Mathematics for Economics

Michael Hoy
John Livernois
Chris McKenna
Ray Rees
Thanasis Stengos

Third Edition

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This book is dedicated to our families:

Maureen, Alexis, and Joshua
Ali and Becky
Jane, James, and Kate
Denny, Zac, and Dan
Bieke, Yanna, Mariana, and Dominiek

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Preface

A major challenge in writing a book on mathematics for economists is to select the appropriate mathematical topics and present them with the necessary clarity. Another challenge is to motivate students of economics to study these topics by convincingly demonstrating their power to deal with economic problems. All this must be done without sacrificing anything in terms of the rigor and correctness of the mathematics itself.

A problem lies in the difference between the logic of the development of the mathematics and the way in which economics progresses from models of individual consumer and firm, through market models and general equilibrium, to macroeconomic models. The primary building blocks, the models of consumer and firm behavior, are based on methods of constrained optimization that, mathematically speaking, are already relatively advanced. In this book we have chosen instead to follow the logic of the mathematics. After a review of fundamentals, concerned primarily with sets, numbers, and functions, we pay careful attention to the development of the ideas of limits and continuity, moving then to the calculus of functions of one variable, linear algebra, multivariate calculus, and finally, dynamics. In the treatment of the mathematics our goal has always been to give the student an understanding of the mathematical concepts themselves, since we believe this understanding is required if he or she is to develop the ability and confidence to tackle problems in economic analysis. We have very consciously sought to avoid a "cookbook" approach.

We have tried to develop the student's problem-solving skills and motivation by working through a large number of examples and economic applications, far more than is usual in this type of book. Although the selection of these, and the order in which they are presented, was determined by the logic of the development of the mathematics rather than that of an economics course, in the end the student will have covered virtually all of the standard undergraduate mathematical economics syllabus.

Many people helped us in the preparation of this book and it is a pleasure to acknowledge our debt to them here. The following individuals read early versions of the manuscript and offered helpful suggestions, a large number of which were freely used:

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Part I Introduction and Fundamentals

Chapter 1 Introduction

Almost for as long as economics has existed as a subject of study, mathematics has played a part in both the exploration and the exposition of economic ideas.[1] It is not simply that many economic concepts are *quantifiable* (examples include prices, quantities of goods, volume of money) but also that mathematics enables us to explore relationships among these quantities. These relationships are explored in the context of *economic models*, and how such models are developed is one of the key themes of this book. Mathematics possesses the accuracy, the rigor, and the capacity to deal clearly with complex systems, which makes it highly valuable as a method for analyzing economic issues.

This book covers a wide range of mathematical techniques and outlines a large number of economic problems to which these techniques may be applied. However, mathematical modeling in economics has some unifying features and conventions that we will summarize here at the outset. Although model details are problem-specific, there are some basic principles in the modeling process that are worth spelling out.

1.1 What Is an Economic Model?

At its most general, a model of anything is a *representation*. As such, a model differs from the original in some way such as scale, amount of detail, or degree of complexity, while at the same time preserving what is important in the original in its broader or most salient aspects. The same is true of an economic model, though unlike model airplanes, our models do not take a physical form. Instead, we think of an economic model as a set of mathematical relationships between economic magnitudes. Knowing how to distill the important aspects of an economic problem into an abstract simplification is part of the formal training of an economist. The model must be convincing and must be capable of addressing the questions that the researcher has set. We now set out the central features of an economic model.

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[1] The more famous early works in economics with a mathematical exposition include A. Cournot, *Récherches sur les principes mathématiques de la théorie des richesses* (1838), W. S. Jevons, *The Theory of Political Economy* (1874), L. Walras, *Eléments d'économie politique pure* (1874), A. Marshall, *Principles of Economics* (1890), and V. Pareto, *Cours d'économie politique* (1896).

Quantities, Magnitudes, and Relationships

We can start by thinking of how we measure things in economics. Numbers represent quantities and ultimately it is this circumstance that makes it possible to use mathematics as an instrument for economic modeling. When we discuss market activity, for example, we are concerned with the quantity traded and the price at which the trade occurs. This is so whether the "quantity" is automobiles, bread, haircuts, shares, or treasury bills. These items possess *cardinality*, which means that we can place a definite number on the quantity we observe. Cardinality is absolute but is not always necessary for comparisons. *Ordinality* is also a property of numbers but refers only to the ordering of items. The difference between these two number concepts may be illustrated by the following two statements.

1. Last year, the economy's growth rate was 3%.
2. The economy's output last year was greater than the year before.

Both of these statements convey *quantitative* information. The first of these is a cardinal property of the change in output. We are able to measure the change and put a definite value on it. The second is an ordinal statement about economic activity in the past year. Last year's output is higher than the year before. This of course is an implication of the first statement, but the first statement cannot be inferred from the second statement.

However, there is a greater difference between cardinality and ordinality, because we can also decide on a *ranking* of items based on their *qualitative* properties rather than on their quantifiable ones. Most statements about preferences are ordinal in this sense, so to say: "I prefer brand A to brand B, and brand B to brand C" is an ordinal statement about how one person subjectively evaluates three brands of a good. If we let larger numbers denote more preferred brands, then we could associate brand A with the number 3, brand B with the number 2, and brand C with the number 1. However, the numbers 10, 8, and 0 would serve equally well in the absence of any other information. This statement may provide useful information, and certain logical and mathematical consequences may follow from it, but it is not a statement about quantities.

Variables and Parameters

When we start to build an economic model, we know that we are not going to be able to explain everything. Some things must be treated as given or as data for our problem. These are the *exogenous variables* and the *parameters* of the model. The *endogenous variables*, then, are those that are going to be explained by the model. A simple example will illustrate.

Suppose that we are trying to determine the equilibrium price and quantity in a market for a homogeneous good. We hypothesize that the quantity demanded of some good may be represented as

$$q^D = a - bp + cy \tag{1.1}$$

which is a simple linear demand function. Each time price, p, increases by one dollar, the quantity demanded, q^D, falls by b times one dollar. A rise in income, y, of one dollar increases quantity demanded by c units. This demand curve may be chosen purely for simplicity, or it may be known, by looking at market data, that the demand curve for this good does take this simple form. Now suppose that the supply of this good is fixed at some amount which we will call $\bar{q}^S$, and suppose that we believe that the prevailing price in this market is the price that equates demand with supply. Then

$$q^D = \bar{q}^S \quad \text{implies} \quad p = \frac{a - \bar{q}^S + cy}{b} \tag{1.2}$$

So here, p is the endogenous variable, a and b are the exogenously given parameters, and $\bar{q}^S$ and y are exogenous variables. For instance, demand is determined by tastes, weather, and many other environmental and social factors, all of which are captured here by a, b, and c. All supply-side considerations are, in this particularly simple case, summarized by the quantity $\bar{q}^S$. Parameters may also incorporate the effects of exogenous variables, which we do not wish to specify explicitly. For example, a may incorporate the effects of prices of other goods on the demand for this one. Finally, in this example, there is one further endogenous variable. Since quantity demanded, q^D, depends partly on p (which is endogenous) it too is endogenous: therefore q^D is only known when the price is known. Substituting equation (1.2) into equation (1.1) gives simply $q^D = \bar{q}^S$ as the value of demand.

In general, as in this simple example, we can use relationships between economic variables and background parameters to reach conclusions or predictions based upon the mathematical solutions of those relationships.

Behavior and Equilibrium

As we have just seen, a further step in building an economic model is to identify the *behavioral* equations, or the equations that describe the economic environment, and to identify the *equilibrium* conditions. In the simple supply-and-demand example above, the behavioral equations are the demand and supply functions describing the relationships between the endogenous variables and exogenous variables. The equilibrium condition determines what the values of the endogenous variables will be. In this case the condition is that supply equals demand. The specification of the equilibrium condition is based on our understanding of how the part of the economy in question works, and embodies the crucial hypothesis of how the endogenous variables are determined.

