m^2\left(\frac{\omega'-\omega}{\omega\omega'}\right)^2\right)\\
&= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+\underbrace{2\,m\,\frac{\omega'-\omega}{\omega\omega'}+m^2\left(\frac{\omega'-\omega}{\omega\omega'}\right)^2+1}_{=\left(1+m\frac{\omega'-\omega}{\omega\omega'}\right)^2=\cos^2\theta}-1\right)\\
&= \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right)\,.
\end{aligned}$$