Quantum Field Theory, problem sheet 9

Solutions to be discussed on 03/12/2024.

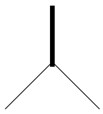
Problem 1: Decay of a scalar particle

Consider the following Lagrangian, involving two real scalar fields ϕ and χ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\chi^2 - \frac{1}{2}\mu\,\chi\phi^2.$$

Suppose that $M > 2m$, so that the decay $\chi \to \phi\phi$ is kinematically possible. Calculate the lifetime of χ to leading order in the coupling μ.

Hints: The $\chi\phi^2$ interaction term gives rise to a vertex attaching to two ϕ propagators (thin lines) and one χ propagator (thick line), for which the Feynman rule is

 $= -i\mu\,.$

From this it is easy to deduce the tree-level matrix element $\mathcal{M}$, which you can insert into the formula for the differential decay width (N is a combinatorial factor given by $n_f!$ for each group of n_f indistinguishable final-state particles)

$$\mathrm{d}\Gamma = \frac{1}{N}\frac{1}{2M}\prod_f \widetilde{\mathrm{d}p'_f}\,|\mathcal{M}|^2\,(2\pi)^4\delta^{(4)}\Big(p_i - \sum_f p'_f\Big)\,.$$

Problem 2: Compton scattering

Consider an $e\gamma \to e\gamma$ scattering process. The four-momenta in the initial state are p_1 for the electron and p_2 for the photon, while in the final state they are p'_2 for the photon and $p'_1 = p_1 + p_2 - p'_2$ for the electron. A tree-level calculation in quantum electrodynamics gives the squared matrix element

$$|\overline{\mathcal{M}}|^2 = 32\pi^2\,\alpha^2\left(\frac{p_1p'_2}{p_1p_2} + \frac{p_1p_2}{p_1p'_2} + 2\,m^2\left(\frac{1}{p_1p_2} - \frac{1}{p_1p'_2}\right) + m^4\left(\frac{1}{p_1p_2} - \frac{1}{p_1p'_2}\right)^2\right)\,.$$

Here α is the fine-structure constant, m is the electron mass, and the bar in $\overline{\mathcal{M}}$ indicates that we have averaged over initial spin and polarization states and summed over final ones.

Starting from this expression, derive the *Klein-Nishina formula*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right)\,,$$

where ω and ω' are the initial and final photon energies, and θ is the scattering angle between the two photons, in a frame where the initial electron is at rest.

Hints: Show that $\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$, and thus

$$\widetilde{\mathrm{d}^3p'_1}\;\widetilde{\mathrm{d}^3p'_2}\;(2\pi)^4\delta^{(4)}(p_1 + p_2 - p'_1 - p'_2) = \frac{1}{8\pi}\mathrm{d}\cos\theta\frac{(\omega')^2}{\omega m}\,.$$

If you get stuck, see Peskin & Schroeder p. 162f.