QFT, SOLUTIONS TO PROBLEM SHEET 2

Problem 1: Annihilation and creation operators for the free real scalar field

1. *Show that $\dot{a} = 0$.*

Evaluating the two-sided derivative gives

$$a = \int \mathrm{d}^3x\, e^{ikx} \left(i\dot{\phi} + k^0\phi\right) = \int \mathrm{d}^3x\, e^{ikx} \left(i\dot{\phi} + \omega\phi\right) .$$

Therefore

$$\dot{a} = \int \mathrm{d}^3x\, e^{ikx} \left(-\omega\dot{\phi} + i\omega^2\phi + i\,\ddot{\phi} + \omega\dot{\phi}\right) = i\int \mathrm{d}^3x\, e^{ikx} \left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\phi$$
$$= i\int \mathrm{d}^3x\, e^{ikx} \left(\frac{\partial^2}{\partial t^2} + \vec{k}^2 + m^2\right)\phi = i\int \mathrm{d}^3x\, e^{ikx} \left(\Box + m^2\right)\phi = 0\,,$$

where in the last step we have used the Klein-Gordon equation.

2. *Show that defining a in this way inverts the Fourier decomposition.*

One has

$$a^* = \int \mathrm{d}^3x\, e^{-ikx} \left(-i\dot{\phi} + k^0\phi\right) ,$$

and so

$$\int \widetilde{\mathrm{d}k}\left(a\, e^{-iky} + a^*\, e^{iky}\right) = \int \widetilde{\mathrm{d}k} \int \mathrm{d}^3x \left(e^{ik(x-y)}\left(i\dot{\phi}(x) + k^0\phi(x)\right) - e^{-ik(x-y)}\left(-i\dot{\phi}(x) - k^0\phi(x)\right)\right)$$
$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega} \int \mathrm{d}^3x \Bigg(e^{i\omega(x^0-y^0)}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\left(i\dot{\phi}(x) + \omega\phi(x)\right)$$
$$- e^{-i\omega(x^0-y^0)}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(i\dot{\phi}(x) - \omega\phi(x)\right)\Bigg)$$

Given that a is time-independent, we may choose $x^0 = y^0$ to obtain

$$\ldots = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega} \int \mathrm{d}^3x \left(e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\left(i\dot{\phi}(x) + \omega\phi(x)\right) - e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(i\dot{\phi}(x) - \omega\phi(x)\right)\right)\Bigg|_{x^0=y^0}$$

Changing the integration variable $\vec{k} \to -\vec{k}$ in the second term doesn't change the value of the integral, so the $\dot{\phi}$ terms cancel:

$$\ldots = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\omega} \int \mathrm{d}^3x\, e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}(2\omega\phi(x))\Bigg|_{x^0=y^0} = \int \mathrm{d}^3x\, \delta^{(3)}(\vec{x}-\vec{y})\phi(x)\Bigg|_{x^0=y^0} = \phi(y)\,.$$

3. *Using the canonical equal-time commutation relations, compute*

$$\left[a(\vec{k}), a^\dagger(\vec{k}\,')\right] .$$

According to part 1., we can write

$$a(\vec{k}) = i\int \mathrm{d}^3y\, e^{iky}\left(\pi(y) - ik^0\phi(y)\right)\,, \quad a^\dagger(\vec{k}') = -i\int \mathrm{d}^3x\, e^{-ik'x}\left(\pi(x) + ik'^0\phi(x)\right)$$

and so

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \int \mathrm{d}^3x \int \mathrm{d}^3y\; e^{iky-ik'x}[\pi(y) - ik^0\phi(y), \pi(x) + ik'^0\phi(x)]\,.$$