Behavioral equations contain hypotheses about the way the individual, market, or economy works. One of the key strengths of mathematical analysis in economics is that it forces us to be precise about our assumptions. If the implications of those behavioral equations prove to be unsubstantiated or ridiculous, then the natural course of action is to look more closely at the assumptions and to attempt to identify those responsible for throwing us off course.

Single-Equation Models and Multiple-Equation Models

Although sometimes the problem we are trying to analyze may be captured in a single-equation model, there are many instances where two or more equations are necessary. Interactions among a number of economic agents or among different sectors of the economy typically cannot be captured in a single equation, and a system of equations must be specified and solved simultaneously. We can extend our earlier example to illustrate this.

Consider first the demand and supply of two goods. We denote the demands by q_1^D and q_2^D, and the supplies by q_1^S and q_2^S, where the subscripts 1 and 2 identify which good we are referring to. Now, as before, we may specify how demands and supplies are related to the prices of the two goods, but this time recognizing that the demand for and supply of good 1 may depend on both its own price and on the price of good 2. Recognition of this fact gives rise to an interdependence between the two markets. For simplicity, suppose that the demand for each good depends on both prices, while the supply of each good depends only on the good's own price. The question we are asking is: If the interdependence between two goods takes this form, what are the consequences for the equilibrium price and quantity traded in each market in equilibrium? Again, as before, we will restrict ourselves to *linear* relationships only. We may write

$$q_1^D = a - b_1 p_1 + b_2 p_2, \qquad b_1, b_2 > 0 \tag{1.3}$$

and

$$q_2^D = \alpha - \beta_1 p_2 + \beta_2 p_1, \qquad \beta_1, \beta_2 > 0 \tag{1.4}$$

Notice that in addition to incorporating the usual negative relationship between the demand for a good and its own price, we have included a specific assumption about the *cross-price effects*, namely that these goods are *substitutes*. If the price of good 1 increases, the demand for good 2 increases, and vice versa. Setting supply equal to an exogenous amount in each market gives us

$$p_1 = \frac{a + b_2 p_2 - \bar{q}_1^s}{b_1} \tag{1.5}$$

$$p_2 = \frac{\alpha + \beta_2 p_1 - \bar{q}_2^s}{\beta_1} \tag{1.6}$$

and solving gives

$$p_1 = \frac{\beta_1 \left(a - \bar{q}_1^s\right) + b_2 \left(\alpha - \bar{q}_2^s\right)}{b_1 \beta_1 - b_2 \beta_2} \tag{1.7}$$

$$p_2 = \frac{\beta_2 \left(a - \bar{q}_1^s\right) + b_1 \left(\alpha - \bar{q}_2^s\right)}{b_1 \beta_1 - b_2 \beta_2} \tag{1.8}$$

Notice that the solutions here express the two prices in terms only of the exogenous variables and the parameters—these are the *reduced-form* solutions. We are now set to consider the implications of our initial model.

Statics and Dynamics

In introductory treatments of economics the time dimension of the problem is often ignored, or supressed for simplicity. In reality, the time dimension is always present. In the examples of market equilibrium already discussed, we should think of the quantities as *flows* per period of time, so that q^D is the quantity demanded of a good *per period*, however short or long that period may be in terms of calendar time. Models where we explicitly or implicitly consider a situation within a single period of time we refer to as *static* models. The economic activities studied in static models do not take into account the *history* of these activities and do not consider the future consequences of these activities. Static problems are solved independently of the passage of time. Put this way, it seems that the static framework for economic analysis is extremely restrictive. However, many useful models of economic behavior have been developed in the static framework, and we will be studying several such examples in the book. Many problems, though, are necessarily dynamic in nature. The theories of economic growth, inflation, and resource depletion, for example, are impossible to model without explicit consideration of the time dimension.

Explicit consideration of time opens up new challenges as well as new opportunities for the modeler, and in the later chapters of the book we will develop the more important techniques for studying dynamic models. It is useful at this point, however, to highlight some important concepts that emerge when we explicitly account for the passage of time.

Quantities and values that recur each period in a dynamic model continue to be referred to as *flows*. Income, investment, saving, production, worker hires, and purchase and consumption of goods are examples. Some of these flows may be stored or *accumulated* into *stocks*. Flows of investment in machinery become the accumulated capital stock. Accumulated savings are assets.

Accumulated hires are the workforce. Unsold production accumulates into inventories. Accumulated deficits are total debt, and so on.

Mathematically we must specify the relationships between the flows and the stocks as part of any dynamic model. The relationship between investment and the capital stock is a useful example. Denote by I_t the amount of investment during period t, and denote the capital stock at the begining of period t by K_t. Then we can define the flow of investment in terms of the change in the capital stock between periods (ignoring the depreciation of capital):

$$I_t = K_{t+1} - K_t$$

The flow of investment is simply the change in the stock of capital. Note that we use time subscripts to date the stocks and the flows that we are interested in.

From the modeling point of view, we have certain choices when we are deciding on the appropriate mathematical structure for a dynamic model. One of these is the modeling of time itself. So far we have thought of time as being divided up into intervals or "periods." In these models of *discrete time*, all relevant economic factors are allowed to change between periods but not within periods. Of course, since the length of a period is arbitrary, this condition is not very restrictive. Different mathematical techniques are required if we wish to think of time as evolving continuously. In *continuous time* models, we date the stocks and flows by instants in time, and we can invoke calculus to define the relationships between flows and successive instantaneous values of stocks. We show how this is done later in the book.

1.2 How to Use This Book

The book is intended to be comprehensive in its coverage of mathematical methods for undergraduate economics programs. The arrangement of the material follows the logic of the development of the mathematical ideas rather than those of economics. Examples and exercises relate both to purely mathematical techniques and to their applications in economics. The book also contains extended discussions of some economic applications. Chapters, sections, exercises, examples, and economic applications that we regard as advanced or that might otherwise be omitted are indicated by an asterisk.

Important concepts are highlighted. Key theorems and definitions are displayed, while keywords are set in bold type. Each section ends with exercises, and each chapter ends with a chapter review, consisting of a list of key concepts, questions for discussion or review, and review exercises. Answers to the odd-numbered questions are given at the end of the book.

The third edition of this book is intended to be used along side the book's website at http://mitpress.mit.edu/math_econ3.

We decided to place some of the material from the second edition, such as some proofs and examples, on the website and to add examples and explanations. This improves the balance of the book and allows us to add material without adding to the length of the book, which keeps costs down. The Web materials are intended to be an integral part of the book. The book may be used alone, but we strongly recommend use of the Web materials and for reason of this integration, we cite examples and figures available on the website in the book.

With the two formats combined, we have reduced the length of the hard copy, preserved useful material from the second edition, and added new examples and explanations. Each topic is self-explanatory for both instructors and students. This enhances the value of the book as a reference, since it has models and examples that are likely to come up in a student's future courses (e.g., the Hotelling location model if the student takes a course in Industrial Organization). Some of the material represents extra examples that a student can use for practice or an instructor might feel is important, whereas some of the material is probably more advanced and/or detailed than most (if not all) instructors would think appropriate (e.g., the intermediate value theorem).

To the Instructor . . .

There is clearly more material here than can be covered in a one-semester course. A first course is unlikely to progress beyond chapter 13 or possibly chapter 14. Chapter 16, on integration, is positioned here as a preparation for the chapters on dynamic methods, though it could be brought forward if integration techniques are to be given a higher priority in the single-semester course. A second-semester course could be built around chapter 15 onward.

