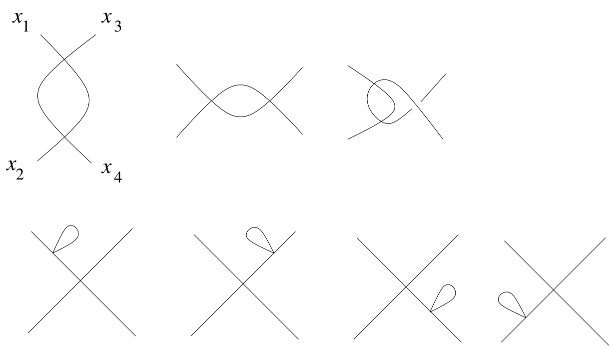
QFT, SOLUTIONS TO PROBLEM SHEET 7

Problem 1: Feynman diagrams

For ϕ^4 theory:

1. Find all connected Feynman diagrams contributing to the four-point function at $\mathcal{O}(\lambda^2)$ and determine their symmetry factors.

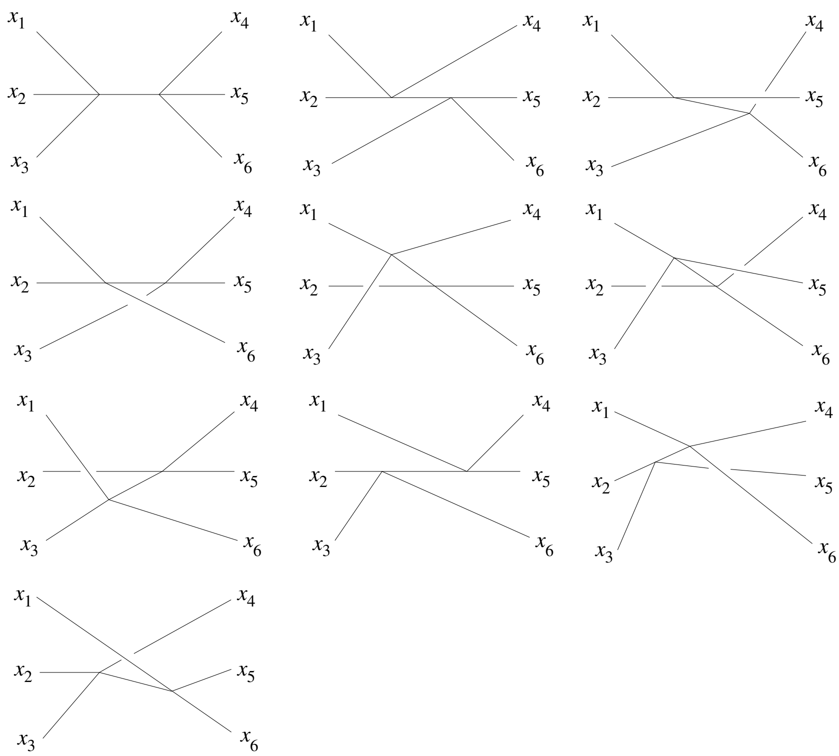
There are seven distinct diagrams, all of which have $S = 2$:



Each diagram in the first line has $S = 2$ because the two loop propagators can be swapped, which gives the same diagram. Each diagram in the second line has $S = 2$ because the two ends of the loop propagator can be exchanged, which gives the same diagram.

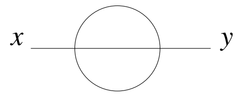
2. Repeat this exercice for the six-point function at $\mathcal{O}(\lambda^2)$.

There are 10 distinct diagrams, all with $S = 1$:



To see why this list is complete, note that the external leg labelled by x_1 must share a vertex with three other lines. One of these is an internal line leading to another vertex (because at $\mathcal{O}(\lambda^2)$ there are precisely two vertices), and the two others are external legs labelled by x_i and x_j. So there is one diagram each for $(i,j) = (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6)$ and $(5,6)$. The remaining three external legs then attach to the second vertex.

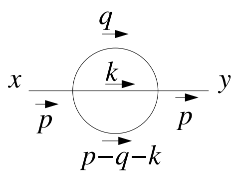
3. State the algebraic expression in momentum space which corresponds to the following Feynman diagram (without evaluating the integrals)



The symmetry factor is $S = 3! = 6$, corresponding to permutations of the three internal lines. The algebraic expression is

$$\frac{1}{6}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{ipx}\frac{i}{p^2-m^2}(-i\lambda)\frac{i}{q^2-m^2}\frac{i}{k^2-m^2}\frac{i}{(p-k-q)^2-m^2}(-i\lambda)\frac{i}{p^2-m^2}e^{-ipy}$$

where we have used momentum conservation at the vertices and parameterized the momenta as follows:

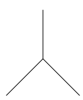


4. Find the Feynman rules for ϕ^3 theory. Draw the connected Feynman diagrams contributing to the 1-point, 2-point and 3-point functions at $\mathcal{O}(\mu^2)$, and determine their symmetry factors.