Since a and $a^\dagger$ are time-independent, we choose $x^0 = y^0 \equiv t$. With $k^0 \equiv \omega$ and $k'^0 \equiv \omega'$ this gives

$$\begin{aligned}
\left[a(\vec{k}), a^\dagger(\vec{k}')\right] &= \int \mathrm{d}^3x \int \mathrm{d}^3y\; e^{i(\vec{k}\cdot\vec{y}-\vec{k}'\cdot\vec{x})}e^{i(\omega'-\omega)t}\Big([\pi(t,\vec{y}), \pi(t,\vec{x})] - i\omega[\phi(t,\vec{y}), \pi(t,\vec{x})] \\
&\qquad + i\omega'[\pi(t,\vec{y}), \phi(t,\vec{x})] + \omega\omega'[\phi(t,\vec{y}), \phi(t,\vec{x})]\Big) \\
&= \int \mathrm{d}^3x \int \mathrm{d}^3y\; e^{i(\vec{k}\cdot\vec{y}-\vec{k}'\cdot\vec{x})}e^{i(\omega'-\omega)t}\left(\omega\,\delta^{(3)}(\vec{x}-\vec{y}) + \omega'\,\delta^{(3)}(\vec{x}-\vec{y})\right) \\
&= \int \mathrm{d}^3x\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}e^{i(\omega'-\omega)t}\,(\omega+\omega') \\
&= \underbrace{\int \mathrm{d}^3x\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')}\, e^{i(\omega'-\omega)t}\,(\omega+\omega') \\
&= 2\omega\,(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')\,.
\end{aligned}$$

In a similar way, one may show that $[a(\vec{k}), a(\vec{k}')] = 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')]$.

Problem 2: Canonical quantisation of the free complex scalar field

1. *Find the canonical momenta conjugate to ϕ and ϕ^*.*

We have $\pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi^*$ and $\pi_{\phi^*} = \frac{\partial\mathcal{L}}{\partial\dot\phi^*} = \dot\phi = \pi^*_\phi$, so from now on we write $\pi \equiv \pi_\phi$ and $\pi^* \equiv \pi_{\phi^*}$.

2. *Promoting ϕ and ϕ^* to operators, imposing canonical equal-time commutation relations for the fields and their conjugate momenta, and writing the mode expansion of ϕ as*

$$\phi(x) = \int \widetilde{\mathrm{d}k}\left(a(\vec{k})e^{-ikx} + b^\dagger(\vec{k})e^{ikx}\right)\,,$$

guess the commutation relations which should be obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. Verify that your guess leads to the correct canonical commutators for ϕ and $\phi^\dagger$.

The nonzero equal-time commutators should be

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,, \qquad [\phi^\dagger(t,\vec{x}), \pi^\dagger(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,.$$

The mode expansion implies that

$$\pi(x) = \int \widetilde{\mathrm{d}\ell}\; i\ell^0\left(a^\dagger(\vec{\ell})e^{i\ell x} - b(\vec{\ell})e^{-i\ell x}\right)$$

and hence we should have, from the first commutation relation,

$$
\begin{aligned}
i\delta^{(3)}(\vec{x}-\vec{y}) &= \int \widetilde{\mathrm{d}k} \int \widetilde{\mathrm{d}\ell}\, i\ell^0 \left[a(\vec{k})e^{-ik^0t+i\vec{k}\cdot\vec{x}} + b^\dagger(\vec{k})e^{ik^0t-i\vec{k}\cdot\vec{x}}, a^\dagger(\vec{\ell})e^{i\ell^0t-i\vec{\ell}\cdot\vec{y}} - b(\vec{\ell})e^{-i\ell^0t+i\vec{\ell}\cdot\vec{y}}\right] \\
&= \int \widetilde{\mathrm{d}k} \int \widetilde{\mathrm{d}\ell}\, i\ell^0 \Bigg(e^{i(\ell^0-k^0)t-i(\vec{\ell}\cdot\vec{y}-\vec{k}\cdot\vec{x})}[a(\vec{k}),a^\dagger(\vec{\ell})] + e^{-i(\ell^0-k^0)t+i(\vec{\ell}\cdot\vec{y}-\vec{k}\cdot\vec{x})}[b(\vec{\ell}),b^\dagger(\vec{k})] \\
&\qquad + e^{i(\ell^0+k^0)t-i(\vec{\ell}\cdot\vec{y}+\vec{k}\cdot\vec{x})}[b^\dagger(\vec{k}),a^\dagger(\vec{\ell})] - e^{-i(\ell^0+k^0)t+i(\vec{\ell}\cdot\vec{y}+\vec{k}\cdot\vec{x})}[a(\vec{k}),b(\vec{\ell})] \Bigg) .
\end{aligned}
$$