To the Student . . .

In any course, you will need to attempt exercises, and other material, that have not been covered in class. The only way to learn mathematics and economics is to *do* mathematics and economics. We have provided many examples and exercises in order to encourage independent study.

1.3 Conclusion

The aim of this book is to present the key mathematical concepts that most frequently prove helpful in analyzing economic problems. However, the approach we take is not simply to provide a recipe book of results and procedures. Our aim is to take students through the model-building process by working out a large number of economic examples. By the end of the book, in addition to acquiring particular mathematical skills, we hope that the student will have some knowledge of the main economic models and their properties.

Chapter 2 Review of Fundamentals

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Some Examples of Power Sets
- Proof of Theorem 2.1.
- The Completeness Property of $\mathbb{R}$
- Proofs, the Necessary and Sufficient Conditions
- Practice Exercises

In this chapter we give a concise overview of some fundamental concepts that underlie everything we do in the rest of the book.

In section 2.1 we present the basic elements of set theory. We then go on to discuss the various kinds of numbers, ending with a concise treatment of the properties of real numbers, and the dimensions of economic variables. We then introduce the idea of point sets, beginning with the simplest case of intervals of the real line, and define their most important properties from the point of view of economics: closedness, boundedness, and convexity. Next we give the general definition of a function, and set out the main properties of the types of functions most frequently encountered in economics. We also define the important properties of concavity, convexity, and quasiconcavity and quasiconvexity. Finally, there is a short discussion of the meaning of necessary conditions and sufficient conditions, and of how proofs are formulated, in the context of an economic example.

2.1 Sets and Subsets

A **set** is any collection of items thought of as a whole. The collection is treated as a single object, to which mathematical operations may be applied. One way of defining a particular set is by enumeration: we simply list the items included in the set—the **elements** of the set. Alternatively, we can state a specific property. If an item possesses that property, it is an element of the set, but if it does not, it is excluded from the set. This latter method is far more generally used because defining a set by enumeration is often very cumbersome and sometimes impossible.

For example, consider the set of even numbers between 1 and 11. In the standard notation for describing sets, we could write

$$S = \{x : x \text{ is an even number between 1 and 11}\} \tag{2.1}$$

or, equivalently,

$$S = \{2, 4, 6, 8, 10\} \tag{2.2}$$

The first way of writing S corresponds to definition by property (the ":" should be read as "given that"); the second to definition by enumeration. The key aspects of this notation are as follows:

- A capital letter denoting the set, here S.
- A lowercase letter denoting a typical element of the set, here x.
- Braces, $\{\ldots\}$, that enclose the elements of the set and emphasize that we treat them as a single entity.

In general, a lowercase letter such as x denotes items that may or may not be elements of some particular set. For example, here x can represent any number whatsoever. Then, if x takes on a value that is in the set, we write

$$x \in S$$

and if it takes on a value that is not in the set, we write

$$x \notin S$$

Also, in general, the definition of a set by property may be written

$$X = \{x : P(x)\}$$

where P is a property that x may or may not have, so $P(x)$ is equivalent to the statement "x possesses the property P."

Consider now three further sets of numbers:

$$Z_+ = \{x : x \text{ is a positive integer}\} \tag{2.3}$$

$$A = \{x \in Z_+ : x \leq 11\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} \tag{2.4}$$

$$B = \{x \in A : x/2 \in Z_+\} = \{2, 4, 6, 8, 10\} \tag{2.5}$$

Note that in the case of A and B we have extended the notation slightly by writing the set to which x must belong (a property of A and B) to the left of the colon.

Example 2.1 Define the set Z_+ in equation (2.3) by enumeration.

Solution

The difficulty here is that the set of positive integers is very large. The way around this is to have a notation to stand for "and so on" once a pattern has been established so that there is little ambiguity about what follows. One suggestion in this example might be

$$Z_+ = \{1, 2, 3, 4, \ldots\}$$

where "..." stands for "and so on." Notice, however, that this notation only suggests the nature of Z_+, since any infinitely long sequence of integers could follow 4. ■

We observe also that all the elements of A are also elements of Z_+ and that all the elements of B are also elements of A. This observation leads to the following important definition:

Definition 2.1 If all the elements of a set X are also elements of a set Y, then X is a **subset** of Y, and we write

$$X \subseteq Y$$

where $\subseteq$ is the set-inclusion relation.

In our examples we have $B \subseteq A$ and $A \subseteq Z_+$. Note also that these two facts lead to $B \subseteq Z_+$. Since A contains B and Z_+ contains A, then Z_+ must contain B.

Is Z_+ a subset of A? Clearly not, because there exist numbers $x \geq 12$ that are in Z_+ but not in A. This observation leads to

Definition 2.2 If all the elements in a set X are in a set Y, but not all the elements of Y are in X, then X is a **proper subset** of Y, and we write

$$X \subset Y$$

So, for our examples, we certainly have $A \subset Z_+$. It must then also be true that $B \subset Z_+$, since $B \subseteq A$. Now consider the relationship between A and B. Is B a proper subset of A? Clearly, $B \subset A$ because the odd integers $1, 3, \ldots, 11$ are in A but not in B.

The equality between two sets is defined by

Definition 2.3 Two sets X and Y are equal if they contain exactly the same elements, and we write

$$X = Y$$

Note that the equality of two sets means that they are identical, as is the case with S and B defined earlier. Formally, we demonstrate that two sets are equal by showing that they are both subsets of each other. That is,

$$X \subseteq Y \text{ and } Y \subseteq X \qquad \underbrace{\Leftrightarrow}_{\text{implies and is implied by}} \qquad X = Y$$

So far we have considered examples that are sets of numbers. It is important to emphasize, though, that we can talk of a huge variety of things that we can collect into sets. Some economic examples are as follows:

- The set of all the firms in an economy.
- The set of firms producing a particular good, usually referred to as the *industry* for that good.
- The set of buyers and sellers of a good, usually referred to as the *market* for that good.
- The set of quantities of goods and services that a consumer is physically capable of consuming, usually called the *consumption set* for the consumer.
- The set of bundles of goods and services that a consumer can afford to buy, usually called the *budget set* of the consumer.
- The set of output quantities a firm is technologically capable of producing and the input quantities required to produce these, usually called the *production set* for the firm.
- The set of output quantities technologically capable of being produced in an economy given the available resources, usually called the *production possibility set* for an economy.

Set Operations

We are now going to define operations on sets that can be thought of as roughly analogous to the basic operations of addition, subtraction, multiplication, and division that we carry out on numbers. To avoid some logical problems that can arise if we continue to assume that we are dealing with any kinds of sets whatsoever, we take it that we have some given set of items of some specific kind, called the **universal set,** U, and we define the set-theoretic operations in terms of subsets

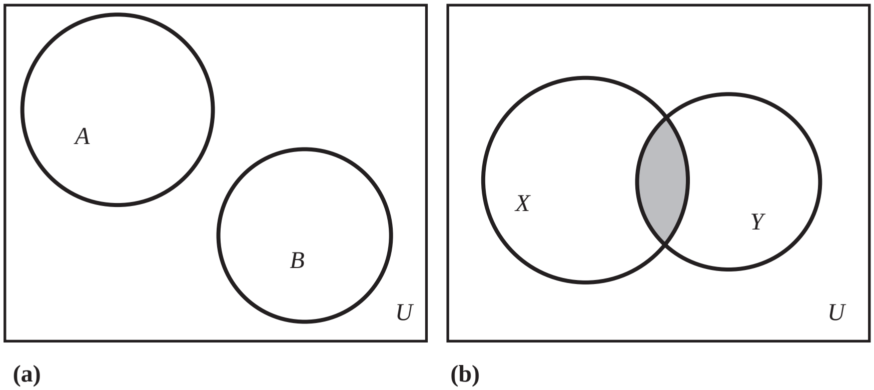


Figure 2.1 Venn diagrams

of U. A **Venn diagram,** shown in figure 2.1, is a useful device for illustrating the relationships between sets and subsets. The rectangle represents the universal set, U, so that any point in U is an item x. The sets we are interested in are shown by collections of points such as A and B or X and Y. Note that this is a purely schematic representation, and U should be thought of as a set of items in general, not necessarily as points in a plane. When we think about the relationship between the sets, we notice a difference between the cases shown in (a) and (b) of figure 2.1. In (b), X and Y overlap, whereas in (a), A and B do not.

