Quantum Field Theory, problem sheet 4

Solutions to be discussed on 08/10/2024

Problem 1: Asymptotic overlap between wave packets

Consider an interacting theory of a real scalar field with mass m. Let $|\psi\rangle$ be a multi-particle wave packet given by

$$|\psi\rangle = \int \mathrm{d}^3p \; \psi(\vec{p}) \, |\lambda_p\rangle \, .$$

Here $\psi(\vec{p})$ is an enveloping function (e.g. a Gaussian) and $|\lambda_p\rangle$ is a multi-particle state of total 4-momentum $p = (p^0, \vec{p})$ and invariant mass $\sqrt{p^2} \equiv M > m$. With $a_1^\dagger(t) = \int \mathrm{d}^3k \, f_1(\vec{k}) a^\dagger(\vec{k}, t)$ defined as in the lecture, show that

$$\lim_{t \to \pm\infty} \langle \psi | a_1^\dagger(t) | 0 \rangle = 0 \, ,$$

i.e. the overlap between a one-particle and a multi-particle wave packet tends to zero at $t \to \pm\infty$.

Hints:

- Express $a^\dagger$ in terms of ϕ and write $\phi(x) = e^{iPx} \phi(0) e^{-iPx}$, where P is the 4-momentum operator.
- Two of the resulting integrals collapse when using $\int \mathrm{d}^3x \; e^{i\vec{q}\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{q})$.
- Estimate the remaining integral with the help of the *stationary phase approximation*: Let $f : \mathbb{R}^n \to \mathbb{C}$ be a suitable test function, $\lambda > 0$, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Suppose that φ has a single critical point p_0 and that this critical point is nondegenerate, i.e. the Hessian matrix $H_\varphi(p_0)$ is nonsingular. Then, with $\sigma = \operatorname{sgn} H_\varphi(p_0) =$ the number of positive minus the number of negative eigenvalues of $H_\varphi(p_0)$,

$$\int \mathrm{d}^n p \; f(p) e^{i\lambda\varphi(p)} = |\det H_\varphi(p_0)|^{-1/2} \, e^{i\lambda\varphi(p_0) + i\pi\sigma/4} \left(\frac{2\pi}{\lambda}\right)^{n/2} f(p_0) + \mathrm{o}(|\lambda|^{-n/2}) \, .$$

Problem 2: Gaussian and oscillatory integrals

We recall the basic Gaussian integral: if $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, then

$$\int_{-\infty}^{\infty} \mathrm{d}x \; e^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

1. Let $a, b \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Show that:

$$\int_{-\infty}^{\infty} \mathrm{d}x \; \exp\left(-\frac{1}{2} a \, x^2 + b \, x\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{1}{2} \frac{b^2}{a}\right) \, .$$

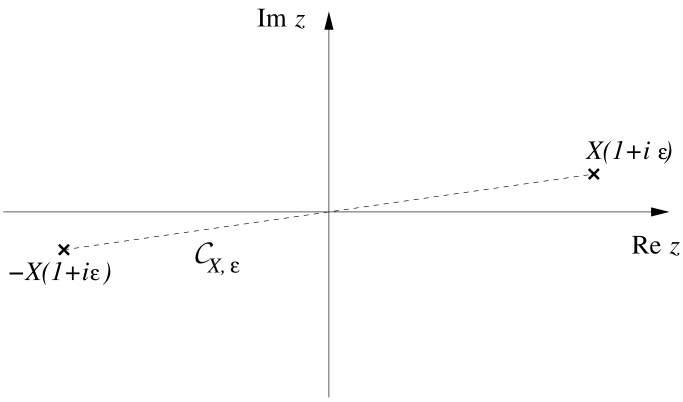
2. Let $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$. We define the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(\frac{i}{2} a\, x^2 + i\, b\, x\right)$$

by a complex line integral whose curve of integration $\mathcal{C}_{X,\epsilon}$ is the straight line connecting the points $-X(1+i\epsilon)$ and $+X(1+i\epsilon)$:

$$I = \lim_{\epsilon \searrow 0} \lim_{X \to \infty} \int_{\mathcal{C}_{X,\epsilon}} \mathrm{d}z \, \exp\left(\frac{i}{2} a\, z^2 + i\, b\, z\right) .$$

For illustration:



Calculate I. Compare your result with the stationary phase approximation given in exercise 1.

3. Let A be a real symmetric positive definite $n \times n$ matrix. Show that

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2} x^T A x\right) = \sqrt{\frac{(2\pi)^n}{\det A}}, \qquad \int \mathrm{d}^n x \exp\left(\frac{i}{2} x^T A x\right) = e^{in\pi/4} \sqrt{\frac{(2\pi)^n}{\det A}}.$$