Quantum Field Theory, problem sheet 10

Solutions to be discussed on 10/12/2024.

Problem 1: The Clifford algebra

1. Given a set of four matrices γ^μ which satisfy the Clifford algebra
$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}\,,$$
show that the matrices $\gamma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the Lorentz algebra:
$$[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] = i\left(g^{\lambda\rho}\gamma^{\kappa\sigma} - g^{\kappa\rho}\gamma^{\lambda\sigma} - g^{\lambda\sigma}\gamma^{\kappa\rho} + g^{\kappa\sigma}\gamma^{\lambda\rho}\right)\,.$$

2. Verify that the Clifford algebra is satisfied by both the *Weyl representation* of γ matrices
$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$
and the *Dirac-Pauli representation*
$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$
and find the unitary transformation that takes one into the other.

3. Defining $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, calculate
$$\{\gamma^5, \gamma^\mu\} \qquad \text{and} \qquad [\gamma^5, \gamma^{\mu\nu}]\,.$$

Problem 2: The Dirac field

1. Show that
$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^\mu\left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu + \mathcal{O}(||\omega||^2)\,,$$
where the $M^{\rho\sigma}$ generate the vector representation of $\mathfrak{so}(1,3)$,
$$(M^{\kappa\lambda})_{\mu\nu} = i\left(\delta^\kappa{}_\mu\delta^\lambda{}_\nu - \delta^\kappa{}_\nu\delta^\lambda{}_\mu\right)\,.$$
Use this result to conclude that the Dirac Lagrangian
$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$$
is invariant under proper orthochronous Lorentz transformations.

2. Find the Euler-Lagrange equations obtained from the Dirac Lagrangian.