Definition 2.4

The **intersection,** W, of two sets X and Y is the set of elements that are in *both* X and Y. We write

$$W = X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

Thus the intersection is the set of common elements. The expression $X \cap Y$ is read "the intersection of X and Y." What about the intersection of the sets A and B in figure 2.1 (a)? Clearly, it does not exist, since there are no common elements shared by A and B. This leads to the idea of a set that has no elements.

Definition 2.5

The **empty set** or the **null set** is the set with no elements. The empty set is always written $\emptyset$.

Since there are no common elements shared by A and B, we can write

$$A \cap B = \emptyset \tag{2.6}$$

and these two sets are said to be **disjoint.** Since all the sets under discussion are in U, we have $\emptyset \subseteq U$. We can also think about the total of elements in a number of sets.

Definition 2.6 The **union** of two sets A and B is the set of elements in one or other of the sets. We write

$$C = A \cup B = \{x : x \in A \text{ or } B\}$$

The expression $A \cup B$ is read "the union of A and B." In figure 2.1 (a), C simply consists of the points in A and the points in B thought of now as a single set. If we let $R = X \cup Y$, then, in figure 2.1 (b), R is the set shaped something like "∞." Note that we must have

$$X \cap Y \subset R, \quad \text{where } R = X \cup Y$$

and if we now define $\emptyset$ to be a proper subset of any nonempty subset of U, we can also write

$$A \cap B \subset C, \quad \text{where } C = A \cup B$$

So we see that intersections of sets are always contained in their unions.

Example 2.2 Take as our universal set the set of positive integers, Z_+, and let

$$X = \{x \in Z_+ : x \le 20 \text{ and } x/2 \in Z_+\}$$

$$Y = \{x \in Z_+ : 10 \le x \le 24 \text{ and } x/2 \in Z_+\}$$

What are $X \cap Y$ and $X \cup Y$?

Solution

The simplest way to answer this question is by enumeration. We have

$$X = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

$$Y = \{10, 12, 14, 16, 18, 20, 22, 24\}$$

Then

$$X \cap Y = \{10, 12, 14, 16, 18, 20\}$$
$$= \{x \in Z_+ : 10 \le x \le 20 \text{ and } x/2 \in Z_+\}$$

$$X \cup Y = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$$
$$= \{x \in Z_+ : x \le 24 \text{ and } x/2 \in Z_+\}$$

Note that when we take the union, we do not count the common elements twice. ■

Example 2.3 What are $X \cap Z_+$, $X \cup Z_+$, $Y \cap Z_+$, $Y \cup Z_+$?

Solution

By referring to figure 2.1 (b) with Z_+ as the universal set, we can quickly establish that

$$X \cap Z_+ = X, \quad X \cup Z_+ = Z_+$$

$$Y \cap Z_+ = Y, \quad Y \cup Z_+ = Z_+$$

This suggests an immediate generalization. If, for two sets S and V we have $S \subseteq V$, then $S \cap V = S$ and $S \cup V = V$. This result is illustrated in figure 2.2. ■

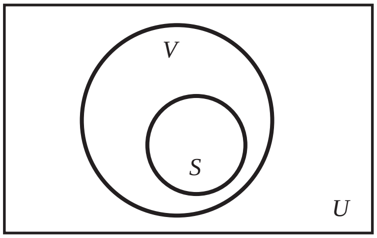


Figure 2.2 $S \cap V = S$ and $S \cup V = V$ when $S \subseteq V$

It is useful to define as a set the elements that are *not* in some given set X.

Definition 2.7 The **complement** of a set X is the set of elements of the universal set U that are not elements of X, and it is written $\bar{X}$. Thus

$$\bar{X} = \{x \in U : x \notin X\}$$

In figures 2.1 and 2.2 the complement of any set is simply the area outside the area denoting the set. Note that we must have

$$\bar{U} = \emptyset, \quad \bar{\emptyset} = U$$

Example 2.4 Given Z_+, X, and Y as defined in example 2.2, what are their complements? What are the complements of $X \cap Y$ and $X \cup Y$, written $\overline{X \cap Y}$ and $\overline{X \cup Y}$? What are $\overline{X \cap Z_+}$ and $\overline{X \cup Z_+}$?

Solution

Since Z_+ is the universal set in this example, we have $\bar{Z}_+ = \emptyset$. Taking our earlier descriptions of X and Y, we have

$$\begin{aligned}
\bar{X} &= \{1, 3, \ldots, 19, 21, 22, \ldots\} \\
&= \{x \in Z_+ : x \leq 20 \text{ and } x/2 \notin Z_+, \text{ or } x > 20\} \\
\bar{Y} &= \{1, 2, \ldots, 9, 11, 13, \ldots, 25, 26, \ldots\} \\
&= \{x \in Z_+ : x < 10, \text{ or } x > 24, \text{ or } 10 \leq x \leq 24 \text{ and } x/2 \notin Z_+\}
\end{aligned}$$

$$\begin{aligned}
\overline{X \cap Y} &= \{1, \ldots, 9, 11, 13, \ldots, 21, 22, \ldots\} \\
&= \{x \in Z_+ : x < 10, \text{ or } x > 20, \text{ or } 10 \leq x \leq 20 \text{ and } x/2 \notin Z_+\}
\end{aligned}$$

$$\begin{aligned}
\overline{X \cup Y} &= \{1, 3, 5, \ldots, 23, 25, 26, \ldots\} \\
&= \{x \in Z_+ : x \leq 24 \text{ and } x/2 \notin Z_+, \text{ or } x > 24\}
\end{aligned}$$

Since $X \cap Z_+ = X$, we have $\overline{X \cap Z_+} = \bar{X}$. Since $X \cup Z_+ = Z_+$, we have $\overline{X \cup Z_+} = \bar{Z}_+ = \emptyset$. ■

These examples show that if we have a set $X \subseteq U$ or $X = \{x \in U : P(x)\}$ so that elements x in X have the property P, then the complement of X is the set of x-values that do not possess property P or $\bar{X} = \{x \in U : \textit{not}\, P(x)\}$.

We can think of the complement of a set $X \subseteq U$ as the difference between the sets X and U. This can be generalized to indicate the difference between any two sets.

Definition 2.8 The **relative difference** of X and Y, denoted $X - Y$, is the set of elements of X that are not also in Y

$$X - Y = \{x \in U : x \in X \text{ and } x \notin Y\}$$

As the Venn diagram in figure 2.1 (b) shows, we can think of $X - Y$ as the part of X remaining when we take out the intersection of X and Y, so we may write

$$X - Y = X \cap \bar{Y}$$

Example 2.5 If $X = Y$, show that $X - Y = Y - X = \emptyset$.

Solution

If $X = Y$, then $X - Y = X - X = X \cap \bar{X} = \emptyset$. If $X = Y$, then $Y - X = Y - Y = Y \cap \bar{Y} = \emptyset$, since no element of a set can be an element of its complement, and vice versa. ■

Example 2.6 If $X \cap Y = \emptyset$, show that $X - Y = X$.

