M2 Cosmos, Champs et Particules — Faculté des sciences de Montpellier

QFT, SOLUTIONS TO PROBLEM SHEET 3

Problem 1: The free scalar field and causality

Show that for any spacelike four-vector x there exists a proper orthochronous Lorentz transformation sending $x^0 \to 0$. Conclude that

$$\Delta(x,y) = 0 \qquad \textit{whenever} \quad (x-y)^2 < 0\,.$$

Here $\Delta(x,y) = [\phi(x), \phi(y)]$ and ϕ is a free real scalar field. What is the corresponding statement for a complex scalar field?

Let $x = (t, \vec{x})$ be a space-like four-vector, $x_\mu x^\mu < 0$. By a rotation we can always transform $\vec{x} \to \begin{pmatrix} x^1 \\ 0 \\ 0 \end{pmatrix}$. Here $|x^1| > |t|$ because x is spacelike. By a boost in the x^1 direction, we can then transform

$$\begin{pmatrix} t \\ x^1 \end{pmatrix} \to \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x^1 \end{pmatrix} = \begin{pmatrix} 0 \\ x'^1 \end{pmatrix}$$

by choosing $\beta = -\frac{t}{x^1}$; this is permissible because $|x^1| > |t|$ and therefore $|\beta| < 1$.
Now let $(x-y)$ be spacelike and choose a Lorentz frame where $x^0 = y^0$. Then we have $[\phi(x), \phi(y)] = 0$ according to the canonical commutation relations.
The corresponding statement for a complex free scalar field is

$$\langle 0 | [\phi^\dagger(x), \phi(y)] | 0 \rangle = [\phi^\dagger(x), \phi(y)] = 0 \qquad \text{if } (x-y)^2 < 0\,.$$

Problem 2: The residue theorem

To obtain the electrostatic potential of a point particle in Exercise 3.4 on Problem Sheet 1, you were given the identity

$$\int_0^\infty \mathrm{d}k\, \frac{k\, \sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)\,.$$

Prove this formula with the help of the residue theorem.

Given the real positive constants x and m, we define the meromorphic functions $f_\pm(z)$ by

$$f_\pm(z) = \frac{z\, e^{\pm izx}}{z^2 + m^2}\,.$$

Both these functions have poles at $z = im$ and at $z = -im$. Moreover, $f_+(z)$ decreases exponentially as $|z| \to \infty$ in the upper half-plane, and $f_-(z)$ decreases exponentially as $|z| \to \infty$ in the lower half-plane. We therefore have

$$\int_{-\infty}^\infty f_+(k)\, \mathrm{d}k = \oint_{\gamma_+} f_+(z) \mathrm{d}z = 2\pi i \,\mathrm{res}(f_+, z = im) = 2\pi i \left. \frac{z e^{izx}}{z + im} \right|_{z=im} = \pi i e^{-mx}$$

where γ_+ is a curve along the real axis and closed along the arc at infinity in the upper half-plane (which includes the pole at $z = im$), and

$$\int_{-\infty}^\infty f_-(k)\, \mathrm{d}k = \oint_{\gamma_-} f_-(z) \mathrm{d}z = -2\pi i \,\mathrm{res}(f_-, z = -im) = -2\pi i \left. \frac{z e^{-izx}}{z - im} \right|_{z=-im} = -\pi i e^{-mx}$$

where γ_- is a curve along the real axis and closed along the arc at infinity in the lower half-plane (which includes the pole at $z = -im$). It follows that

$$\int_0^\infty \frac{1}{2i}\left(f_+(k) - f_-(k)\right) \mathrm{d}k = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{2i}\left(f_+(k) - f_-(k)\right) \mathrm{d}k = \frac{\pi}{2}e^{-mx}$$

which is the desired identity.

