QFT, solutions to problem sheet 5

Problem 1: The interaction picture

1. *Show that $\dot{\phi}_I(t,\vec{x}) = \pi_I(t,\vec{x})$.*
   By the product rule,

$$\begin{aligned}\dot{\phi}_I(t,\vec{x}) &= i\,H_0\,e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t} + e^{iH_0t}\phi(0,\vec{x})e^{-iH_0t}(-i\,H_0) \\ &= e^{iH_0t}i\,[H_0,\phi(0,\vec{x})]e^{-iH_0t}\,.\end{aligned} \tag{1}$$

   In the commutator, only the π-dependent part of H_0 contributes because $[\phi,\phi]=0$ at equal times. Using the canonical commutation relations:

$$[H_0,\phi(0,\vec{x})] = \int \mathrm{d}^3y\,\frac{1}{2}[\pi^2(0,\vec{y}),\phi(0,\vec{x})] = \pi(0,\vec{x})$$

   Inserting into (1) gives the result.

2. *Starting from an expression for $\ddot{\phi}_I$, show that ϕ_I obeys the Klein-Gordon equation, and hence is a free field.*
   We have

$$\begin{aligned}\ddot{\phi}_I(t,\vec{x}) &= \partial_t\pi_I(t,\vec{x}) = e^{iH_0t}i[H_0,\pi(0,\vec{x})]e^{-iH_0t} \\ &= e^{iH_0t}\left(\frac{i}{2}\int \mathrm{d}^3y\,\left([(\vec{\nabla}\phi)^2(0,\vec{y}),\pi(0,\vec{x})] + m^2[\phi^2(0,\vec{y}),\pi(0,\vec{x})]\right)\right)e^{-iH_0t}\,.\end{aligned}$$

   Here we have used that $[\pi,\pi]=0$ at equal times. With

$$[(\vec{\nabla}\phi)^2(0,\vec{y}),\pi(0,\vec{x})] = 2i\,(\vec{\nabla}\phi(0,\vec{y}))\cdot\vec{\nabla}\,\delta^{(3)}(\vec{x}-\vec{y})$$

   and with the action of the differential operator $\vec{\nabla}$ on the distribution $\delta^{(3)}(\vec{x}-\vec{y})$ defined by integration by parts, i.e. $(\vec{\nabla}\phi(0,\vec{y}))\cdot\vec{\nabla}\,\delta^{(3)}(\vec{x}-\vec{y}) = -(\nabla^2\phi(0,\vec{x}))\,\delta^{(3)}(\vec{x}-\vec{y})$, this becomes

$$\ddot{\phi}_I(t,\vec{x}) = \nabla^2\phi_I(t,\vec{x}) - m^2\phi_I(t,\vec{x})$$

   which is the Klein-Gordon equation.

3. *Show that $U(t)\equiv e^{iH_0t}e^{-iHt}$ is unitary, and that $\phi(x) = U^\dagger(t)\phi_I(x)U(t)$.*
   This is straightforward:

$$U^\dagger = e^{iHt}e^{-iH_0t} \;\Rightarrow\; U^\dagger U = e^{iHt}e^{-iH_0t}e^{iH_0t}e^{-iHt} = \mathbb{1}\,.$$

$$U^\dagger\phi_I(x)U = e^{iHt}\underbrace{e^{-iH_0t}\phi_I(x)e^{iH_0t}}_{\phi(0,\vec{x})}e^{-iHt} = \phi(x)\,.$$

4. *We would like to express $U(t)$ entirely in terms of ϕ_I. To this end, start by showing that $U(t)$ obeys the Schrödinger equation*

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H_I(t)U(t)$$

where H_I is the interaction Hamiltonian in the interaction picture, $H_I(t) = e^{iH_0t}H_{\text{int}}e^{-iH_0t}$, with the boundary condition $U(0) = \mathbb{1}$.
It is obvious that $U(0) = e^0 = \mathbb{1}$. Moreover,

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}U(t) &= i\frac{\mathrm{d}}{\mathrm{d}t}e^{iH_0t}e^{-iHt} = ie^{iH_0t}\,(iH_0)\,e^{-iHt} + ie^{iH_0t}\,(-iH)\,e^{-iHt} \\ &= e^{iH_0t}H_{\text{int}}e^{-iHt} = \underbrace{e^{iH_0t}H_{\text{int}}e^{-iH_0t}}_{H_I(t)}\underbrace{e^{iH_0t}e^{-iHt}}_{U(t)} \end{aligned}$$

Then show that, for $t > 0$,

$$U(t) = \mathrm{T}\ \exp\left(-i\int_0^t \mathrm{d}t'\,H_I(t')\right)$$

also solves this Schrödinger equation and satisfies the same boundary condition. Therefore both expressions must be equal.
We have

$$i\frac{\mathrm{d}}{\mathrm{d}t}U = i\frac{\mathrm{d}}{\mathrm{d}t}\ \mathrm{T}\ \exp\left(-i\int_0^t \mathrm{d}t'\,H_I(t')\right) = \mathrm{T}\ H_I(t)\exp\left(-i\int_0^t \mathrm{d}t'\,H_I(t')\right)$$

and since t is the latest time appearing on the right-hand side, we can pull the factor $H_I(t)$ to the left of the time-ordering symbol:

$$i\frac{\mathrm{d}}{\mathrm{d}t}U = H_I(t)\ \mathrm{T}\ \exp\left(-i\int_0^t \mathrm{d}t'\,H_I(t')\right) = H_I(t)\,U\,.$$

Again, $U(0) = \mathbb{1}$ is trivially satisfied.

