# Quantum Field Theory, problem sheet 11

Solutions to be discussed on 20/12/2023.

## Problem 1: The Grassmann algebra

1. Find the vector space dimension of the Grassmann algebra generated by $\mathbb{1}$ and $\{\theta_i\}_{i=1\ldots n}$.

2. Let $U = (U_{ij})$ be a unitary matrix of commuting numbers ($i, j = 1 \ldots n$). Show that the Grassmann integral

$$\int \mathrm{d}^n\theta^* \, \mathrm{d}^n\theta \, f(\theta_i, \theta_i^*)$$

   is invariant under the change of variables $\theta_i \to U_{ij}\theta_j$, $\theta_i^* \to U_{ij}^*\theta_j^*$.

3. Let $A$ be a hermitian $n \times n$ matrix of commuting numbers. Show that

$$\int \mathrm{d}^n\theta^* \, \mathrm{d}^n\theta \, e^{-\theta^\dagger A \, \theta} = \det A \, .$$

## Problem 2: Yukawa theory

Consider the Yukawa Lagrangian

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - M)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - y\,\phi\overline{\psi}\psi,$$

where $\psi$ is a Dirac field, $\phi$ is a real scalar field, and $y$ is a coupling.

1. Compute the matrix element $\mathcal{M}_{\mathrm{fi}}$ for $\overline{\psi}\psi \to \phi\phi$ scattering to lowest order in perturbation theory.

   (If you are motivated: Use this result to further calculate the total cross section as a function of the Mandelstam variables $s$, $t$, and $u$, assuming that the incoming particles are unpolarized — but be warned, this calculation is long and tedious, and best done with the help of a computer algebra system.)

2. Draw the Feynman diagrams contributing to all the 1-point, 2-point, 3-point, and 4-point functions in this theory at the one-loop level. Which of them do you expect to diverge? (There is no need to evaluate them in detail.) Thus, what other terms should we have included in the above Lagrangian and why?