Connexions de Galois, Treillis de Galois, Treillis de Concepts

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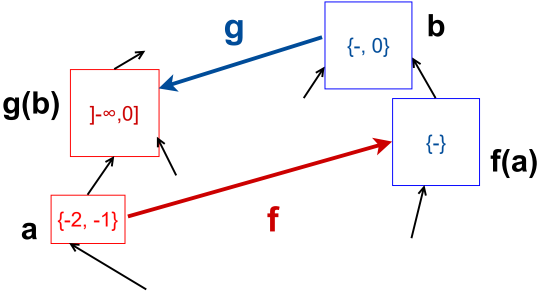
Galois connection *in* Lattice theory



Georges David Birkhoff, 1940; Øystein Ore, 1944

A monotone Galois connection between $(A, \leq)$ and $(B, \leq)$ is a pair (f, g) s.t. f, g monotone and $f : A \to B$ and $g : B \to A$ $\forall a \in A, b \in B$, $a \leq g(b) \Leftrightarrow f(a) \leq b$

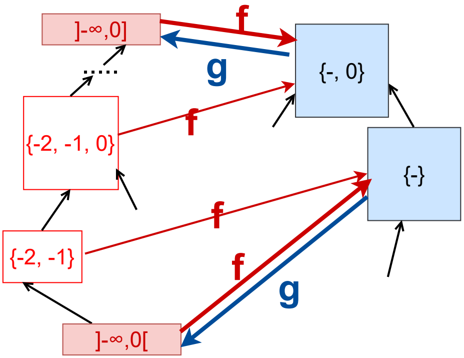
$(A, \leq)$
Sets of integers ordered by inclusion



$(B, \leq)$
Set of signs (+, - 0) ordered by inclusion

Galois connection *in* Lattice theory

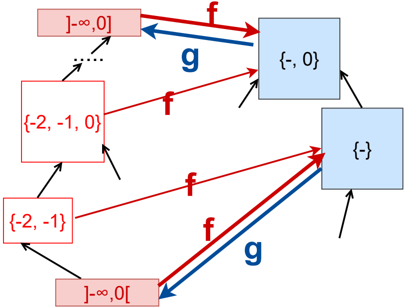
(f, g) Galois connection *implies* $f \circ g$ and $g \circ f$ are closure operators
e.g. for $f \circ g$: isotone ($X \leq Y \Rightarrow f \circ g(X) \leq f \circ g(Y)$), extensive ($X \leq f \circ g(X)$, and idempotent ($f \circ g(f \circ g(X)) = f \circ g(X)$)



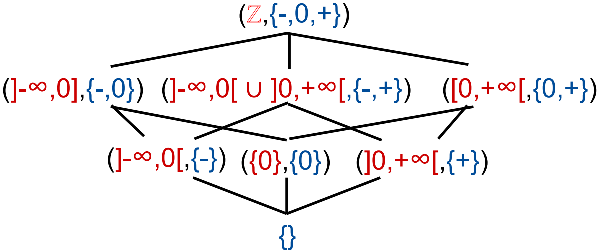
Closed elements for $g \circ f$
$g \circ f(x) = x$
Ex. $]-\infty, 0]$
$g \circ f(]-\infty, 0]) =]-\infty, 0]$
Counter Ex. $\{-2, -1, 0\}$
$g \circ f(\{-2, -1, 0\}) =]-\infty, 0]$

Closed elements for $f \circ g$
$f \circ g(x) = x$
Ex. $\{-, 0\}$

Galois lattices *in* Lattice theory



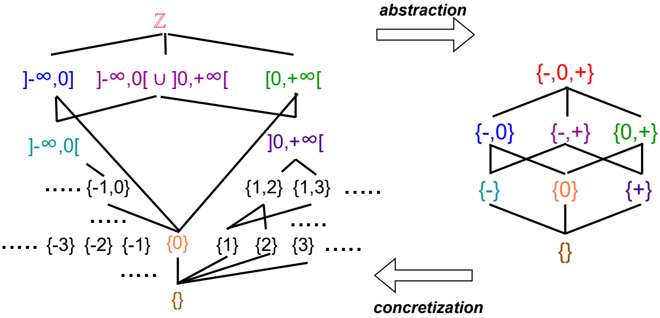
A few corresponding closed elements



The Galois lattice is the ordered set of pairs/assemblies of corresponding closed elements

