Quantum Field Theory, problem sheet 10

Solutions to be discussed on 13/12/2023.

Problem 1: The Clifford algebra

1. Given a set of four matrices γ^μ which satisfy the Clifford algebra
$$\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}\,,$$
show that the matrices $\gamma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the Lorentz algebra:
$$[\gamma^{\kappa\lambda}, \gamma^{\rho\sigma}] = i\left(g^{\lambda\rho}\gamma^{\kappa\sigma} - g^{\kappa\rho}\gamma^{\lambda\sigma} - g^{\lambda\sigma}\gamma^{\kappa\rho} + g^{\kappa\sigma}\gamma^{\lambda\rho}\right)\,.$$

2. Verify that the Clifford algebra is satisfied by both the *Weyl representation* of γ matrices
$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}.$$
and the *Dirac-Pauli representation*
$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$
and find the unitary transformation that takes one into the other.

3. Defining $\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$, calculate
$$\{\gamma^5, \gamma^\mu\} \qquad \text{and} \qquad [\gamma^5, \gamma^{\mu\nu}]\,.$$

Problem 2: The Dirac field

1. Show that
$$\left(\mathbb{1} + \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right)\gamma^\mu\left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}\gamma^{\rho\sigma}\right) = \left(\mathbb{1} - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu + \mathcal{O}(||\omega||^2)\,,$$
where the $M^{\rho\sigma}$ generate the vector representation of $\mathfrak{so}(1,3)$,
$$(M^{\kappa\lambda})_{\mu\nu} = i\left(\delta^\kappa{}_\mu\delta^\lambda{}_\nu - \delta^\kappa{}_\nu\delta^\lambda{}_\mu\right)\,.$$
Use this result to conclude that the Dirac Lagrangian
$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$$
is invariant under proper orthochronous Lorentz transformations.

2. Find the Euler-Lagrange equations obtained from the Dirac Lagrangian.

Problem 3: Hamiltonian of the Dirac field

1. Let $p = (p^0, \vec{p})$ be a momentum 4-vector, $p^2 = m^2$, and let $\bar{p} = (p^0, -\vec{p})$ be the same 4-vector with the sign of the spatial components reversed. Prove that

$$\gamma^0 \not{p} + \not{p} \gamma^0 = 2\, p^0 \,, \qquad \gamma^0 \not{p} - \not{\bar{p}} \gamma^0 = 0 \,.$$

2. Prove that

$$\begin{aligned} \bar{u}_s(\vec{p})\gamma^0 u_r(\vec{p}) &= 2\, p^0\, \delta_{sr} \,, \quad \bar{v}_s(\vec{p})\gamma^0 v_r(\vec{p}) = 2\, p^0\, \delta_{sr} \,, \\ \bar{u}_s(-\vec{p})\gamma^0 v_r(\vec{p}) &= 0 \,, \quad \bar{v}_s(-\vec{p})\gamma^0 u_r(\vec{p}) = 0 \,. \end{aligned}$$

   Hint: Use the result of 1., remembering that $u_s(\vec{p})$ and $v_s(\vec{p})$ satisfy

$$\begin{aligned} (\not{p} - m) u_s(\vec{p}) &= 0 \,, & (\not{p} + m) v_s(\vec{p}) &= 0 \,, \\ \bar{u}_s(\vec{p})(\not{p} - m) &= 0 \,, & \bar{v}_s(\vec{p})(\not{p} + m) &= 0 \,, \\ \bar{u}_s(\vec{p}) u_r(\vec{p}) &= 2m\, \delta_{rs} \,, & \bar{v}_s(\vec{p}) v_r(\vec{p}) &= -2m\, \delta_{rs} \,. \end{aligned}$$

3. Starting from the Fourier mode expansion of a free Dirac field

$$\begin{aligned} \psi(x) &= \sum_{s=+,-} \int \widetilde{\mathrm{d}p} \left(a_s(\vec{p}) u_s(\vec{p}) e^{-ipx} + b_s^\dagger(\vec{p}) v_s(\vec{p}) e^{ipx} \right) \,, \\ \overline{\psi}(x) &= \sum_{s=+,-} \int \widetilde{\mathrm{d}p} \left(b_s(\vec{p}) \bar{v}_s(\vec{p}) e^{-ipx} + a_s^\dagger(\vec{p}) \bar{u}_s(\vec{p}) e^{ipx} \right) \,, \end{aligned}$$

   and using the relations derived in 2., show that the Dirac Hamiltonian $H = \int \mathrm{d}^3x \, ((\partial \mathcal{L}/\partial \dot{\psi})\dot{\psi} - \mathcal{L})$ can be written as

$$H = \sum_s \int \widetilde{\mathrm{d}p}\, \omega_{\vec{p}} \left(a_s^\dagger(\vec{p}) a_s(\vec{p}) - b_s(\vec{p}) b_s^\dagger(\vec{p}) \right) \,.$$

It follows that the Hamiltonian can be made positive definite by imposing canonical anticommutation relations (rather than commutation relations).