# Quantum Field Theory, problem sheet 4

Solutions to be discussed on 11/10/2023

## Problem 1: Asymptotic overlap between wave packets

Consider an interacting theory of a real scalar field with mass $m$. Let $|\psi\rangle$ be a multi-particle wave packet given by

$$|\psi\rangle = \int \mathrm{d}^3p\ \psi(\vec{p})\,|\lambda_p\rangle\,.$$

Here $\psi(\vec{p})$ is an enveloping function (e.g. a Gaussian) and $|\lambda_p\rangle$ is a multi-particle state of total 4-momentum $p = (p^0,\,\vec{p})$ and invariant mass $\sqrt{p^2} \equiv M > m$. With $a_1^\dagger(t) = \int \mathrm{d}^3k\, f_1(\vec{k}) a^\dagger(\vec{k},\,t)$ defined as in the lecture, show that

$$\lim_{t\to\pm\infty} \langle\psi|a_1^\dagger(t)|0\rangle = 0\,,$$

i.e. the overlap between a one-particle and a multi-particle wave packet tends to zero at $t \to \pm\infty$.

*Hints:*

- Express $a^\dagger$ in terms of $\phi$ and write $\phi(x) = e^{iPx}\phi(0)e^{-iPx}$, where $P$ is the 4-momentum operator.
- Two of the resulting integrals collapse when using $\int \mathrm{d}^3x\ e^{i\vec{q}\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{q})$.
- Estimate the remaining integral with the help of the *stationary phase approximation*: Let $f : \mathbb{R}^n \to \mathbb{C}$ be a suitable test function, $\lambda > 0$, and $\varphi : \mathbb{R}^n \to \mathbb{R}$ a smooth function. Suppose that $\varphi$ has a single critical point $p_0$ and that this critical point is nondegenerate, i.e. the Hessian matrix $H_\varphi(p_0)$ is nonsingular. Then, with $\sigma = \operatorname{sgn} H_\varphi(p_0) =$ the number of positive minus the number of negative eigenvalues of $H_\varphi(p_0)$,

$$\int \mathrm{d}^np\ f(p)e^{i\lambda\varphi(p)} = |\det H_\varphi(p_0)|^{-1/2}\, e^{i\lambda\varphi(p_0)+i\pi\sigma/4} \left(\frac{2\pi}{\lambda}\right)^{n/2} f(p_0) + \mathrm{o}(|\lambda|^{-n/2})\,.$$

## Problem 2: Gaussian and oscillatory integrals

We recall the basic Gaussian integral: if $a \in \mathbb{C}$ with $\operatorname{Re} a > 0$, then

$$\int_{-\infty}^{\infty} \mathrm{d}x\ e^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

1. Let $a, b \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Show that:

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \exp\left(-\frac{1}{2}a\,x^2 + b\,x\right) = \sqrt{\frac{2\pi}{a}}\exp\left(\frac{1}{2}\frac{b^2}{a}\right)\,.$$

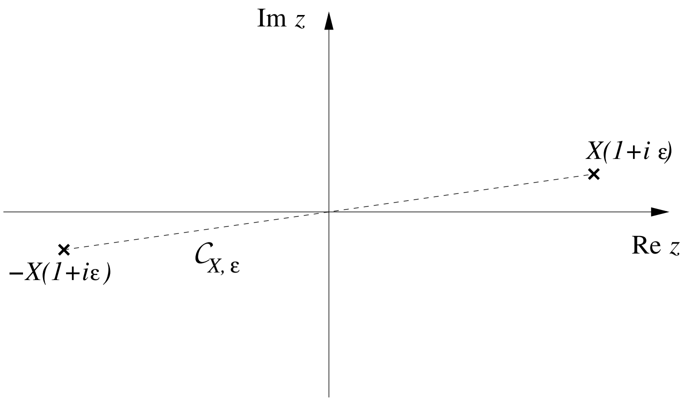
2. Let $a \in \mathbb{R}_+^*$ and $b \in \mathbb{C}$. We define the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(\frac{i}{2} a \, x^2 + i \, b \, x\right)$$

by a complex line integral whose curve of integration $\mathcal{C}_{X,\epsilon}$ is the straight line connecting the points $-X(1+i\epsilon)$ and $+X(1+i\epsilon)$:

$$I = \lim_{\epsilon \searrow 0} \lim_{X \to \infty} \int_{\mathcal{C}_{X,\epsilon}} \mathrm{d}z \, \exp\left(\frac{i}{2} a \, z^2 + i \, b \, z\right) .$$

For illustration:




Calculate $I$. Compare your result with the stationary phase approximation given in exercise 1.

3. Let $A$ be a real symmetric positive definite $n \times n$ matrix. Show that

$$\int \mathrm{d}^n x \exp\left(-\frac{1}{2} x^T A x\right) = \sqrt{\frac{(2\pi)^n}{\det A}} , \qquad \int \mathrm{d}^n x \exp\left(\frac{i}{2} x^T A x\right) = e^{in\pi/4} \sqrt{\frac{(2\pi)^n}{\det A}} .$$