Galois connection *in* Lattice theory

This lattice is used in abstract interpretation (Cousot & Cousot) to reason on signs of variables whose values are integer sets



Galois lattices *in* Lattice theory

Particular case: Galois connection associated with a binary relation

O objects, A attributes, $R \subseteq O \times A$

	flies	nocturnal	feathered	migratory	red-bill	elytra	sea-habitat	wood-habitat	six-legged	eats-fish	water-habitat
ladybird	×					×			×		
bat	×	×									
ostrich			×								
greater-flamingo	×		×	×			×			×	×
silver-gull	×		×		×		×			×	×
little-tern	×		×	×			×			×	×
great-auk	×		×				×			×	×
wood-pecker	×		×					×			
giant-otter										×	×
arctic-tern	×		×	×	×		×			×	×

Galois lattices *in* Lattice theory

f associates an object set with their shared attributes

$f : P(O) \rightarrow P(A) \quad X \longmapsto f(X) = \{y \in A \mid \forall x \in X, (x, y) \in R\}$

	flies	nocturnal	feathered	migratory	red-bill	elytra	sea-habitat	wood-habitat	six-legged	eats-fish	water-habitat
ladybird	×					×			×		
bat	×	×									
ostrich			×								
greater-flamingo	×		×	×			×			×	×
silver-gull	×		×		×		×			×	×
little-tern	×		×	×			×			×	×
great-auk	×		×				×			×	×
wood-pecker	×		×					×			
giant-otter										×	×
arctic-tern	×		×	×	×		×			×	×

Galois lattices *in* Lattice theory

g associates an attribute set with the objects sharing them

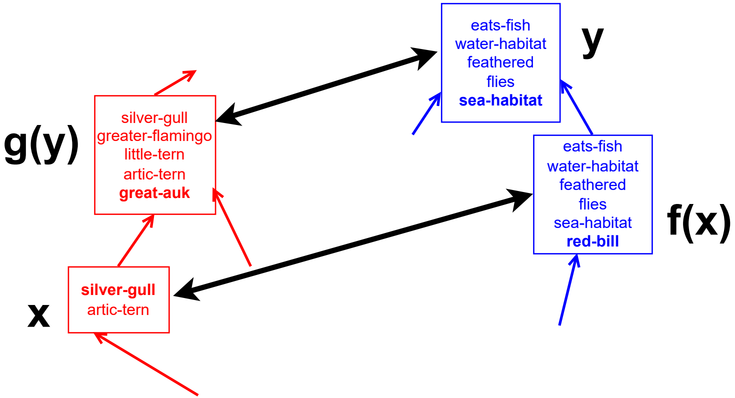
$$g : P(A) \rightarrow P(O) \qquad Y \longmapsto g(Y) = \{x \in O \mid \forall y \in Y, (x, y) \in R\}$$

	flies	nocturnal	feathered	migratory	red-bill	elytra	sea-habitat	wood-habitat	six-legged	eats-fish	water-habitat
ladybird	×					×			×		
bat	×	×									
ostrich			×								
greater-flamingo	×		×	×			×			×	×
silver-gull	×		×		×		×			×	×
little-tern	×		×	×			×			×	×
great-auk	×		×				×			×	×
wood-pecker	×		×					×			
giant-otter										×	×
arctic-tern	×		×	×	×		×			×	×

Galois lattices *in* Lattice theory

(f, g) is a Galois connection between $(2^O, \subseteq)$ and $(2^A, \supseteq)$

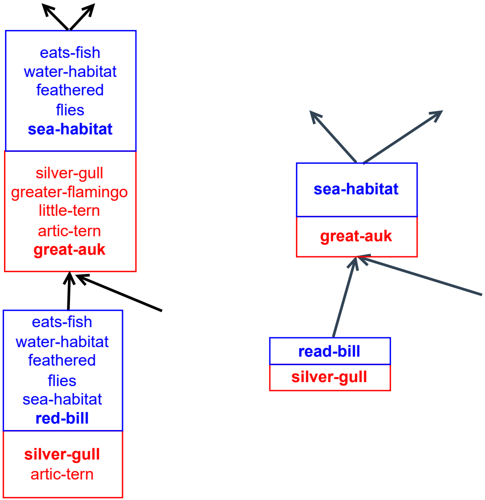
Closed sets are maximal sets of objects sharing maximal set of attributes (and reversely)



