

## QUANTUM FIELD THEORY, PROBLEM SHEET 6

Solutions to be discussed on 25/10/2023.

### Problem 1: The path integral for the harmonic oscillator

Consider a quantum mechanical harmonic oscillator with the Hamiltonian (setting  $m = 1$ )

$$H = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{q}^2.$$

1. By retracing the steps we followed in the lecture for a free field theory, show that the vacuum-to-vacuum transition amplitude in the presence of a source  $j(t)$  can be written as

$$\langle 0|0\rangle_j = \exp\left(-\frac{1}{4\omega} \int dt dt' j(t)e^{-i\omega|t-t'|}j(t')\right).$$

2. For a harmonic oscillator which is in its ground state at  $t \rightarrow -\infty$  and which is subject to an external force

$$j(t) = \begin{cases} j_0 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

(where  $j_0 > 0$  is a constant), calculate that the probability that it is still in its ground state for  $t \rightarrow +\infty$ .

### Problem 2: Photon propagator

Recall that the action for classical electrodynamics with a fictitious photon mass term and an external source  $J$  is (see problem sheet 1)

$$S[A, J] = \int d^4x \left( -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - A_\mu J^\mu \right).$$

In quantum theory we define the generating functional via the path integral:

$$Z[J] = \int \mathcal{D}A \exp(iS[A, J]).$$

1. Show that

$$(g_{\mu\nu} - k_\mu k_\nu / m^2) ((k^2 - m^2)g^{\nu\lambda} - k^\nu k^\lambda) = (k^2 - m^2)\delta_\mu^\lambda.$$

2. Using this result, find the function  $\tilde{D}_{\mu\nu}(k, m)$  which satisfies

$$Z[J] = Z[0] \exp\left(\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{J}^\mu(k) \tilde{D}_{\mu\nu}(k, m) \tilde{J}^\nu(-k)\right).$$

Here  $\tilde{J}^\mu(k)$  is the Fourier transform of  $J^\mu(x)$ .

3. Assuming that  $J^\nu$  is a conserved current, what does this imply for  $k_\nu \tilde{J}^\nu(k)$ ? What can we therefore take for  $\tilde{D}_{\mu\nu}(k, m)$  as  $m \rightarrow 0$  (the limit which corresponds to a realistic massless photon)?