# Quantum Field Theory, problem sheet 5

Solutions to be discussed on 15/10/2024

## Problem 1: The interaction picture

In this exercise we will derive a method for calculating $n$-point functions for the real scalar field purely based on canonical quantisation, without using the path integral. We split the Hamiltonian of an interacting real scalar field into a free part $H_0$ and an interaction part $H_{\text{int}}$, hence $H = H_0 + H_{\text{int}}$ with

$$H_0 = \int \mathrm{d}^3x \left( \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) ,$$
$$H_{\text{int}} = \int \mathrm{d}^3x \, \mathcal{V}_{\text{int}}(\phi) .$$

Denote by $|0\rangle$ the ground state of $H$ and by $|\emptyset\rangle$ the ground state of $H_0$. We add a constant to $H$ such that $H_0|\emptyset\rangle = 0$. The Heisenberg-picture field operator and its canonical momentum are

$$\phi(t, \vec{x}) = e^{iHt}\phi(0, \vec{x})e^{-iHt} , \qquad \pi(t, \vec{x}\,') = e^{iHt}\pi(0, \vec{x}\,')e^{-iHt} ,$$

and as usual they satisfy $[\phi(t, \vec{x}), \pi(t, \vec{x}\,')] = i\delta^{(3)}(\vec{x} - \vec{x}\,')$. We define the *interaction-picture operators* by

$$\phi_I(t, \vec{x}) = e^{iH_0t}\phi(0, \vec{x})e^{-iH_0t} , \qquad \pi_I(t, \vec{x}) = e^{iH_0t}\pi(0, \vec{x})e^{-iH_0t} .$$

1. Show that $\dot{\phi}_I(t, \vec{x}) = \pi_I(t, \vec{x})$.
2. Starting from an expression for $\ddot{\phi}_I$, show that $\phi_I$ obeys the Klein-Gordon equation, and hence is a free field.

   *Hint:* Differential operators are defined to act on distributions (such as the Dirac delta) by integration by parts: $\int \mathrm{d}x \, f(x) \frac{\partial}{\partial x}\delta(x) \equiv -\int \mathrm{d}x \, \frac{\partial f}{\partial x}\delta(x)$.
3. Show that $U(t) \equiv e^{iH_0t}e^{-iHt}$ is unitary, and that $\phi(x) = U^\dagger(t)\phi_I(x)U(t)$.
4. We would like to express $U(t)$ entirely in terms of $\phi_I$. To this end, start by showing that $U(t)$ obeys the Schrödinger equation

   $$i\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H_I(t)U(t)$$

   where $H_I$ is the interaction Hamiltonian in the interaction picture, $H_I(t) = e^{iH_0t}H_{\text{int}}e^{-iH_0t}$, with the boundary condition $U(0) = \mathbb{1}$. Then show that, for $t > 0$,

   $$U(t) = \mathrm{T} \, \exp\left( -i \int_0^t \mathrm{d}t' \, H_I(t') \right)$$

   also solves this Schrödinger equation and satisfies the same boundary condition. Therefore both expressions must be equal.

5. Define $U(t_2, t_1) = U(t_2)U^\dagger(t_1)$. By a similar argument as used in 4., show that, for $t_2 > t_1$,
$$U(t_2, t_1) = \mathrm{T}\,\exp\left(-i\int_{t_1}^{t_2} \mathrm{d}t'\, H_I(t')\right) .$$

6. Show that $U(t_1, t_3) = U(t_1, t_2)U(t_2, t_3)$, and that $U^\dagger(t_1, t_2) = U(t_2, t_1)$.

7. We would like to find a relation between the free vacuum $|\emptyset\rangle$ and the interacting vacuum $|0\rangle$. Let $E_0$ be the vacuum energy of the interacting theory, $H|0\rangle = E_0|0\rangle$, and assume that $\langle 0|\emptyset\rangle \neq 0$. By inserting a complete set $\{|n\rangle\}$ of energy eigenstates of the interacting theory,
$$e^{-iHT}|\emptyset\rangle = e^{-iHT}\sum_n |n\rangle\langle n|\emptyset\rangle$$
show that
$$|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|\emptyset\rangle}{e^{-iE_0T}\langle 0|\emptyset\rangle} = \lim_{T\to\infty(1-i\epsilon)} \frac{U(0,-T)|\emptyset\rangle}{e^{-iE_0T}\langle 0|\emptyset\rangle} .$$
Similarly, show that
$$\langle 0| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle\emptyset|U(T,0)}{e^{-iE_0T}\langle\emptyset|0\rangle} .$$
*Hint:* Split the sum over energy eigenstates into the ground state and the excited states, then use the fact that $E_n > E_0$ for $n \neq 0$.

8. Finally, use the results of 5., 6. and 7. to show that
$$\langle 0|\mathrm{T}\,\phi(x)\phi(y)|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle\emptyset|\mathrm{T}\,\phi_I(x)\phi_I(y)\, e^{-i\int_{-T}^{T}\mathrm{d}t\, H_I(t)}|\emptyset\rangle}{\langle\emptyset|\mathrm{T}\, e^{-i\int_{-T}^{T}\mathrm{d}t\, H_I(t)}|\emptyset\rangle} .$$

It is straightforward to demonstrate that higher correlation functions are given by the obvious generalisation of this formula (more factors of $\phi$ on the left and of $\phi_I$ on the right), the *Gell-Mann-Low formula.* It can be evaluated perturbatively by expanding the exponentials on the right-hand side, and using the fact that $\phi_I$ is a free field and therefore enjoys a free Fourier mode expansion.

*Remark:* When trying to define interacting quantum field theory in a mathematically rigorous way, one comes across a problem known as *Haag's theorem*: It can be shown that the interaction picture does not actually exist. There is strictly no unitary transformation $U$ which maps operators on the free-theory Hilbert space onto operators on the Hilbert space of the interacting theory. Therefore, the approach you just derived heuristically cannot be made mathematically exact.
However, this caveat mostly bothers axiomatic field theorists; practitioners of quantum field theory tend to ignore it because it does lead to an accurate description of experimentally observed phenomena. In fact, finding soundly defined interacting quantum field theories in four dimensions continues to be an active research topic in mathematical physics, even though the working rules for QFT were established many decades ago.