5. *Define $U(t_2, t_1) = U(t_2)U^\dagger(t_1)$. By a similar argument as used in 4., show that, for $t_2 > t_1$,*

$$U(t_2, t_1) = \mathrm{T}\ \exp\left(-i\int_{t_1}^{t_2} \mathrm{d}t'\,H_I(t')\right)\,.$$

This follows from the fact that both expressions for $U(t_2, t_1)$ satisfy the Schrödinger equation

$$i\frac{\partial U(t_2, t_1)}{\partial t_2} = H_I(t_2)\,U(t_2, t_1)$$

and the boundary condition $U(t_1, t_1) = \mathbb{1}$. The calculation is the same as in 4.

6. *Show that $U(t_1, t_3) = U(t_1, t_2)U(t_2, t_3)$, and that $U^\dagger(t_1, t_2) = U(t_2, t_1)$.*
This is again straightforward:

$$\begin{aligned} U(t_1, t_2)U(t_2, t_3) &= e^{iH_0t_1}e^{-iH(t_1-t_2)}e^{-iH_0t_2}e^{iH_0t_2}e^{-iH(t_2-t_3)}e^{-iH_0t_3} \\ &= e^{iH_0t_1}e^{-iH(t_1-t_3)}e^{iH_0t_3} = U(t_1, t_3)\,. \end{aligned}$$

$$U^\dagger(t_1, t_2) = e^{iH_0t_2}e^{-iH(t_2-t_1)}e^{-iH_0t_1} = U(t_2, t_1)\,.$$

7. *We would like to find a relation between the free vacuum $|\emptyset\rangle$ and the interacting vacuum $|0\rangle$. Let E_0 be the vacuum energy of the interacting theory, $H|0\rangle = E_0|0\rangle$. Show that (assuming $\langle 0|\emptyset\rangle \neq 0$):*

$$|0\rangle = \lim_{T\to\infty(1-i\epsilon)}\frac{e^{-iHT}|\emptyset\rangle}{e^{-iE_0T}\langle 0|\emptyset\rangle} = \lim_{T\to\infty(1-i\epsilon)}\frac{U(0,-T)|\emptyset\rangle}{e^{-iE_0T}\langle 0|\emptyset\rangle}\,.$$

and that

$$\langle 0| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle\emptyset|U(T,0)}{e^{-iE_0T}\langle\emptyset|0\rangle}\,.$$

The reasoning is similar to the lecture for the path integral derivation of the generating functional. We inset a complete set of eigenstates of the interacting theory $\{|n\rangle\}$:

$$e^{-iHT}|\emptyset\rangle = e^{-iHT}\sum_n |n\rangle\langle n|\emptyset\rangle = e^{-iE_0T}|0\rangle\langle 0|\emptyset\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|\emptyset\rangle$$

As $T \to \infty(1-i\epsilon)$, the terms in the sum on the RHS are exponentially decaying, and only the vacuum contributes:

$$\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|\emptyset\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0T}|0\rangle\langle 0|\emptyset\rangle \quad\Rightarrow\quad |0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{e^{-iHT}|\emptyset\rangle}{e^{-iE_0T}\langle 0|\emptyset\rangle}$$

where

$$e^{-iHT}|\emptyset\rangle = \underbrace{U(0)}_{\mathbb{1}}\, e^{iH(-T)}\,\underbrace{e^{-iH_0(-T)}|\emptyset\rangle}_{|\emptyset\rangle} = U(0,-T)|\emptyset\rangle\,.$$

The calculation for the other identity is similar.

8. *Finally, use the results of 5., 6. and 7. to show that*

$$\langle 0|\mathrm{T}\,\phi(x)\phi(y)|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle\emptyset|\mathrm{T}\,\phi_I(x)\phi_I(y)\,e^{-i\int_{-T}^{T}\mathrm{d}t\,H_I(t)}|\emptyset\rangle}{\langle\emptyset|\mathrm{T}\,e^{-i\int_{-T}^{T}\mathrm{d}t\,H_I(t)}|\emptyset\rangle}\,.$$

We take $x^0 > y^0$ without loss of generality (otherwise reverse the roles of x and y):

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0T}|\langle\emptyset|0\rangle|^2}\langle\emptyset|\,U(T,0)\,\phi(x)\phi(y)\,U(0,-T)|\emptyset\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0T}|\langle\emptyset|0\rangle|^2}\langle\emptyset|\,U(T,0)U(0,x^0)\phi_I(x)U(x^0,0)U(0,y^0)\phi_I(y)U(y^0,0)U(0,-T)|\emptyset\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0T}|\langle\emptyset|0\rangle|^2}\langle\emptyset|\,U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|\emptyset\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0T}|\langle\emptyset|0\rangle|^2}\langle\emptyset|\,\mathrm{T}\,\phi_I(x)\phi_I(y)U(T,-T)|\emptyset\rangle\,.
\end{aligned}$$

In the last equality we have used that $x^0 > y^0$, so all the operators in the next-to-last line are in time order. Plugging in the normalization condition

$$1 = \langle 0|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{1}{e^{-2iE_0T}|\langle\emptyset|0\rangle|^2}\langle\emptyset|U(T,-T)|\emptyset\rangle$$

and representing $U(T,-T)$ by the expression derived in 5., one finally obtains the Gell-Mann-Low formula.