Quantum Field Theory, solutions to problem sheet 13

Problem 1: Furry's theorem

1. *Charge conjugation* is a linear operation C exchanging particles with antiparticles without involving any of their space-time transformation properties (such as spin or parity). For the Dirac field, $\psi \to C\overline{\psi}^T$ with $C = i\gamma^0\gamma^2$ in the Weyl basis. For the photon, $A_\mu \to -A_\mu$. Show that the QED Lagrangian is invariant under charge conjugation.

2. How does the electromagnetic current j_μ transform under charge conjugation? Show that all correlation functions for an odd number of currents are zero,

$$\langle 0|\mathrm{T}\, j_{\mu_1}(x_1)\ldots j_{\mu_{2n+1}}(x_{2n+1})|0\rangle = 0$$

and that, consequently, all $(2n+1)$-photon amplitudes are zero. This is *Furry's theorem.*

Solution: 1. We first collect some useful relations which are valid in the Weyl basis of γ matrices:

$$(\gamma^0)^T = \gamma^0\,, \quad (\gamma^1)^T = -\gamma^1\,, \quad (\gamma^2)^T = \gamma^2\,, \quad (\gamma^3)^T = -\gamma^3\,. \tag{1}$$

Furthermore, we explicitly write out the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$:

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad (\mu \neq \nu)\,, \qquad (\gamma^0)^2 = \mathbb{1}\,, \qquad (\gamma^i)^2 = -\mathbb{1} \quad (i = 1, 2, 3)\,. \tag{2}$$

Finally, explicit calculation shows that

$$C^\dagger = C^T = C^{-1} = -C\,. \tag{3}$$

From the transformation $\psi \to C\overline{\psi}^T$, one deduces that $\overline{\psi} \to \psi^T C$. Therefore,

$$\overline{\psi}\psi \to \psi^{\mathrm{T}} C C \overline{\psi}^T = -\psi^T\overline{\psi}^T = \overline{\psi}\psi\,.$$

(the minus sign in the last equality is from anticommuting the spinors) and so the electron mass term $m\overline{\psi}\psi$ is invariant.
Moreover, we have

$$\overline{\psi}\gamma^\mu\psi = \psi^T C\gamma^\mu C\overline{\psi}^T\,.$$

Using the relations (1) and (2), it is easy to show (for each $\mu = 0, 1, 2, 3$ separately) that

$$C\gamma^\mu C = (\gamma^\mu)^T\,.$$

Therefore,

$$\overline{\psi}\gamma^\mu\psi \to \psi^T(\gamma^\mu)^T\overline{\psi}^T = -\overline{\psi}\gamma^\mu\psi\,,$$

so the electron-photon coupling $eA_\mu\overline{\psi}\gamma^\mu\psi$ is invariant *if A_μ is taken to transform as* $A_\mu \to -A_\mu$ (note that this leaves the photon kinetic term invariant because it is quadratic in A_μ).

Finally, ∂_μ does not change under charge conjugation, so the kinetic term for the electron transforms as

$$\overline{\psi}\gamma^\mu\partial_\mu\psi \;\to\; -(\partial_\mu\overline{\psi})\gamma^\mu\psi = \overline{\psi}\gamma^\mu\partial_\mu\psi + \;(\text{total derivative})\,.$$

2. We showed already that $j^\mu \to -j^\mu$. Therefore,

$$\langle 0|\mathrm{T}\, j^{\mu_1}(x_1)\ldots j^{\mu_{2n+1}}(x_{2n+1})|0\rangle \;\to\; -\langle 0|\mathrm{T}\, j^{\mu_1}(x_1)\ldots j^{\mu_{2n+1}}(x_{2n+1})|0\rangle\,.$$

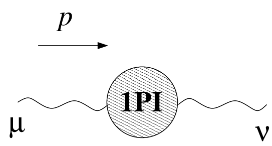
Since charge conjugation is a symmetry of QED, the correlation functions should not change under charge conjugation, hence

$$\langle 0|\mathrm{T}\, j^{\mu_1}(x_1)\ldots j^{\mu_{2n+1}}(x_{2n+1})|0\rangle = 0\,.$$

The k-photon amplitude is proportional to the k-current amplitude since each external photon attaches to a current. Therefore, for k odd also the k-photon amplitude is zero.

