Exercise sheet 3

Exercise 1: Second-order Runge-Kutta method

Compute the solution of the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t\,(x(t)^2 + 1)$$

with the initial condition $x(0) = 1$ on the interval $[0,\, 1]$, using the Runge-Kutta method of the second order. Plot the solution for $N = 100$. In a separate figure, plot the difference between your numerical solution for $N = 10, 20, 100$ and the exact solution $x(t) = \tan(t^2/2 + \pi/4)$.

Exercise 2: Lorenz equations

The Lorenz equations were originally written down to describe convective phenomena in fluid dynamics in a certain approximation. They are famous as an example of *chaotic dynamics*:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y(t) - x(t))\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \rho\, x(t) - y(t) - x(t)z(t)\,, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = x(t)y(t) - \beta\, z(t)\,.$$

1. Solve the Lorenz equations for $\sigma = 10$, $\rho = 28$, $\beta = \frac{8}{3}$ between $t = 0$ and $t = 50$, using the RK4 method. The initial conditions are $x = 0$, $y = 1$, $z = 0$.
2. Plot $x(t)$ and $z(x)$. What do you observe?

Exercise 3: Ballistic trajectory

Consider a spherical cannon ball subject to the gravitational force, $\mathbf{F}_g = -mg\mathbf{e}_y$, and to a friction force due to air drag,

$$F_d = \frac{1}{2}\rho_{\mathrm{a}} S_{\mathrm{b}} C v^2\,.$$

Here $m = 2$ kg is the cannon ball's mass, $g = 9.81$ m s^{-2} is the gravitational acceleration, $C = 0.47$ is the drag coefficient of a sphere, $\rho_{\mathrm{a}} = 1.22$ kg m^{-3} is the density of air, $S_{\mathrm{b}} = \pi r_{\mathrm{b}}^2$ is the cross-sectional area of a sphere, $r_b = 4$ cm is its radius and v is the velocity. The drag force is always opposed to the direction of movement.

1. Show that the equations of motion can be written

$$\ddot{x} = -\alpha\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}\,, \qquad \ddot{y} = -g - \alpha\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$

   where α is a constant. Express α in terms of the given quantities.

2. Transform the equations of motion into a system of four first-order equations.
3. Compute the solution $x(t)$ and $y(t)$ with the RK4 method for an initial velocity of 250 ms^{-1} and a shooting angle of $\theta = 20°$. Plot the trajectory $y(x)$.
4. Upon increasing the mass m of the cannon ball while leaving all other parameters unchanged, how does the range of the cannon change? Why?

Exercise 4: Cometary orbit

The trajectories of comets are ellipses that can be highly excentric. When a comet is close to the sun, it accelerates; to calculate its orbit numerically with good precision, one therefore needs rather small time increments. Far away from the sun, the comet is slow and its acceleration is much smaller, hence the step size can be increased to save computing time.

In this exercise you will use the adaptive RK4 method to compute the trajectory. Neglect all other celestial objects, as well as the gravitational attraction exercised by the comet on the sun, and choose coordinates where the sun is at the center and the motion takes place in the (x, y) plane. Newton's gravitational constant is $G = 6.67408 \times 10^{-11}$ m^3 kg^{-1} s^{-2} and the solar mass is $M = 1.989 \times 10^{30}$ kg.

1. Find the two equations of motion for the Cartesian coordinates $x(t)$ and $y(t)$ of the comet's position.
2. Transform them into four first-order equations.
3. Write a program which solves them for the initial conditions $x(0) = 4{\times}10^9$ km, $y(0) = 0$, $\dot{x}(0) = 0$, $\dot{y}(0) = 500$ m s^{-1} with the fixed-step RK4 method. Plot $x(y)$; choose the increment h sufficiently small, such that the orbit does not visibly change between two periods.
4. Write a second program doing the same computation but with the adaptive RK4 method. The target accuracy is δ =1 km/year. Compare with the fixed-step method.

Exercise 5: Shooting method

Consider again the cannon of Exercise 3. Write a program to compute the initial shooting angle θ such that the cannon ball hits a target which is exactly 1200 m away (and at the same height as the cannon, i.e. on the ground in the plane).

Exercise 6: Gravitational n-body problem (mini-project)

Write a code for solving the equations of motion of an arbitrary number of mass points in the gravitational field created by themselves. The velocities and positions are assumed to be given at $t = 0$. Use the adaptive RK4 method.

Instructions:

- Start by a pen-and-paper calculation to find analytic expressions for the distance between two mass points with given Cartesian coordinates, and for the gravitational force exerted by one on the other.
- It is convenient to implement a function returning all of the right-hand sides of the first-order equations, transformed into standard form, in a `NumPy` array.
- You might consider using orient-objected programming in order to better organize your code, if you are familiar with it.
- The desired accuracy will depend on the concrete problem under study.

Applications:

1. Recalculate the cometary orbit of exercise 4.

2. Compute the trajectories of five mass points of mass $M = M_{\text{sun}} = 2 \times 10^{30}$ kg. At $t = 0$ their positions are randomly chosen between -5 au and 5 au, and their velocities are random and of the order of 0.01 au / day. Plot the trajectories over a few years and/or create an animation.

3. Compute the trajectory of Voyager 2, a space probe that was launched on 20/8/1977 and subsequently passed close to Jupiter, Saturn, Uranus and Neptune using the "gravitational slingshot" principle of picking up speed by exploiting the planets' gravitational fields. The file `data.txt` contains the initial data for the most relevant celestial objects as provied by NASA's public data base HORIZON (use the function `numpy.loadtxt` to read it). Plot the (x, y)-plane projection of the trajectories of Earth, the major planets, and of Voyager 2. According to your computation, does the probe reach Jupiter? Saturn? Uranus? Neptune? If not, what effects could be included to increase the precision of the computation?

Units: 1 au $= 1.495978707 \times 10^{11}$ m, 1 day $= 8.64 \times 10^4$ s, $G = 1.488136 \times 10^{-34}$ au^3 kg^{-1} day^{-2}.