Solution

First note that if $X \cap Y = \emptyset$, then $X \cap \bar{Y} = X$. It then follows immediately that

$$X - Y = X \cap \bar{Y} = X$$ ■

We have considered only individual subsets of the universal set U. We now consider two important collections of subsets.

Definition 2.9 A **partition** of the universal set U is a collection of disjoint subsets of U, the union of which is U.

Thus, if we have n subsets X_i, $i = 1, \ldots, n$, such that

$$X_i \cap X_j = \emptyset, \qquad i, j = 1, \ldots, n$$

and

$$X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_n = U$$

then these n subsets form a partition of U. This result is illustrated in figure 2.3. The key point is that each element of U lies in one and only one of the subsets. Let $\mathcal{S}$ denote this collection of subsets, and let the union of the n subsets be denoted by $\bigcup_{i=1}^{n} X_i$. Then we have

$$\mathcal{S} = \left\{ X_i \subseteq U : \bigcup_{i=1}^{n} X_i = U \text{ and } X_i \cap X_j = \emptyset, \;\; i, j = 1, \ldots, n, i \neq j \right\}$$

is a partition of U.

Example 2.7 Show that $\{X, \bar{X}\}$, for $X \subseteq U$, is a partition of U.

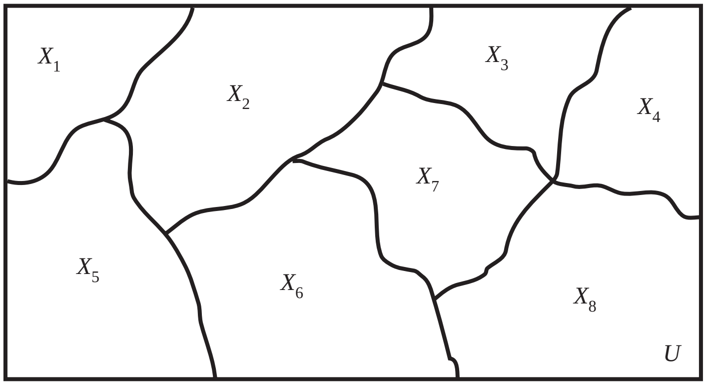


Figure 2.3 A partition of U

Solution

By the definition of a complement, we know that $X \cup \bar{X} = U$. But also, $X \cap \bar{X} = \emptyset$. Thus $\{X, \bar{X}\}$ is a partition of U. ■

Example 2.8 Do the sets X and Y of example 2.2 form a partition of Z_+?

Solution

No, since $X \cap Y \neq \emptyset$ and $X \cup Y \subset Z_+$. ■

Example 2.9 Consider the collection of subsets of Z_+ defined as follows:

$$X_i = \{x \in Z_+ : 10(i-1) < x \leq 10i, \;\; i \in Z_+\}$$

Does the collection of these X_i form a partition of Z_+?

Solution

The first three subsets are

$$X_1 = \{x \in Z_+ : 0 < x \leq 10\}$$

$$X_2 = \{x \in Z_+ : 10 < x \leq 20\}$$

$$X_3 = \{x \in Z_+ : 20 < x \leq 30\}$$

Our intuition immediately tells us that this collection does form a partition of Z_+. A proof goes as follows: First, we have to prove that the subsets are disjoint, so we take X_i and X_j with $j > i$ (and $j \geq i + 1$). If $x \in X_i$, then $x \leq 10i \leq 10(j - 1)$, and so $x \notin X_j$. If $x \in X_j$, then $x > 10(j - 1) \geq 10i$ and so $x \notin X_i$. Thus the subsets are disjoint. To prove that any element of Z_+ is in one of these sets, take any $x \in Z_+$ and note that either x is a multiple of 10 or it is not. If it is, let $i = x/10$. Then we have $x \in X_i$. If it is not, $x/10$ can be written $a + (b/10)$, where $a \in Z_+$ and $(b/10) < 1$. Then set $i = a + 1$, and it is straightforward to show that $x \in X_i$. ■

Definition 2.10

The **power set** of a set X is the set of *all* subsets of X; it is written $\mathcal{P}(X)$. That is, $\mathcal{P}(X) = \{A : A \subseteq X\}$.

Example 2.10

Find the power set of $X = \{1, 2, 3\}$.

Solution

Note first that $\emptyset$ is a subset of every set $X \subseteq U$. By convention, we take X to be a subset of X. We also have $\{1\}, \{2\}, \{3\}$, and the sets of pairs to give

$$\mathcal{P}(X) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X, \emptyset\}$$ ■

Note that $\mathcal{P}(X)$ in this example has exactly $2^3 = 8$ elements. For any set X with a finite number n of elements, it can be shown that its power set has exactly 2^n elements. In other words, a set with n elements has 2^n subsets.

EXERCISES

1. Explain the difference between the statements

$$x \in X$$

and

$$A \subset X, \qquad \text{where } A = \{x\}$$

2. Define the relationships ($\subseteq, \subset, =$), if any, among the following sets:

$$A = \{x : 0 \leq x \leq 1\}$$

$$B = \{x : 0 < x < 1\}$$

$$C = \{x : 0 \leq x < 1\}$$

$$D = \{x : 0 \leq x^2 \leq 1\}$$

$$E = \{x : 0 \leq x < 1/2 \text{ and } 1/2 \leq x \leq 1\}$$

3. Given the set $S = \{1, 2, 3, 4, 5\}$, define all its possible subsets. How many of them are there?

4. Given any set X, it must be true that $X \subseteq X$ (though *not* $X \subset X$). Explain the difference between the statement "X is a subset of itself" and the statement "X is an element of itself."

5. Are the two sets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ equal?

6. Prove that the set $X = \{x : x^3 > 0 \text{ and } x < 0\}$ equals the empty set.

7. A consumer's consumption set is given by

$$C = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$$

and her budget set is given by

$$B = \{(x_1, x_2) : p_1 x_1 + p_2 x_2 \leq M\}$$

where x_1 and x_2 are quantities of goods, p_1, $p_2 > 0$ are prices, and $M > 0$ is income.

Illustrate in a diagram the sets

(a) B, C

(b) $B \cup C$

(c) $B \cap C$

and interpret each of these.

8. Given two subsets X and Y of a universal set U, prove that

(a) $\overline{X \cap Y} = \bar{X} \cup \bar{Y}$

(b) $\overline{X \cup Y} = \bar{X} \cap \bar{Y}$

(c) $X - Y = X \cap \bar{Y}$

(d) $X \subseteq Y$ implies $\bar{Y} \subseteq \bar{X}$

(e) $X \subseteq Y$ implies $X \cup (Y - X) = Y$

(f) $X - Y \subseteq X \cup Y$

(g) $X \cap Y = \emptyset$ implies $Y \cap \bar{X} = Y$

Illustrate each case on a Venn diagram.

9. A firm's production set is given by

$$P = \{(x, y) : 0 \leq y \leq \sqrt{x}, 0 \leq x \leq \bar{x}\}$$

where y is output and x is labor input. Sketch and interpret P in economic terms. How would you interpret $\bar{x}$?

10. Prove that for subsets X, Y, and Z of a given universal set U

(a) $X - Y = X - (X \cap Y) = (X \cup Y) - Y$

(b) $(X - Y) - Z = X - (Y \cup Z)$

(c) $X - (Y - Z) = (X - Y) \cup (X \cap Z)$

(d) $(X \cup Y) - Z = (X - Z) \cup (Y - Z)$

(e) $X - (Y \cup Z) = (X - Y) \cap (X - Z)$

Illustrate each case on a Venn diagram.