Problem 2: Photon self-energy

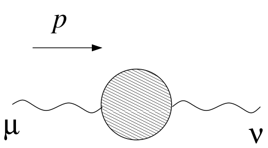
Show that, if $i\Pi^{\mu\nu}(p)$ is defined to be the sum of all 1-particle irreducible insertions into the photon propagator,

$i\Pi^{\mu\nu}(p) =$ 

then $\Pi^{\mu\nu}(p)$ can be written as

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\,\Pi(p^2)$$

for a suitable function $\Pi(p^2)$. Thus, show that the full photon propagator takes the form



$$= \frac{-i}{p^2(1-\Pi(p^2))}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) - i\xi\frac{p^\mu p^\nu}{(p^2)^2}\,.$$

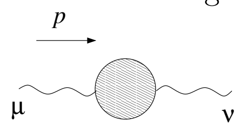
Conclude that the photon mass remains zero to all orders in perturbation theory.

Solution: The only Lorentz-covariant two-index object that can be formed from p is of the form $A\,p^\mu p^\nu + Bg^{\mu\nu}$, where A and B are scalars (functions of p^2 only). By the Ward identity, $p_\mu\Pi^{\mu\nu} = 0$, which fixes $B = -A\,p^2$. We can therefore write

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\,\Pi(p^2) = \Delta^{\mu\nu}\,p^2\Pi(p^2)$$

where we have defined $\Delta^{\mu\nu} = g^{\mu\nu} - p^\mu p^\nu/p^2$. (In principle, for a complete proof one should also show that the function $\Pi(p^2)$ is nonsingular.)

Using this notation, we can write down the series obtained from successive insertions of 1PI diagrams into the tree-level propagator to obtain the exact propagator:



$$
\begin{aligned}
&= \frac{-i}{p^2}\left(\Delta^{\mu\nu} + \xi\frac{p^\mu p^\nu}{p^2}\right) \\
&\quad + \frac{-i}{p^2}\left(\Delta^{\mu\lambda} + \xi\frac{p^\mu p^\lambda}{p^2}\right)\left(i\Delta_{\lambda\kappa}p^2\Pi(p^2)\right)\frac{-i}{p^2}\left(\Delta^{\kappa\nu} + \xi\frac{p^\kappa p^\nu}{p^2}\right) \\
&\qquad + \frac{-i}{p^2}\left(\Delta^{\mu\lambda} + \xi\frac{p^\mu p^\lambda}{p^2}\right)\left(i\Pi(p^2)p^2\Delta_{\lambda\kappa}\right)\frac{-i}{p^2}\left(\Delta^{\kappa\rho} + \xi\frac{p^\kappa p^\rho}{p^2}\right)\left(i\Pi(p^2)p^2\Delta_{\rho\sigma}\right)\frac{-i}{p^2}\left(\Delta^{\sigma\nu} + \xi\frac{p^\sigma p^\nu}{p^2}\right) \\
&\qquad\quad + \dots \\
&= \frac{-i}{p^2}\left(\Delta^{\mu\lambda} + \xi\frac{p^\mu p^\lambda}{p^2}\right)\left(\sum_{n=0}^{\infty}\left(\Pi(p^2)\Delta\right)^n\right)^{\nu}_{\lambda} \\
&= \frac{-i}{p^2(1-\Pi(p^2))}\Delta^{\mu\nu} - i\xi\frac{p^\mu p^\nu}{(p^2)^2}\,.
\end{aligned}
$$

Here we have repeatedly used $\Delta^{\mu\nu}\Delta_\nu{}^\lambda = \Delta^{\mu\lambda}$ (which is easy to show by direct calculation) and $\Delta^{\mu\nu}p_\nu = 0$, and summed the geometric series in the last step. According to the Ward identity the terms proportional to $p_\mu p_\nu$ cannot contribute to matrix elements (they must in fact be unphysical since they depend on the gauge parameter ξ). The pole is at $p^2 = 0$, so the photon remains massless.