Quantum Field Theory, problem sheet 2

Solutions to be discussed on 27/09/2022

Problem 1: Annihilation and creation operators for the free real scalar field

1. In the lecture, the Fourier coefficients $a(\vec{k})$ appearing in the Fourier decomposition of a real scalar field were defined by

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\vec{k}) e^{-ikx} + a^*(\vec{k}) e^{ikx} \right) .$$

Show that

$$a(\vec{k}) = i \int \mathrm{d}^3 x \, e^{ikx} \overleftrightarrow{\partial_0} \phi(x) , \qquad \text{where } f \overleftrightarrow{\partial} g \equiv f \partial g - (\partial f) g .$$

Here the right-hand side is to be evaluated at some arbitrary time x^0.

Hint: It is easiest to work backwards, i.e., starting from the expression for $a(\vec{k})$, verify that the one for $\phi(x)$ is satisfied.

2. Using the canonical equal-time commutation relations

$$[\phi(t,\vec{x}), \pi(t,\vec{x}\,')] = i\delta^{(3)}(\vec{x} - \vec{x}\,') , \qquad [\phi(t,\vec{x}), \phi(t,\vec{x}\,')] = 0 = [\pi(t,\vec{x}), \pi(t,\vec{x}\,')] ,$$

verify as many of the following relations as you wish:

$$\left[a(\vec{k}), a^\dagger(\vec{k}\,') \right] = 2\omega_{\vec{k}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}\,') , \quad \left[a(\vec{k}), a(\vec{k}\,') \right] = 0 = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}\,') \right] .$$

Problem 2: Canonical quantisation of the free complex scalar field

Consider again the free complex scalar field $\phi(x)$ of Problem Sheet 1. The Lagrangian density is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 .$$

1. Find the canonical momenta conjugate to ϕ and ϕ^*.

2. Promoting ϕ and ϕ^* to operators, imposing canonical equal-time commutation relations for the fields and their conjugate momenta, and writing the mode expansion of ϕ as

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\vec{k}) e^{-ikx} + b^\dagger(\vec{k}) e^{ikx} \right) ,$$

guess the commutation relations which should be obeyed by $a(\vec{k})$ and $b(\vec{k})$ and their hermitian conjugates. Verify that your guess leads to the correct canonical commutators for ϕ and $\phi^\dagger$.

3. Express the Hamiltonian in terms of $a(\vec{k})$, $b(\vec{k})$, and their conjugates.