With the ansatz

$$
[a(\vec{k}),a^\dagger(\vec{\ell})] = 2\,k^0\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{\ell}) = [b(\vec{k}),b^\dagger(\vec{\ell})]\,,
$$

$$
[a(\vec{k}),b(\vec{\ell})] = 0 = [a^\dagger(\vec{k}),b^\dagger(\vec{\ell})]
$$

(which can be guessed by analogy with the real scalar field case) the desired relation is easily seen to be satisfied:

$$
\begin{aligned}
[\phi(t,\vec{x}),\pi(t,\vec{y})] &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\,k^0} \int \frac{\mathrm{d}^3\ell}{(2\pi)^3}\frac{1}{2\,\ell^0} i\ell^0 \Bigg(e^{i(\ell^0-k^0)t-i(\vec{\ell}\cdot\vec{y}-\vec{k}\cdot\vec{x})} 2\,k^0\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{\ell}) \\
&\qquad + e^{-i(\ell^0-k^0)t+i(\vec{\ell}\cdot\vec{y}-\vec{k}\cdot\vec{x})} 2\,k^0\,(2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{\ell}) \Bigg) \\
&= i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\left(\frac{1}{2}e^{-i\vec{k}\cdot(\vec{y}-\vec{x})} + \frac{1}{2}e^{i\vec{k}\cdot(\vec{y}-\vec{x})}\right) = i\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
$$

The other commutators can be checked in a similar way.

3. *Express the Hamiltonian in terms of $a(\vec{k})$, $b(\vec{k})$, and their conjugates.*

The Hamiltonian density is obtained by a Legendre transform from the Lagrangian density:

$$
\mathcal{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L} = |\pi|^2 + |\vec{\nabla}\phi|^2 + m^2\,|\phi|^2\,.
$$

The Hamiltonian is therefore

$$
\begin{aligned}
H = \int \mathrm{d}^3x\,\mathcal{H} &= \int \mathrm{d}^3x \int \widetilde{\mathrm{d}k} \int \widetilde{\mathrm{d}\ell} \Bigg(k^0\ell^0 \left(a(\vec{k})e^{-ikx} - b^\dagger(\vec{k})e^{ikx}\right)\left(a^\dagger(\vec{\ell})e^{i\ell x} - b(\vec{\ell})e^{-i\ell x}\right) \\
&\qquad + \vec{k}\cdot\vec{\ell}\left(a^\dagger(\vec{k})e^{ikx} - b(\vec{k})e^{-ikx}\right)\left(a(\vec{\ell})e^{-i\ell x} - b^\dagger(\vec{\ell})e^{i\ell x}\right) \\
&\qquad + m^2\left(a^\dagger(\vec{k})e^{ikx} + b(\vec{k})e^{-ikx}\right)\left(a(\vec{\ell})e^{-i\ell x} + b^\dagger(\vec{\ell})e^{i\ell x}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \int \mathrm{d}^3x \int \widetilde{\mathrm{d}k} \int \widetilde{\mathrm{d}\ell} \\
&\Bigg(k^0\ell^0 \left(a(\vec{k})a^\dagger(\vec{\ell})e^{-i(k-\ell)x} + b^\dagger(\vec{k})b(\vec{\ell})e^{i(k-\ell)x} - a(\vec{k})b(\vec{\ell})e^{-i(k+\ell)x} - b^\dagger(\vec{k})a^\dagger(\vec{\ell})e^{i(k+\ell)x}\right) \\
&+ \vec{k}\cdot\vec{\ell}\left(a^\dagger(\vec{k})a(\vec{\ell})e^{i(k-\ell)x} + b(\vec{k})b^\dagger(\vec{\ell})e^{-i(k-\ell)x} - b(\vec{k})a(\vec{\ell})e^{-i(k+\ell)x} - a^\dagger(\vec{k})b^\dagger(\vec{\ell})e^{i(k+\ell)x}\right) \\
&+ m^2\left(a^\dagger(\vec{k})a(\vec{\ell})e^{i(k-\ell)x} + b^\dagger(\vec{k})b(\vec{\ell})e^{-i(k-\ell)x} + b(\vec{k})a(\vec{\ell})e^{-i(k+\ell)x} + a^\dagger(\vec{k})b^\dagger(\vec{\ell})e^{i(k+\ell)x}\right)\Bigg)
\end{aligned}
$$

