Quantum Field Theory, problem sheet 2

Solutions to be discussed on 18/09/2024

Problem 1: Annihilation and creation operators for the free real scalar field

Let $\phi(x) = \phi(t, \vec{x})$ be a free real scalar field which obeys the Klein-Gordon equation. We set

$$a(t, \vec{k}) = i \int \mathrm{d}^3 x \, e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} \overleftrightarrow{\partial_t} \phi(t, \vec{x})$$
$$\text{where } \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}\,, \; f \overleftrightarrow{\partial} g \equiv f \partial g - (\partial f) g\,.$$

1. Show that $\dot{a}(t, \vec{k}) = 0$. The t dependence in $a(t, \vec{k})$ is therefore spurious; from now on we write simply $a(\vec{k})$.

2. Show that the $a(\vec{k})$ defined in this way are in fact the Fourier coefficients which appear in the decomposition

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right) \Big|_{k^0 = \omega_{\vec{k}}} \,.$$

3. Using the canonical equal-time commutation relations

$$[\phi(t, \vec{x}), \pi(t, \vec{x}\,')] = i\delta^{(3)}(\vec{x} - \vec{x}\,')\,, \qquad [\phi(t, \vec{x}), \phi(t, \vec{x}\,')] = 0 = [\pi(t, \vec{x}), \pi(t, \vec{x}\,')]\,,$$

compute $\left[a(\vec{k}), a^\dagger(\vec{k}\,')\right]$. (If you are motivated, compute also $\left[a(\vec{k}), a(\vec{k}\,')\right]$.)

Problem 2: Canonical quantisation of the free complex scalar field

Consider again a free complex scalar field $\phi(x)$, whose Lagrangian density is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2\,.$$

1. Find the canonical momenta conjugate to ϕ and ϕ^*.

2. Now promoting ϕ and ϕ^* to operators, imposing canonical equal-time commutation relations for the fields and their conjugate momenta, and writing the mode expansion of ϕ as

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx} \right) \Big|_{k^0 = \omega_{\vec{k}}}\,,$$

guess the commutation relations that should be obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. Verify that your guess leads to the correct canonical commutators for ϕ and $\phi^\dagger$.

3. Express the Hamiltonian in terms of $a(\vec{k})$, $b(\vec{k})$, and their conjugates.