2.2 Numbers

The most basic and familiar kinds of numbers are **natural numbers,** the elements of the set

$$Z_+ = \{1, 2, 3, \ldots\}$$

They arise naturally in counting objects of all kinds. What does it mean to *count* a set of objects, say a pile of dollar bills? When we count dollar bills, we take each element in the set of dollar bills and pair it with an element of Z_+, starting with 1 and moving successively through the set. When we have exhausted the elements of the first set (of dollar bills), the element of Z_+ that we have reached in this process gives us the number of dollar bills. This number is called the **cardinality** of the set of dollar bills: the cardinality of a set is the number of objects it contains. As we saw in section 2.1, Z_+ has an infinite number of elements. Given any positive integer, we can always find a larger one. We now define some properties of natural numbers and show that other types of numbers arise from the operations of addition, multiplication, subtraction, and division.

Consider the addition or multiplication of any two elements, a and b, of Z_+. These two operations are closely related, since multiplying a and b is simply adding a to itself b times or b to itself a times. The first thing we note is that the result of this is itself always an element of Z_+:

$$a + b \in Z_+, \quad ab \in Z_+$$

This is formally expressed by saying that the set Z_+ is *closed under the operations of addition and multiplication*.

Is Z_+ closed under the operations of subtraction and division? Clearly not. If $a \leq b$, we have

$$a - b \leq 0 \notin Z_+$$

so Z_+ is not closed under subtraction. It is easy to find cases where a/b is not an integer.

The fact that Z_+ is not closed under subtraction leads naturally to the definition of the set of **integers**

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

where the three dots indicate that we go out to infinity in each direction. Z is a somewhat more abstract set than Z_+, since it is hard to imagine observing a set containing, say -3 objects. Nevertheless, it is impossible in general to consider solving the simplest kind of equation

$$x + b = a$$

unless we have negative numbers available. Intuitively we can think of a negative number as "something owing"—a kind of debt. It is also useful to represent Z as in figure 2.4. Along a horizontal line we mark off intervals of equal length, choose a central point to represent zero, and measure the positive integers in ascending value to the right and negative integers in descending value to the left (so that $-a < -b$ if $a > b > 0$). It is also clear that $Z_+ \subset Z$.

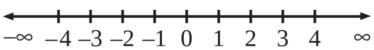


Figure 2.4 The set Z

The set of integers is closed under addition, subtraction, and multiplication. However, neither Z nor Z_+ is closed under division, since, for example, $a/b \notin Z$ if $a = -2$ and $b = 3$, though both a and b are in Z. This means, for example, that we could not in all cases find $x \in Z$ such that

$$bx = a, \quad a, b \in Z$$

This leads to the definition of the set Q of **rational numbers**

$$Q = \left\{ \frac{a}{b} : a \in Z, b \in Z - \{0\} \right\}$$

Note that $Z \subset Q$, since we could clearly choose $a = kb$ for $k \in Z$. Note also that we rule out division by zero. We say that any expression involving zero in the denominator is *undefined*. The reason for this can be seen from the equation $bx = a$. If $b = 0$, then no x exists such that $bx = a$ for any $a \neq 0$, and so ruling out division by zero recognizes this fact. (The term "rational number" comes from the fact these numbers are ratios of integers.) Figure 2.5 shows the set of rational numbers and indicates that these include points between the integers. Take any two points on the line that give distinct rational numbers. There is an infinity of other rational numbers between those points. To see this, consider the two rational numbers 1 and 2, and note that $1 + (2 - 1)/c$ with $c \in Z_+$, must be a rational number between 1 and 2. Each value of c gives a different rational number, and since Z_+ has an infinite number of elements, there must be an infinite number of rational numbers between 1 and 2. Replacing 1 by a and 2 by $b > a$, where a and b are any elements in Q, shows that this must be true in general.

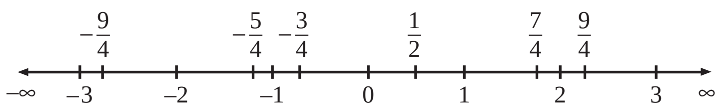


Figure 2.5 The set Q

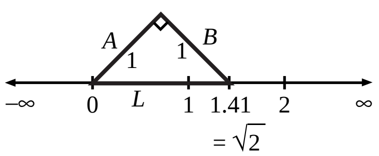


Figure 2.6 $\sqrt{2}$ is irrational

An important observation is that the segment between the numbers 1 and 2 (or between any two rational numbers) is not entirely composed of rational numbers. Other numbers exist. In figure 2.6 we have constructed a right-angle triangle with sides A and B of length 1 and hypotenuse of length L lying along the line. From the Pythagorean theorem we know that the square of the distance L along the line is given by

$$L^2 = 1^2 + 1^2 = 2$$

so $L = \sqrt{2}$. The following theorem, the proof of which was well known to the ancient Greeks, shows that $\sqrt{2}$ is *not* a rational number. This tells us that Q is not closed under the operation of taking square roots. Alternatively, it tells us that there must be numbers other than rational numbers. These are **irrational** numbers, which cannot be expressed as the ratio of two integers.

Theorem 2.1 The number $\sqrt{2}$ is not a rational number, that is, $\sqrt{2} \notin Q$.

The proof can be found on the website.

The Real Numbers and Their Properties

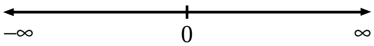


Figure 2.7 The set of real numbers, $\mathbb{R}$

The union of the sets of rational and irrational numbers is the set of **real numbers.** We think of the set of real numbers, $\mathbb{R}$, as extending along a line to infinity in each direction having no breaks or gaps, as in figure 2.7. We refer to this as the **real line.**

The properties of $\mathbb{R}$ define the basic operations that we can carry out on the elements of $\mathbb{R}$. Consider three (not necessarily distinct) elements of $\mathbb{R}$: a, b, and c. We can postulate the following properties:

1. **Closure** If $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$. So $\mathbb{R}$ is closed under addition and multiplication.
2. **Commutative laws** For all $a, b \in \mathbb{R}$

$$a + b = b + a \quad \text{and} \quad ab = ba$$

which simply says that the order in which we add or multiply two real numbers does not affect the outcome.

3. **Associative laws** For all $a, b, c \in \mathbb{R}$,

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

so that the order in which we add or multiply three real numbers does not matter.

4. **Distributive law** For all $a, b, c \in \mathbb{R}$,

$$a(b + c) = ab + ac$$

5. **Zero** The element $0 \in \mathbb{R}$ is defined as having the property that for all $a \in \mathbb{R}$,

$$a + 0 = a \quad \text{and} \quad a0 = 0$$

6. **One** The element $1 \in \mathbb{R}$ is defined as having the property that

$$1(a) = a$$

7. **Negation** For each $a \in \mathbb{R}$, there is a number $-a \in \mathbb{R}$ defined as having the property

$$a + (-a) = 0$$

8. **Reciprocals** For each element $a \in \mathbb{R} - \{0\}$, there is an element $1/a \in \mathbb{R}$ defined as having the property

$$a\left(\frac{1}{a}\right) = 1$$

For $a = 0$, the reciprocal is undefined.

Example 2.11 For all $a, b \in \mathbb{R}$, if $a + b = 0$, show that $b = -a$.

Solution

Start with $a + b = 0$ and add $-a$ to both sides to obtain

$$\begin{aligned} -a + a + b &= -a \\ (-a + a) + b &= -a && \text{(from property 3)} \\ 0 + b &= -a && \text{(from property 7)} \\ b &= -a && \text{(from property 5)} \end{aligned}$$

■

The following are also the more obvious consequences of these properties:

(a) For all $a \in \mathbb{R}$, $-a = (-1)a$.
(b) For all $a, b \in \mathbb{R}$, $-(a + b) = (-a) + (-b)$.
(c) For all $a \in \mathbb{R}$, $-(-a) = a$.
(d) For all $a, b \in \mathbb{R}$, $(-a)(-b) = ab$.