Galois lattices *in* Lattice theory

Concepts are assemblies of closed sets of objects and attributes

Galois lattices *in* Lattice theory



Detail of the Galois lattice of animals

Elements are assemblies of closed sets of objects and attributes

Simplified view (detail): Top-down inherited attributes, bottom-up inherited objects are removed

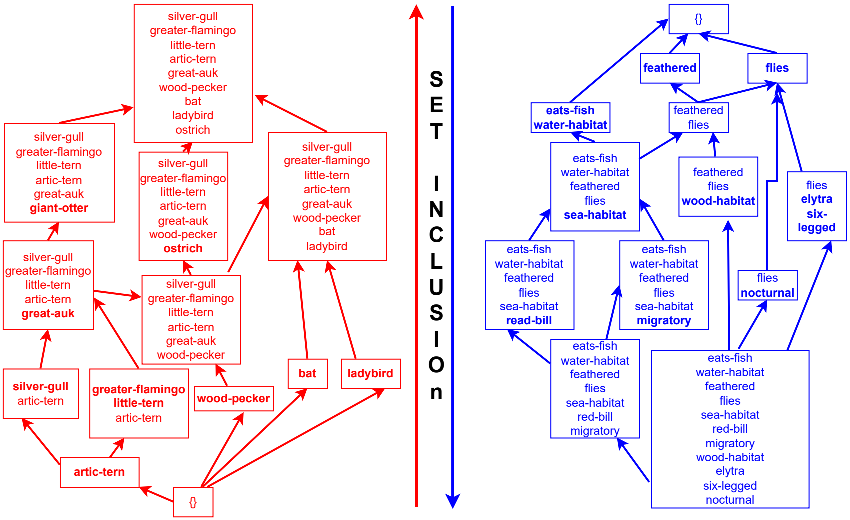
Concept ordering

Concepts can be ordered through the following partial order $\leq_s$:
$(E_1, I_1) \leq_s (E_2, I_2) \Leftrightarrow E_1 \subseteq E_2$, or equivalently $I_2 \subseteq I_1$
(E_1, I_1) sub-concept of (E_2, I_2); (E_2, I_2) super-concept of (E_1, I_1)

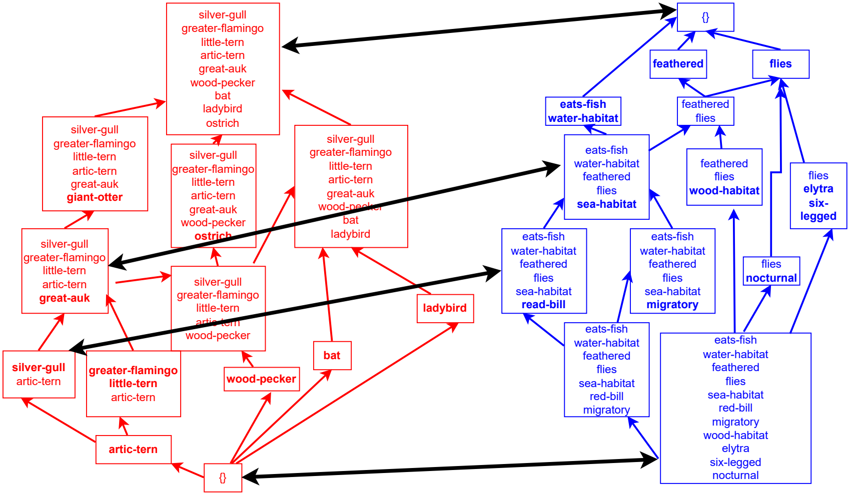
$C_{great-auk}$:
$X_{10} = \{great-auk, silver-gull, greater-flamingo, little-tern, arctic-tern\}$
$Y_{10} = \{sea-habitat, eats-fish, water-habitat, feathered, flies\}$

$C_{silver-gull}$:
$X_8 = \{silver-gull, arctic-tern\}$
$Y_8 = \{sea-habitat, eats-fish, water-habitat, feathered, flies, red-bill\}$
$C_{silver-gull} \leq_s C_{great-auk}$, as $X_8 \subseteq X_{10}$ (and $Y_{10} \subseteq Y_8$).

