M2 Cosmos, Champs et Particules — Faculté des Sciences de Montpellier

Quantum Field Theory, problem sheet 10

Problem 1: Compton scattering

Consider an $e\gamma \to e\gamma$ scattering process. The four-momenta in the initial state are p_1 for the electron and p_2 for the photon, while in the final state they are p_2' for the photon and $p_1' = p_1 + p_2 - p_2'$ for the electron. A tree-level calculation in quantum electrodynamics gives the squared matrix element

$$|\overline{\mathcal{M}}|^2 = 32\pi^2\,\alpha^2 \left(\frac{p_1 p_2'}{p_1 p_2} + \frac{p_1 p_2}{p_1 p_2'} + 2\,m^2 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right) + m^4 \left(\frac{1}{p_1 p_2} - \frac{1}{p_1 p_2'} \right)^2 \right) .$$

Here α is the fine-structure constant, m is the electron mass, and the bar in $\overline{\mathcal{M}}$ indicates that we have averaged over initial spin and polarization states and summed over final ones.

Starting from this expression, derive the *Klein-Nishina formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right) ,$$

where ω and ω' are the initial and final photon energies, and θ is the scattering angle between the two photons, in a frame where the initial electron is at rest.

Hints: Show that $\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$, and thus

$$\widetilde{\mathrm{d}^3p_1'}\;\widetilde{\mathrm{d}^3p_2'}\;(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2') = \frac{1}{8\pi}\mathrm{d}\cos\theta\frac{(\omega')^2}{\omega m} .$$

If you get stuck, see Peskin & Schroeder p. 162f.

Problem 2: The Clifford algebra

1. Given a set of four matrices γ^μ which satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu} ,$$

   show that the matrices $\gamma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the Lorentz algebra:

$$[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] = i\left(g^{\lambda\rho}\gamma^{\kappa\sigma} - g^{\kappa\rho}\gamma^{\lambda\sigma} - g^{\lambda\sigma}\gamma^{\kappa\rho} + g^{\kappa\sigma}\gamma^{\lambda\rho} \right) .$$

2. Verify that the Clifford algebra is satisfied by both the *Weyl representation* of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$

   and the *Dirac-Pauli representation*

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

   and find the unitary transformation that takes one into the other.

3. Defining $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, calculate

$$\{\gamma^5, \gamma^\mu\} \qquad \text{and} \qquad [\gamma^5, \gamma^{\mu\nu}]\,.$$

Problem 2: The Dirac field

1. Show that

$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^\mu\left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu + \mathcal{O}(||\omega||^2)\,,$$

where the $M^{\rho\sigma}$ generate the vector representation of $\mathfrak{so}(1,3)$,

$$(M^{\kappa\lambda})_{\mu\nu} = i\left(\delta^\kappa{}_\mu\delta^\lambda{}_\nu - \delta^\kappa{}_\nu\delta^\lambda{}_\mu\right)\,.$$

Use this result to conclude that the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$$

is invariant under proper orthochronous Lorentz transformations.

2. Find the Euler-Lagrange equations obtained from the Dirac Lagrangian.