Quantum Field Theory, problem sheet 12

Solutions to be discussed on 17/01/2024.

Problem 1: Schwinger-Dyson equations and Ward-Takahashi identities

In this exercise you will study to what extent the classical equations of motion hold as operator equations in quantum theory, and investigate the quantum equivalent of the conservation of the Noether current associated to a continuous symmetry.

1. As warm-up in finitely many dimensions, show that

$$\int \mathrm{d}^d x \, e^{-s(x)-j\cdot x} \left(\nabla s(x) + j\right) \cdot z = 0$$

   where $s(x)$ is a real function (assumed to tend to infinity at $|x| \to \infty$ sufficiently quickly for the integral to converge), and $j, z \in \mathbb{R}^d$ are constant vectors. *Hint*: Consider the change of variables $x \to x + z$ in $\int \mathrm{d}^d x \, e^{-s(x)-j\cdot x}$.

Now let ϕ be a real scalar field with Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mathcal{V}(\phi)$, and consider a continuous transformation $\phi(x) \to \phi'(x) = \phi(x) + \alpha\,\zeta(x) + \mathcal{O}(|\alpha|^2)$ which leaves the path integral measure invariant: $\mathcal{D}\phi = \mathcal{D}\phi'$.

2. Derive the functional equivalent of the identity you proved in 1. by replacing $\phi \to \phi'$ in the generating functional $Z[J]$:

$$\int \mathcal{D}\phi \int \mathrm{d}^4 y \left(\frac{\delta S}{\delta\phi(y)} + J(y)\right) \zeta(y)\, e^{i(S[\phi]+\int J\phi)} = 0\,.$$

3. By setting $\zeta(y) = \delta^{(4)}(y - x)$ for some fixed x, conclude that

$$\begin{aligned}\Box_x \langle 0|\, \mathrm{T}\, \phi(x)\phi(x_1)\ldots\phi(x_n)|0\rangle + \langle 0|\, \mathrm{T}\, \mathcal{V}'(\phi(x))\, \phi(x_1)\ldots\phi(x_n)|0\rangle \\ = i\sum_{k=1}^{n} \langle 0|\, \mathrm{T}\, \phi(x_1)\ldots\phi(x_{k-1})\delta^{(4)}(x - x_k)\phi(x_{k+1})\ldots\phi(x_n)|0\rangle\,.\end{aligned}$$

These are the *Schwinger-Dyson equations*: Correlation functions of $\phi(x)$ in the quantum theory satisfy the classical equations of motion, up to the *contact terms* on the right-hand side, which are nonzero only when some of the space-time points in the correlation function coincide.

4. Write down the Schwinger-Dyson equation explicitly for the case of $n = 1$ and a free field. Does this look familiar?

5. Using the LSZ reduction formula, convince yourself that contact terms can never contribute to the invariant matrix elements $\mathcal{M}_{\mathrm{fi}}$.

We now assume that the theory possesses a continuous symmetry under which $\phi(x) \to \phi(x) + \alpha\, \delta\phi(x) + \mathcal{O}(|\alpha|^2)$, $\mathcal{D}\phi \to \mathcal{D}\phi$, $S \to S$. Recall that the corresponding Noether current j^μ satisfies

$$\partial_\mu j^\mu(x) = -\frac{\delta S}{\delta\phi(x)}\, \delta\phi(x)\,,$$

where the right-hand side is classically zero by the equations of motion.

6. By setting $\zeta(y) = \delta\phi(x)\, \delta^{(4)}(x-y)$ in the result of 2., show that

$$\int \mathcal{D}\phi\, \big(\partial_\mu j^\mu(x) - J(x)\, \delta\phi(x)\big)\, e^{i(S[\phi]+\int J\phi)} = 0$$

and deduce that

$$\begin{aligned}
&\frac{\partial}{\partial x^\mu}\langle 0|\,\mathrm{T}\; j^\mu(x)\phi(x_1)\ldots\phi(x_n)|0\rangle \\
&\qquad = i\sum_{k=1}^{n}\langle 0|\,\mathrm{T}\;\phi(x_1)\ldots\phi(x_{k-1})\;\delta\phi(x_k)\;\delta^{(4)}(x-x_k)\;\phi(x_{k+1})\ldots\phi(x_n)|0\rangle\,.
\end{aligned}$$

These are the *Ward-Takahashi identities*: Classical current conservation $\partial_\mu j^\mu = 0$ is valid as an operator equation up to the contact terms on the right-hand side.

Going beyond real scalar fields, by the same reasoning one can derive the Ward-Takahashi identities associated to the U(1) gauge symmetry of QED. The symmetry transformation on the fermionic fields is $\psi \to (1 + ie\alpha)\psi + \mathcal{O}(|\alpha|^2)$, the Noether current is $j^\mu = e\overline{\psi}\gamma^\mu\psi$, and the Ward-Takahashi identities read

$$\begin{aligned}
&\partial_\mu\langle 0|\mathrm{T}\; j^\mu(x)\psi(x_1)\ldots\psi(x_n)\overline{\psi}(x_1')\ldots\overline{\psi}(x_n')|0\rangle \\
&\qquad = -e\sum_{k=1}^{n}\delta^{(4)}(x-x_k)\langle 0|\,\mathrm{T}\;\psi(x_1)\ldots\psi(x_n)\overline{\psi}(x_1')\ldots\overline{\psi}(x_n')|0\rangle \\
&\qquad\quad + e\sum_{k=1}^{n}\delta^{(4)}(x-x_k')\langle 0|\,\mathrm{T}\;\psi(x_1)\ldots\psi(x_n)\overline{\psi}(x_1')\ldots\overline{\psi}(x_n')|0\rangle\,.
\end{aligned}$$

(They remain valid when inserting photon fields in the correlation functions.) Consider now a QED scattering amplitude involving some incoming and outgoing fermions as well as external photons, one of which has four-momentum k and polarisation $\varepsilon^\mu(k)$. In Lorenz gauge the LSZ formula reads

$$\langle f|i\rangle = i\varepsilon^\mu(k)\int \mathrm{d}^4x\; e^{-ikx}\Box_x\ldots\langle 0|\mathrm{T}\; A_\mu(x)\ldots|0\rangle$$

where the ellipses stand for terms pertaining to other photons and to fermion fields.

7. Use the Schwinger-Dyson equation for $A_\mu(x)$ and the result of part 5. to show that

$$\langle f|i\rangle = -i\varepsilon^\mu(k)\int \mathrm{d}^4x\; e^{-ikx}\ldots\langle 0|\mathrm{T}\; j_\mu(x)\ldots|0\rangle\,.$$

8. Use the Ward-Takahashi identities to prove the *Ward identity* of QED: If $\mathcal{M}_{\mathrm{fi}}$ is the corresponding invariant matrix element and $\mathcal{M}_\mu(k)$ is defined by $\mathcal{M}_{\mathrm{fi}} = \varepsilon^\mu(k)\mathcal{M}_\mu(k)$, then

$$k^\mu\mathcal{M}_\mu(k) = 0\,.$$

Hence, replacing a photon polarisation vector in a QED matrix element by the corresponding momentum gives zero. The Ward identity is very important for proving the renormalisability of QED, and can also be used to simplify calculations of QED scattering cross sections in perturbation theory. It can be regarded as a quantum manifestation of gauge invariance.