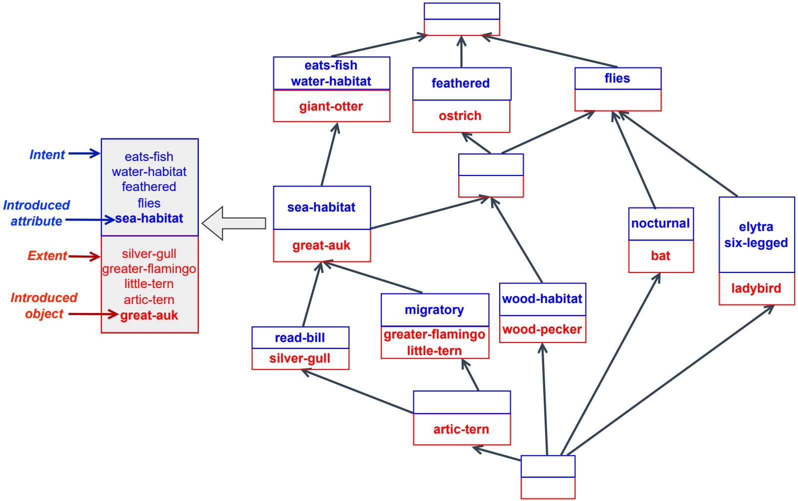
Concept lattices: Assembly of two isomorphic lattices



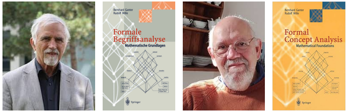
Assembly: through a Galois connection



Galois lattices *in* Lattice theory



FCA *from* philosophy *to* math. (1982-1999)



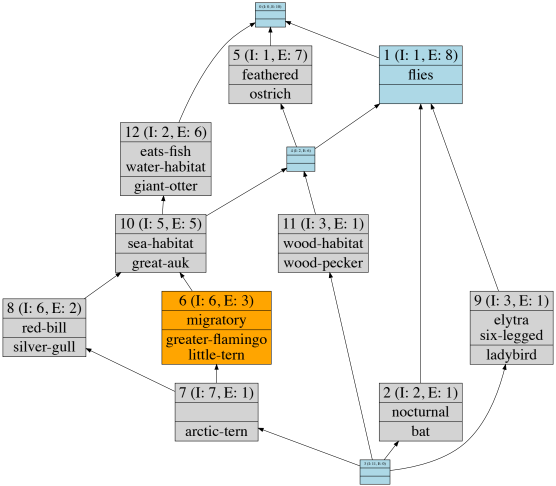
Rudolf Wille, 1982, Bernhard Ganter & Rudolf Wille, 1999

Formal Concept Analysis

Another vocabulary

(O, A, R)	Formal context
$f \quad g \quad f \circ g \quad g \circ f$	$' \quad ' \quad '' \quad ''$
Closed set of objects	Extension, extent
Closed set of attributes	Intension, intent
Assembly of corresponding closed sets	Concept
Partial order on assembled closed sets	Partial order subconcept / superconcept
Galois lattice	Concept lattice

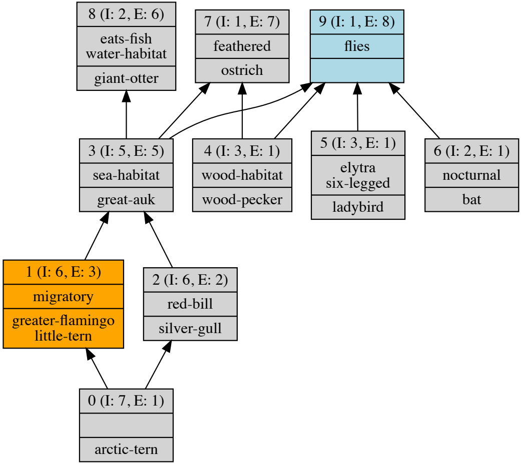
Conceptual structures: Concept lattice



Worst case:
#concepts $< 2^{min(|A|,|O|)}$

Reached with the lattice of all subsets of E
where $E = O$ if $|O| = min(|A|, |O|)$
(otherwise, $E = A$)

