Quantum Field Theory, problem sheet 6

Solutions to be discussed on 22/10/2024.

Problem 1: The path integral for the harmonic oscillator

Consider a quantum mechanical harmonic oscillator with the Hamiltonian (setting $m = 1$)

$$H = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{q}^2 \,.$$

1. By retracing the steps we followed in the lecture for a free field theory, show that the vacuum-to-vacuum transition amplitude in the presence of a source $j(t)$ can be written as

$$\langle 0|0\rangle_j = \exp\left(-\frac{1}{4\omega}\int \mathrm{d}t\,\mathrm{d}t'\; j(t)e^{-i\omega|t-t'|}j(t')\right) \,.$$

2. For a harmonic oscillator which is in its ground state at $t \to -\infty$ and which is subject to an external force

$$j(t) = \begin{cases} j_0 & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(where $j_0 > 0$ is a constant), calculate that the probability that it is still in its ground state for $t \to +\infty$.

Problem 2: Photon propagator

Recall that the action for classical electrodynamics with a ficticious photon mass term and an external source J is (see problem sheet 1)

$$S[A,\, J] = \int \mathrm{d}^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - A_\mu J^\mu\right) \,.$$

In quantum theory we define the generating functional via the path integral:

$$Z[J] = \int \mathcal{D}A\, \exp\left(iS[A,\, J]\right) \,.$$

1. Show that

$$\left(g_{\mu\nu} - k_\mu k_\nu/m^2\right)\left((k^2 - m^2)g^{\nu\lambda} - k^\nu k^\lambda\right) = (k^2 - m^2)\delta^\lambda_\mu \,.$$

2. Using this result, find the function $\widetilde{D}_{\mu\nu}(k,\, m)$ which satisfies

$$Z[J] = Z[0]\, \exp\left(\frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{J}^\mu(k)\widetilde{D}_{\mu\nu}(k,m)\tilde{J}^\nu(-k)\right) \,.$$

Here $\tilde{J}^\mu(k)$ is the Fourier transform of $J^\mu(x)$.

3. Assuming that J^ν is a conserved current, what does this imply for $k_\nu\tilde{J}^\nu(k)$? What can we therefore take for $\widetilde{D}_{\mu\nu}(k,m)$ as $m \to 0$ (the limit which corresponds to a realistic massless photon)?