Problem 3: Propagators

1. *Show that any $iD_{\mathcal{C}}(x-y)$ is a Green function for the Klein-Gordon operator $\Box + m^2$:*
$$(\Box_x + m^2)D_{\mathcal{C}}(x-y) = -i\delta^4(x-y)\,.$$

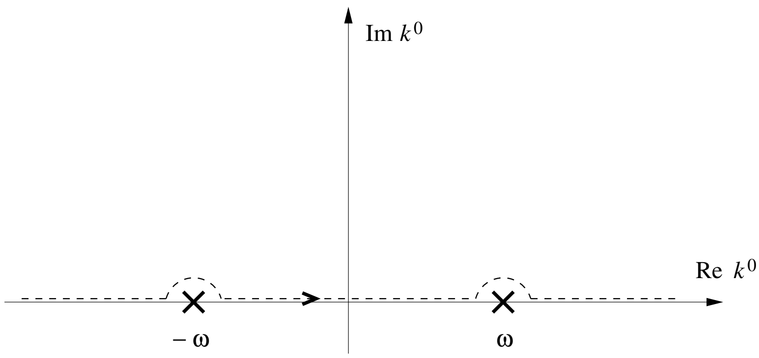
$$(\Box_x + m^2)\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik(x-y)} = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}(\Box_x + m^2)e^{-ik(x-y)}$$
$$= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}(-k^2+m^2)e^{-ik(x-y)} = -i\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)} = -i\delta^{(4)}(x-y)\,.$$

2. *Express $D_R(x-y)$ and $D_A(x-y)$ in terms of Θ functions and of vacuum expectation values of products of $\phi(x)$ and $\phi(y)$. Here $D_R(x-y)$ is defined to avoid both poles in the upper half-plane, and $D_A(x-y)$ is defined to avoid both poles in the lower half-plane.*

We need to calculate the integral

$$\int_{-\infty}^{\infty} f(k^0)\,\mathrm{d}k^0\,, \qquad f(k^0) = \frac{1}{2\pi}\frac{i\,e^{-ik^0(x^0-y^0)}}{(k^0-\omega)(k^0+\omega)}\,.$$

The "retarded" contour $\mathcal{C}_R$ avoids both poles in the upper half-plane:



- If $x^0 > y^0$, then $e^{-ik^0(x^0-y^0)}$ diverges exponentially as $|k^0| \to \infty$ for Im $k^0 > 0$, and decays exponentially as $|k^0| \to \infty$ for Im $k^0 < 0$. The integration contour therefore needs to be closed in the lower half-plane where Im $k^0 < 0$, and includes both poles with winding number -1. We have

$$\oint_{\mathcal{C}_R} f(k^0)\,\mathrm{d}k^0 = 2\pi i\left(-\mathrm{res}\;(f,\omega) - \mathrm{res}\;(f,-\omega)\right)$$
$$= -2\pi i\left(\frac{i\,e^{-i\omega(x^0-y^0)}}{(2\pi)(2\omega)} + \frac{i\,e^{i\omega(x^0-y^0)}}{(2\pi)(-2\omega)}\right) = \frac{1}{2\omega}\left(e^{-i\omega(x^0-y^0)} - e^{i\omega(x^0-y^0)}\right)$$

and it follows that

$$\int_{\mathcal{C}_R} \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i\, e^{-ik(x-y)}}{k^2 - m^2} = \int \underbrace{\frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega}}_{= \widetilde{\mathrm{d}k}} \left(e^{-i\omega(x^0-y^0)} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\omega(y^0-x^0)} \underbrace{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}_{= e^{i\vec{k}\cdot(\vec{y}-\vec{x})} \text{ under } \mathrm{d}^3k} \right)$$

$$= \int \widetilde{\mathrm{d}k} \left(e^{-ik(x-y)} - e^{-ik(y-x)} \right) = D(x-y) - D(y-x) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \,.$$