Conceptual structures: AOC poset

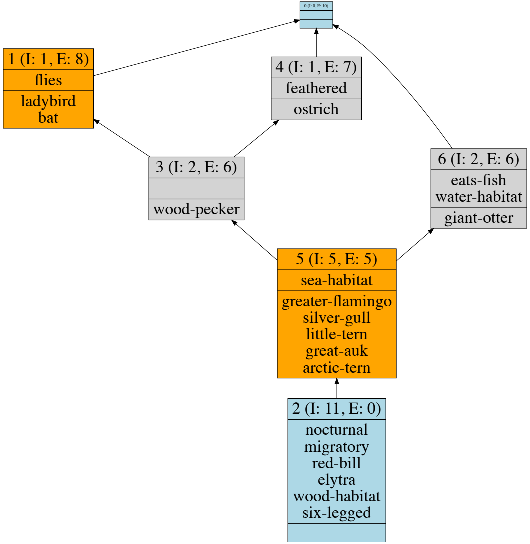


Sub-order of introducer concepts

Worst case complexity:#concepts $< |A| + |O|$

Reached when $|A| = |O|$ and every attribute is shared by several distinct objects (ex. bipartite crown graph)

Conceptual structures: Iceberg



Concepts with frequent intent or frequent extent

Here:
concepts with frequent extent ($\geq$ 50%)

Applications (A few domains)

- Environment, biology, chemistry, health
- Linguistics, Text understanding
- Software engineering
- Communities, social network

Part of them in: Jonas Poelmans, Dmitry I. Ignatov, Sergei O. Kuznetsov, Guido Dedene: Formal concept analysis in knowledge processing: A survey on applications. Expert Syst. Appl. 40(16): 6538-6560 (2013) + significant new work since

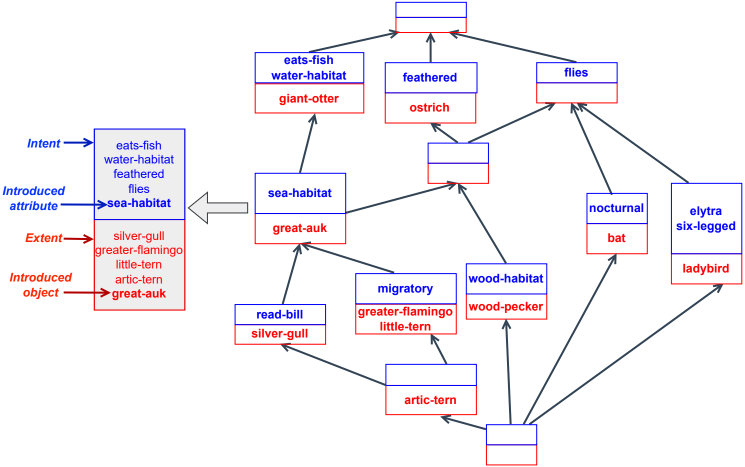
Tools

More than 50 web applications, downloadable software or plugins at Uta Priss webpage `https://upriss.github.io/fca/fcasoftware.html`

- Algorithms: Conexp family, ToscanaJ, fcaR, GALACTIC
- Extensions: RCA software (Galicia, RCAexplore, FCA4J), Fuzzy (fcaR), polyadic (FCA Tools Bundle)
- Search/Query engines and IR: Credo family, Search Sleuth family, Camelis, Sparklis family
- Visualization+navigation: Latviz, RV-xplorer, ConceptCloud, RCAviz

Workshops on tools @ICFCA or @CLA

Exercice : donner les extensions et intensions complètes des concepts



Exercice : construire le treillis de concepts puis l'AOC-poset associé à la table

Plant	lauraceae	asteraceae	aromatic	comestible	toxic	evergreen	contraceptive	antidysenteric	applicOil	applicEssentialOil	applicExtract
cinnamomumZeylanicum	×		×	×		×			×		
chromolaenaOdorata		×	×		×	×				×	×
aspiliaAfricana		×	×		×	×	×			×	×
ageratumConyzoides		×	×		×	×		×		×	×

Exercice : construire le treillis de concepts puis l'AOC-poset associé aux deux tables

Sushis2Weights	20g	25g	30g
california salmon 20g	x		
california salmon 25g		x	
california tuna 20g	x		
california tuna 25g		x	
maki cheese 20g	x		
maki cheese 30g			x
maki tobiko 20g	x		
maki tobiko 30g			x

Sushis2Ingredients	seaweed	salad	salmon	tuna	avocado	rice	tobiko	cucumber	cream-cheese
california salmon 20g	x	x	x		x	x			
california salmon 25g	x	x	x		x	x			
california tuna 20g	x	x		x	x	x			
california tuna 25g	x	x		x	x	x			
maki cheese 20g		x			x	x			x
maki cheese 30g		x			x	x			x
maki tobiko 20g	x				x	x	x	x	
maki tobiko 30g	x				x	x	x	x	