Indeed all the standard rules of algebra follow from properties 1 through 8.

The Order Properties of $\mathbb{R}$

First we define two important subsets of $\mathbb{R}$.

Definition 2.11 The set $\mathbb{R}_{++} \subset \mathbb{R}$ consists of the strictly positive real numbers with the characteristics that

(i) $\mathbb{R}_{++}$ is closed under addition and multiplication.

(ii) For any $a \in \mathbb{R}$, exactly one of the following is true:

$$a \in \mathbb{R}_{++} \quad \text{or} \quad a = 0 \quad \text{or} \quad -a \in \mathbb{R}_{++}$$

Definition 2.12 The set $\mathbb{R}_{+} = \mathbb{R}_{++} \cup \{0\}$ is the set of **nonnegative** real numbers.

Diagrammatically, the set $\mathbb{R}_{++}$ is the right half of the real line in figure 2.7, excluding zero, while $\mathbb{R}_{+}$ is that half including zero. We may similarly identify the left half of the real line, excluding zero, as the complement $\mathbb{R}_{\underline{+}}$ (note that this is not closed under multiplication) and including zero as $\mathbb{R}_{++} = \mathbb{R}_{+} \cup \{0\}$.

The inequality symbols "$>$" and "$\geq$" may be given formal meaning in terms of $\mathbb{R}_{++}$ and $\mathbb{R}_{+}$.

Definition 2.13 Given any $a, b \in \mathbb{R}$:

(i) If $a - b \in \mathbb{R}_{++}$, then $a > b$.
(ii) If $-(a - b) \in \mathbb{R}_{++}$, then $b > a$.
(iii) If $a - b \in \mathbb{R}_{+}$, then $a \geq b$.
(iv) If $-(a - b) \in \mathbb{R}_{+}$, then $b \geq a$.

We refer to "$>$" as the strict inequality and "$\geq$" as the weak inequality, since it permits $a = b$. The properties of these inequalities are stated in the following theorem.

Theorem 2.2 For any $a, b, c \in \mathbb{R}$:

(i) **Completeness** Exactly one of the following is true:

$$a > b \quad \text{or} \quad a = b \quad \text{or} \quad a < b$$

(ii) **Transitivity** If $a > b$ and $b > c$ then $a > c$, and if $a \geq b$ and $b \geq c$ then $a \geq c$.

(iii) **Reflexivity** $a \geq a$

(iv) **Equality** If $a \geq b$ and $b \geq a$ then $a = b$.

Dimensions of Economic Variables

The variables in economic models are usually represented by real numbers corresponding to quantities such as cost, profit, and price or to amounts of goods and services such as coal, bread, and haircuts. These variables are measured in particular kinds of units that we refer to as the **dimensions** of these variables. For example, cost, revenue, and profit are measured in units of money (dollars); a price is measured in terms of units of money *per* physical unit of a good (dollars/unit quantity); coal, potatoes, wine will be measured by weight (pounds, kilograms) or volume (liters, gallons), and so on.

Economists, unlike engineers and physicists, are often rather careless about defining the way in which their variables are measured. The standard rules of arithmetic apply not only to the variables themselves but also to the units in which they are measured—you cannot add apples and pears! Take, as an example, total revenue measured in dollars. This is the product of the total sales of physical output, whose units depend on the output concerned, say tons, and the price per ton. Thus, corresponding to the identity

$$\text{Revenue} = \text{price} \times \text{quantity}$$

we have the identity in terms of dimensions

$$\text{Dollars} = \frac{\text{dollars}}{\text{ton}} \times \text{tons}$$

An equation involving economic variables that does not satisfy the arithmetic relationship among the dimensions of the variables cannot be a valid equation. For example, suppose that after a complicated bit of mathematical analysis we derive the solution

$$\text{Revenue} = \text{marginal cost}$$

Since revenue is measured in dollars and marginal cost in dollars *per* unit, we are proposing the equation

$$\text{Dollars} = \frac{\text{dollars}}{\text{ton}}$$

which cannot be true. Maybe what we should have obtained was

$$\text{Marginal revenue} = \text{marginal cost}$$

Dimensionally, this equation is correct, with units of dollars per ton on each side.

A **pure number** is a number that does not have a dimension. For example, consider profit expressed as a rate of return on sales revenue:

$$\text{Rate of return} = \frac{\text{profit}}{\text{revenue}}$$

Since both profit and revenue are measured in dollars, the rate of return has dimension dollars/dollars and so is a pure number.

EXERCISES

1. Demonstrate the boundedness or unboundedness of the following sets:

(a) $Z_+ = \{1, 2, 3, \ldots\}$

(b) $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(c) $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$

(d) $\bar{\mathbb{R}}_+ = \mathbb{R} - \mathbb{R}_+ = \{x \in \mathbb{R} : x < 0\}$

(e) $S = \{x \in \mathbb{R} : 0 < x < \sqrt{2}\}$

2. We could state a "completeness property" for Q as: Every nonempty subset of Q that has an upper bound has a supremum in Q.

Show by choosing a suitable counterexample that this statement is false.

3. Give the dimension of the variable λ in each of the following expressions:

(a) $\dfrac{\text{wage rate}}{\text{marginal product}} = \lambda$

(b) change in national income $= \lambda \times$ change in investment

(c) $\dfrac{\text{profit}}{\text{amount of labor used}} = \lambda$

(d) tax per unit of a good $= \dfrac{1}{\lambda} \times$ elasticity of demand for the good

(e) $\dfrac{\text{change in profit}}{\text{change in import quota of a good}} = \lambda$

4. Give the dimensions of the following economic quantities:

(a) gross national product

(b) average cost of producing a good

(c) marginal propensity to consume

(d) demand for a good

(e) rate of inflation

(f) marginal product of capital in producing a good

5. A subset of $\mathbb{R}$ *has a maximum* if it contains its supremum. This supremum is then the maximum of the set. Give examples of subsets of $\mathbb{R}$ (including bounded subsets) that do and do not have a maximum.

6. The interest on a loan of \$1 is the amount of money that must be paid after a specified period of time over and above the repayment of the \$1 loan. Show that the *rate of interest* is a number expressed only in time units, varying inversely with time. How would you express a rate of interest per year as a rate of interest per week?

2.3 Some Properties of Point Sets in $\mathbb{R}^n$

In section 2.2 we saw that a point on the real line always corresponds to a real number. We now place two real lines at right angles so that they intersect at the number 0, as in figure 2.8. Any point in the **coordinate system** formed by these two lines can be defined as an **ordered pair** of numbers (x, y) by assigning to x the real number vertically below it on the horizontal axis, and assigning to y the real number horizontally across from it on the vertical axis. The figure shows some examples. The two axes, which go off to infinity in the four directions, can be thought of as defining an entire space of points or ordered pairs (x, y), a two-dimensional space that we refer to as $\mathbb{R}^2$. Note that we refer to these points as *ordered* pairs because the order of the numbers in the pair is essential. The pair, or point $(2, 3)$ is quite different from the point $(3, 2)$, as figure 2.8 makes clear.