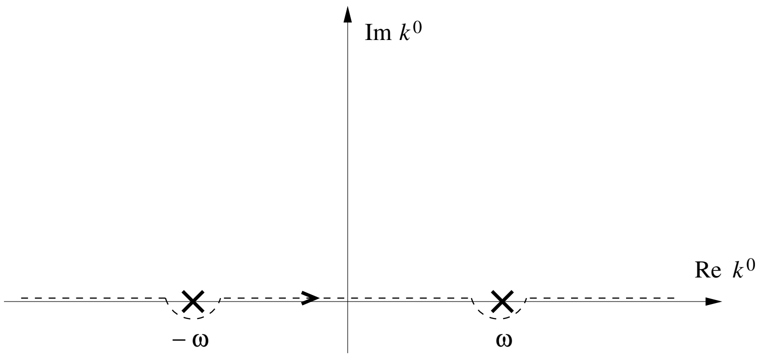
- If $x^0 < y^0$, then $e^{-ik^0(x^0-y^0)}$ diverges exponentially as $|k^0| \to \infty$ for $\mathrm{Im}\, k^0 < 0$, and decays exponentially as $|k^0| \to \infty$ for $\mathrm{Im}\, k^0 > 0$. The integration contour therefore needs to be closed in the upper half-plane, and does not include any pole. Hence

$$\oint_{\mathcal{C}_R} f(k^0)\, \mathrm{d}k^0 = 0 \,.$$

- Altogether we find the following expression for the retarded propagator:

$$D_R(x-y) = \theta(x^0 - y^0)\, \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \,.$$

The "advanced" contour $\mathcal{C}_A$ avoids both poles in the lower half-plane:



Following the same line of reasoning, one finds

$$D_A(x-y) = -\theta(y^0 - x^0)\, \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \,.$$

3. *Starting from the expression of the lecture*

$$D_F(x-y) = \int \widetilde{\mathrm{d}k}\ \left(e^{-ik(x-y)} \Theta(x^0 - y^0) + e^{ik(x-y)} \Theta(y^0 - x^0) \right) \Big|_{k^0 = \omega_{\vec{k}}}$$

and evaluating the integral, write the Feynman propagator for $(x-y)^2 \neq 0$ *explicitly in terms of the modified Bessel function of the second kind* $K_1(z)$. *You can use the identity (see Gradshteyn & Ryzhik, "Table of integrals, series and products", eq. 3.914/9)*

$$\int_0^\infty \frac{x e^{-\beta\sqrt{\gamma^2 + x^2}}}{\sqrt{\gamma^2 + x^2}} \sin bx\, \mathrm{d}x = \frac{\gamma b}{\sqrt{\beta^2 + b^2}} K_1\left(\gamma \sqrt{\beta^2 + b^2} \right) \,.$$

Assume that $x^0 > y^0$ (otherwise exchange x and y). With $z \equiv x - y$ we then have $z^0 > 0$, and

$$D_F(z) = \int \widetilde{\mathrm{d}k}\, e^{-ikz} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\omega} e^{-i\omega z^0 + i\vec{k}\cdot\vec{z}} = \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \frac{k^2 \mathrm{d}k\, \mathrm{d}\cos\theta\, \mathrm{d}\phi}{(2\pi)^3\, 2\omega} e^{-i\omega z^0 + ikz\cos\theta}$$

where in the last term, and from now on, k and z no longer denote four-vectors but $k \equiv |\vec{k}|$ and $z \equiv |\vec{z}|$. The $\mathrm{d}\phi$ integral gives a factor 2π, whereas the $\mathrm{d}\cos\theta$ integral gives

$$\ldots = \frac{1}{8\pi^2} \int_0^\infty \mathrm{d}k\, \frac{k^2}{\omega} \frac{1}{ikz} \left(e^{ikz} - e^{-ikz}\right) e^{-i\omega z^0} = \frac{1}{4\pi^2 z} \int_0^\infty \mathrm{d}k \frac{k}{\sqrt{k^2 + m^2}} \sin(kz) e^{-i\sqrt{k^2+m^2}\, z^0}$$

Using the given representation of K_1 with $\beta = iz^0$, $\gamma = m$, and $b = z$, this is

$$\ldots = \frac{1}{4\pi^2 z} \frac{mz}{\sqrt{-(z^0)^2 + z^2}}\, K_1\left(m\sqrt{-(z^0)^2 + z^2}\right)$$

and finally plugging back the original variables gives

$$D_F(x-y) = \frac{m}{4\pi^2 \sqrt{-(x-y)^2}}\, K_1\left(m\sqrt{-(x-y)^2}\right) .$$

Note the exponentially decaying behaviour for $(x-y)^2 < 0$, while for timelike separations $(x-y)$ the Feynman propagator is complex-valued and oscillatory.