QUANTUM FIELD THEORY, PROBLEM SHEET 2

Problem 1: Annihilation and creation operators for the free real scalar field

Let $\phi(x) = \phi(t, \vec{x})$ be a free real scalar field which obeys the Klein-Gordon equation. We set

$$a(t, \vec{k}) = i \int d^3x \, e^{i\omega_{\vec{k}} \, t - i\vec{k} \cdot \vec{x}} \, \overleftrightarrow{\partial_t} \, \phi(t, \vec{x})$$
where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$, $f \overleftrightarrow{\partial} g \equiv f \partial g - (\partial f) g$.

- 1. Show that $\dot{a}(t, \vec{k}) = 0$. The t dependence in $a(t, \vec{k})$ is therefore spurious; from now on we write simply $a(\vec{k})$.
- 2. Show that the $a(\vec{k})$ defined in this way are in fact the Fourier coefficients which appear in the decomposition

$$\phi(x) = \int \widetilde{dk} \left(a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx} \right) \Big|_{k^0 = \omega_{\vec{k}}}.$$

3. Using the canonical equal-time commutation relations

$$[\phi(t,\vec{x}),\pi(t,\vec{x}')] = i\delta^{(3)}(\vec{x}-\vec{x}') , \qquad [\phi(t,\vec{x}),\phi(t,\vec{x}')] = 0 = [\pi(t,\vec{x}),\pi(t,\vec{x}')] ,$$
 compute $\left[a(\vec{k}),a^{\dagger}(\vec{k}')\right]$. (If you are motivated, compute also $\left[a(\vec{k}),a(\vec{k}')\right]$.)

Problem 2: Canonical quantisation of the free complex scalar field

Consider again a free complex scalar field $\phi(x)$, whose Lagrangian density is

$$\mathcal{L} = \partial_{\mu} \phi^* \partial^{\mu} \phi - m^2 |\phi|^2.$$

- 1. Show that the action is invariant under a phase rotation $\phi(x) \to e^{i\alpha}\phi(x)$, where α is a real constant. Calculate the associated conserved Noether current.
- 2. Find the canonical momenta conjugate to ϕ and ϕ^* .
- 3. Now promoting ϕ and ϕ^* to operators, imposing canonical equal-time commutation relations for the fields and their conjugate momenta, and writing the mode expansion of ϕ as

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\vec{k}) e^{-ikx} + b^{\dagger}(\vec{k}) e^{ikx} \right) \Big|_{k^0 = \omega_{\vec{r}}},$$

guess the commutation relations that should be obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. Verify that your guess leads to the correct canonical commutators for ϕ and ϕ^{\dagger} .

4. Express the Hamiltonian in terms of $a(\vec{k})$, $b(\vec{k})$, and their conjugates.