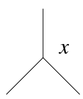
Now the interaction potential is $\mathcal{V}_{\mathrm{int}} = \frac{\mu}{6}\phi^3$, and the generating functional can be written

$$Z[J] = N\sum_{V=0}^{\infty}\frac{1}{V!}\left(\frac{-i\mu}{3!}\int\mathrm{d}^4x\frac{1}{i^3}\frac{\delta^3}{\delta J(x)^3}\right)^V\sum_{P=0}^{\infty}\frac{1}{P!}\left(\frac{1}{2}\int\mathrm{d}^4y\mathrm{d}^4z\, iJ(y)\, D_F(y-z)\, iJ(z)\right)^P .$$

A diagram with E external sources has $E = 2P - 3V$ and can contribute to the E-point function, which is obtained by E-fold differentiation w.r.t. iJ and then setting $J = 0$. In analogy with ϕ^4 theory, one finds that diagrams are built out of propagators, sources and vertices, where every vertex now attaches to three propagators:

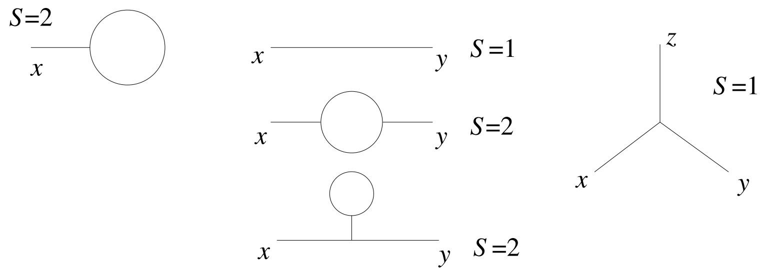
 $= -i\mu\int\mathrm{d}^4x\,\frac{1}{i^3}\frac{\delta^3}{\delta J(x)^3}\,.$

The remaining Feynman rules are the same as for ϕ^4 theory. For position-space diagrams,

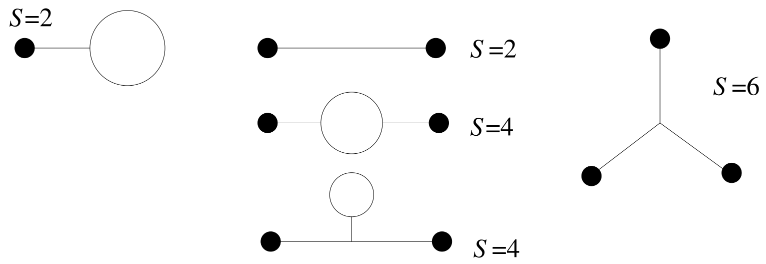


$= -i\mu\int\mathrm{d}^4x$

while in momentum space, there is momentum conservation at each vertex (and also a factor $-i\mu$, analogously to the $-i\lambda$ in ϕ^4 theory).
The Feynman diagrams are

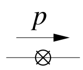


Note that, if instead of the Feynman diagrams contributing to the n-point functions, the question had been about Feynman diagrams corresponding to the generating functional with n external sources, then the symmetry factors would be different (because the sources are indistinguishable):



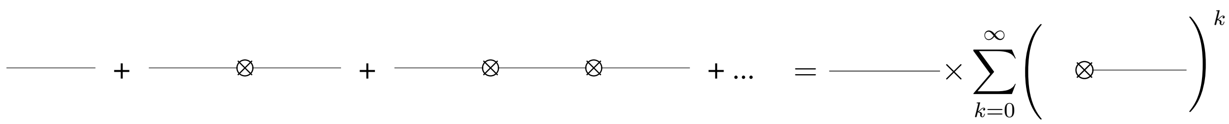
Problem 2: Counterterms in ϕ^4 theory

In ϕ^4 theory in renormalised perturbation theory, show that the Feynman rule for the "2-point vertex" associated to the counterterms δ_Z and δ_{m^2} is



$$= i(p^2\delta_Z - \delta_{m^2})\,.$$

We have



$$= \frac{i}{p^2 + m^2 - i\epsilon} \sum_{k=0}^{\infty} \left(\frac{-p^2\delta_Z + \delta_{m^2}}{p^2 - m^2 + i\epsilon} \right)^k$$

If this is to converge, one should obtain the usual limit of the geometric series, $\sum_n x^n = (1-x)^{-1}$, hence

$$\ldots = \frac{i}{p^2 - m^2 + i\epsilon} \frac{1}{1 - \frac{-p^2\delta_Z + \delta_{m^2}}{p^2 - m^2 + i\epsilon}} = \frac{i}{p^2 - m^2 + i\epsilon + p^2\delta_Z - \delta_{m^2}} = \frac{i}{Zp^2 - (m^2 + \delta_{m^2}) + i\epsilon}$$