Quantum Field Theory, problem sheet 3

Solutions to be discussed on 04/10/2023

Problem 1: The free scalar field and causality

Recall from quantum mechanics that, if two observables are represented by operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with $[\mathcal{O}_1, \mathcal{O}_2] \neq 0$, then a measurement of $\mathcal{O}_1$ will influence a subsequent measurement of $\mathcal{O}_2$. However, in a Lorentz invariant quantum field theory, two events with spacelike separation should not affect each other in order to preserve causality.

Show that for any spacelike four-vector x there exists a proper orthochronous Lorentz transformation sending $x^0 \to 0$. Conclude that

$$\Delta(x, y) = 0 \qquad \text{whenever} \quad (x-y)^2 < 0\,.$$

Here $\Delta(x, y) = [\phi(x), \phi(y)]$ and ϕ is a free real scalar field. What is the corresponding statement for a complex scalar field?

Problem 2: The residue theorem

To obtain the electrostatic potential of a point particle in Exercise 3.4 on Problem Sheet 1, you were given the identity

$$\int_0^\infty \mathrm{d}k\, \frac{k\,\sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)\,.$$

Prove this formula with the help of the residue theorem.

Problem 3: Propagators

You have seen in the lecture that the expression

$$D_{\mathcal{C}}(x - y) = \int_{\mathcal{C}} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik\cdot(x-y)}$$

depends on the curve $\mathcal{C}$ in the plane of complex k^0 along which the poles at $\pm\omega$ are avoided. For example, choosing to circumvent the pole at $k^0 = -\omega$ in the lower half-plane and the pole at $k^0 = +\omega$ in the upper half-plane yields the Feynman propagator

$$\begin{aligned} D_F(x - y) &= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} = \langle 0|\mathrm{T}\,\phi(x)\phi(y)|0\rangle \\ &= \Theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle\,. \end{aligned}$$

1. Show that any $iD_{\mathcal{C}}(x-y)$ is a Green function for the Klein-Gordon operator $\Box + m^2$:
$$(\Box_x + m^2) D_{\mathcal{C}}(x-y) = -i\delta^4(x-y)\,.$$

2. Express $D_R(x-y)$ and $D_A(x-y)$ in terms of Θ functions and of vacuum expectation values of products of $\phi(x)$ and $\phi(y)$. Here $D_R(x-y)$ is defined to avoid both poles in the upper half-plane, and $D_A(x-y)$ is defined to avoid both poles in the lower half-plane.

3. Starting from the expression of the lecture
$$D_F(x-y) = \int \widetilde{\mathrm{d}k}\; \left(e^{-ik(x-y)}\Theta(x^0-y^0) + e^{ik(x-y)}\Theta(y^0-x^0)\right)\Big|_{k^0=\omega_{\vec{k}}}$$
and evaluating the integral, write the Feynman propagator for $(x-y)^2 \neq 0$ explicitly in terms of the modified Bessel function of the second kind $K_1(z)$. You can use the identity (see Gradshteyn & Ryzhik, "Table of integrals, series and products", eq. 3.914/9)
$$\int_0^\infty \frac{xe^{-\beta\sqrt{\gamma^2+x^2}}}{\sqrt{\gamma^2+x^2}} \sin bx\,\mathrm{d}x = \frac{\gamma b}{\sqrt{\beta^2+b^2}} K_1\left(\gamma\sqrt{\beta^2+b^2}\right)\,.$$