Quantum Field Theory, problem sheet 8

Solutions to be discussed on 29/11/2023.

Problem 1: Feynman parameters

Prove the following identities ($A \neq 0$ and $B \neq 0$ are real constants):

1.
$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{(xA + (1-x)B)^2}$$

2.
$$\frac{1}{A^n B} = \int_0^1 \mathrm{d}x \, \frac{n x^{n-1}}{(xA + (1-x)B)^{n+1}}$$

3.
$$\frac{1}{A_1 \cdots A_n} = \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \, \delta(1 - x_1 - \ldots - x_n) \, \frac{(n-1)!}{(x_1 A_1 + \ldots + x_n A_n)^n}$$

Problem 2: The path integral and the semiclassical limit

We reinstate $\hbar$ for this exercise and work with the Wick-rotated Euclidean generating functional

$$Z_E[J] = N \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S_E[\phi, J]}$$

where N is a normalisation constant, S_E is the Euclidean action

$$S_E[\phi, J] = \int \mathrm{d}^4x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V_{\text{int}}(\phi) - J\phi \right) ,$$

and the path integral measure is normalised as $\mathcal{D}\phi = \prod_i \frac{\mathrm{d}\phi_i}{\sqrt{2\pi\hbar}}$.

1. State the classical equation of motion for ϕ in the presence of a source J.
2. Let $f : \mathbb{R} \to \mathbb{R}_+$ with a minimum at x_0. Assume that $f(x)$ increases sufficiently steeply at $x \to \pm\infty$. Demonstrate the *saddle point approximation*:
$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-f(x)} \approx e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}} .$$
3. Let $f : \mathbb{R}^n \to \mathbb{R}_+$ with a minimum at x_0. Assume that $f(x)$ increases sufficiently steeply at $|x| \to \infty$. Use the result of Ex. 4.2.3 to show that
$$\int \mathrm{d}^n x \, e^{-f(x)} \approx e^{-f(x_0)} \sqrt{\frac{(2\pi)^n}{\det H_f(x_0)}}$$
where H_f is the Hessian matrix of f, $(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

4. Apply this approximation to the generating functional $Z_E[J]$. By boldly generalising your previous result to functional calculus in infinitely many dimensions, prove the formal identity

$$Z_E[J] = N \exp\left(-\frac{1}{\hbar}\left(S_E[\phi_0, J] + \frac{\hbar}{2}\operatorname{Tr}\log\left(-\Box_E + m^2 + V''_{\text{int}}(\phi_0)\right) + \mathcal{O}(\hbar^2)\right)\right)$$

where ϕ_0 obeys the classical equation of motion (and therefore implicitly depends on J), $\Box_E \equiv \partial_t^2 + \nabla^2$, and the trace Tr of an operator with a continous spectrum is defined to be the integral over its eigenvalues (which is generally divergent and requires some regularisation, hence "formal identity").

We now specialise to the case $V_{\text{int}}(\phi) = \frac{\lambda}{24}\phi^4$.

5. Expanding ϕ_0 in powers of λ,

$$\phi_0 = \phi^{(0)} + \lambda\phi^{(1)} + \mathcal{O}(\lambda^2)\,,$$

show that

$$\phi^{(0)}(x) = \int \mathrm{d}^4y\; D(x-y)J(y)$$

where $D(x-y)$ is a Green function of the Wick-rotated Klein-Gordon operator, $(-\Box_E + m^2)D(x-y) = \delta^4(x-y)$.

6. Use this result to show that

$$\phi^{(1)}(x) = -\frac{1}{6}\int \mathrm{d}^4y\,\mathrm{d}^4u\,\mathrm{d}^4v\,\mathrm{d}^4w\; D(x-y)D(y-u)D(y-v)D(y-w)\; J(u)J(v)J(w)\,.$$

7. Show that

$$S_E[\phi_0, J] = \int \mathrm{d}^4x\,\left(-\frac{\lambda}{24}\phi_0^4 - \frac{1}{2}J\phi_0\right)\,.$$

Thus, using the results of 5. and 6., calculate the connected part of the four-point function $\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_c$ to leading order in $\hbar$. Finally, calculate its Fourier transform

$$G_E(p_1,p_2,p_3,p_4) = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\; e^{-i\sum_{k=1}^4 x_k p_k}\langle 0|\mathrm{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_c$$

and read off the leading contribution to the $\phi\phi \to \phi\phi$ scattering amplitude, using the LSZ formula.