We have

$$\int \mathrm{d}^3x \; e^{i(k\pm\ell)x} = e^{i(k^0\pm\ell^0)t}\int \mathrm{d}^3x \; e^{-i(\vec{k}\pm\vec{\ell})\cdot\vec{x}} = e^{i(k^0\pm k^0)t}(2\pi)^3\delta^{(3)}(\vec{k}\pm\vec{\ell})$$

$$\int \mathrm{d}^3x \; e^{-i(k\pm\ell)x} = e^{-i(k^0\pm\ell^0)t}\int \mathrm{d}^3x \; e^{i(\vec{k}\pm\vec{\ell})\cdot\vec{x}} = e^{-i(k^0\pm k^0)t}(2\pi)^3\delta^{(3)}(\vec{k}\pm\vec{\ell})$$

which allows to get rid of one of the momentum integrals. Setting $k^0 \equiv \omega$ (which is also equal to ℓ^0 under both $\delta^{(3)}(\vec{k}-\vec{\ell})$ and $\delta^{(3)}(\vec{k}+\vec{\ell})$):

$$\begin{aligned} H = \int \widetilde{\mathrm{d}k}\; \frac{1}{2\omega}\Bigg(&\omega^2\left(a(\vec{k})a^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k}) - a(\vec{k})b(-\vec{k})e^{-2i\omega t} - b^\dagger(\vec{k})a^\dagger(-\vec{k})e^{2i\omega t}\right) \\ &+ |\vec{k}|^2\left(a^\dagger(\vec{k})a(\vec{k}) + b(\vec{k})b^\dagger(\vec{k}) + b(\vec{k})a(-\vec{k})e^{-2i\omega t} + a^\dagger(\vec{k})b^\dagger(-\vec{k})e^{2i\omega t}\right) \\ &+ m^2\left(a^\dagger(\vec{k})a(\vec{k}) + b(\vec{k})b^\dagger(\vec{k}) + b(\vec{k})a(-\vec{k})e^{-2i\omega t} + a^\dagger(\vec{k})b^\dagger(-\vec{k})e^{2i\omega t}\right)\Bigg) . \end{aligned}$$

Finally we use that $|\vec{k}|^2 + m^2 = \omega^2$, that $[a,b] = 0 = [a^\dagger, b^\dagger]$, and we change the integration variable from $\vec{k}$ to $-\vec{k}$ for the last two terms in the first line. This makes all terms proportional to the exponentials cancel. We are left with

$$H = \frac{1}{2}\int \widetilde{\mathrm{d}k}\;\omega\left(a(\vec{k})a^\dagger(\vec{k}) + b^\dagger(\vec{k})b(\vec{k}) + a^\dagger(\vec{k})a(\vec{k}) + b(\vec{k})b^\dagger(\vec{k})\right) .$$

This expression can be normal ordered, at the expense of introducing a divergent zero-point energy E_0:

$$H = \int \widetilde{\mathrm{d}k}\;\omega\left(a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})b(\vec{k})\right) + E_0 .$$