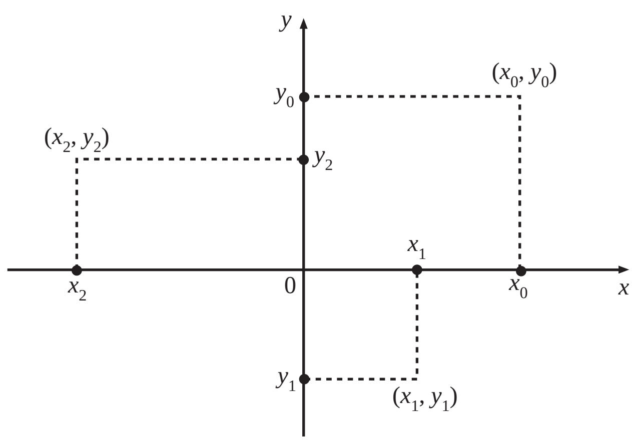


Figure 2.8 $\mathbb{R}^2$

This idea can be put more formally as follows. The **Cartesian product** of two sets X and Y, written $X \otimes Y$, is the set of ordered pairs formed by taking in turn each element in X and associating with it each element in Y. For example, the Cartesian product of the sets $\{1, 2, 3\}$ and $\{a, b\}$ is

$$\{1, 2, 3\} \otimes \{a, b\} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

We can then define the Cartesian product of the set of real numbers $\mathbb{R}$ with itself as the set of ordered pairs:

$$\mathbb{R} \otimes \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\} = \mathbb{R}^2$$

Then figure 2.8 gives a picture of $\mathbb{R}^2$.

But why stop there? We can define the set of ordered triples

$$\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} = \{(x, y, z) : x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\} = \mathbb{R}^3$$

and a picture of this is shown in figure 2.9.

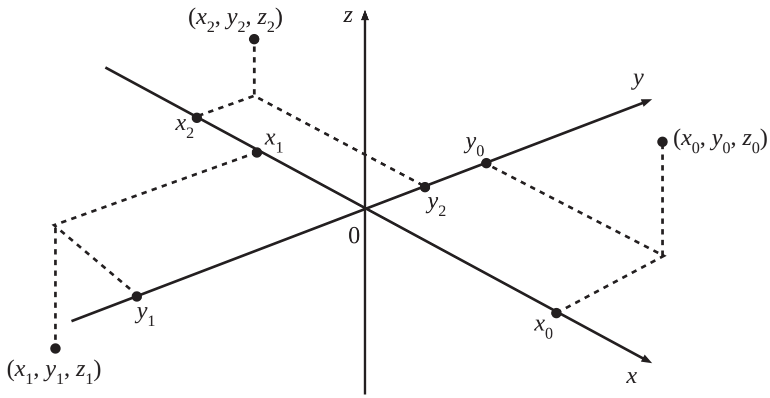


Figure 2.9 $\mathbb{R}^3$

Although figures 2.8 and 2.9 exhaust the set of possibilities for diagrammatic representation, we can define the general set of ordered n-tuples

$$\mathbb{R} \otimes \mathbb{R} \otimes \cdots \otimes \mathbb{R} = \{(x, y, \ldots, z) : x \in \mathbb{R}, y \in \mathbb{R}, \ldots, z \in \mathbb{R}\} = \mathbb{R}^n$$

of which $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ are special cases. In chapter 10 we will consider the algebra of these n-tuples, or vectors. Here we are interested only in some properties of subsets of $\mathbb{R}^n$, which we refer to as **point sets.**

Some important point sets of $\mathbb{R}$ are **intervals.** Given $a, b \in \mathbb{R}$ with $b > a$, we define

- the **closed interval:** $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- the **half-open interval:** $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- the **half-open interval:** $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- the **open interval:** $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

The defining characteristic of the closed interval is that it contains both its boundary points a and b, while the open interval contains neither of these points. Note the convention for using "round" brackets for "open" and "square" brackets for "closed." All the above four intervals are **bounded,** in contrast to which we have the intervals

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$
$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$

which are unbounded above, and

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$
$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

which are unbounded below. An interval is bounded if it is impossible to go off to infinity while remaining inside it, and it is unbounded otherwise. Notice that closedness and boundedness are separate ideas. The intervals $[a, \infty)$ and $(-\infty, b]$ are closed because they contain those boundary points they possess, even though they are unbounded. An interval that is both closed and bounded is called **compact.** Of the above intervals, only $[a, b]$ is compact.

We can distinguish between an *interior point* and a *boundary point* of an interval such as $[a, b]$ in the following way. It is clearly true of a boundary point such as b that *every* interval around it, however small, must contain points that are in $[a, b]$, and points that are not. For example, consider the interval $(b - \epsilon, b + \epsilon)$ shown in figure 2.10. The points that satisfy $b - \epsilon < x < b$ lie inside $[a, b]$, while points that satisfy $b < x < b + \epsilon$ lie outside $[a, b]$ no matter how small ϵ. In the case of an interior point, on the other hand, it will always be possible to find an interval around it that is entirely in $[a, b]$. For example, if x_0 is an interior point of $[a, b]$, then the interval $(x_0 - \epsilon, x_0 + \epsilon)$ certainly lies completely inside $[a, b]$, if we choose ϵ to be the smaller of $(x_0 - a)/2$ and $(b - x_0)/2$. Because of the completeness property of $\mathbb{R}$, such an ϵ can always be defined, however close x_0 is to a or b. Figure 2.10 illustrates this result.

Figure 2.10 Boundary and interior points

Finally, we consider the property of **convexity** in this simple context of intervals of $\mathbb{R}$. A property that the interval in figure 2.10 clearly possesses is the following: Take any two points in the interval, say x_0 and b, and note that all the points on the line segment between these two points also lie in the interval. This is the property of convexity of the interval. More formally, note that if x and x' are any two points in an interval with $x < x'$, then the number

$$\bar{x} = \lambda x + (1 - \lambda)x', \qquad \lambda \in [0, 1]$$

corresponds to a point lying between x and x'. For $\lambda \in (0, 1)$, $\bar{x}$ lies *strictly* between x and x'. Perhaps the easiest way to see this is to rewrite the equation as

$$\bar{x} = x' - \lambda(x' - x), \qquad \lambda \in [0, 1]$$

Thus we find $\bar{x}$ by subtracting some proportion λ of the difference between x and x' from the higher value x'. The lowest value $\bar{x}$ can therefore take is x and the highest value is x'. As λ varies from 1 to 0, it generates all the points in $[x, x']$.

We call $\bar{x}$ the convex combination of x and x', and the property of convexity of an interval can be stated as follows: If x, x' belong to some interval $X \subset \mathbb{R}$, then their convex combination $\bar{x} = \lambda x + (1 - \lambda)x'$, $\lambda \in [0, 1]$, also belongs to this interval.

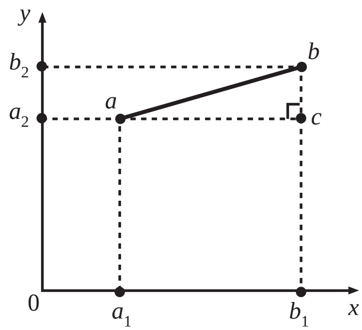


Figure 2.11 Euclidean distance in $\mathbb{R}^2$

We now want to extend these ideas of boundedness, closedness, and convexity to points in $\mathbb{R}^2$ (ordered pairs), in $\mathbb{R}^3$ (ordered triples), or $\mathbb{R}^n$ (ordered n-tuples). To do so we need to introduce the idea of the **distance** between two points in $\mathbb{R}^n$. In $\mathbb{R}$ we think intuitively of the distance between two points a and b with $a < b$, simply as the difference $b - a$. In $\mathbb{R}^2$ we use the concept of Euclidean distance between two points, defined as the length of a line segment between those two points (see figure 2.11).

To find an expression for the length of the line segment ab, we note that it is the hypotenuse of the right-angled triangle abc. By the Pythagorean theorem we then have

$$(ab)^2 = (ac)^2 + (bc)^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

We denote the distance between a and b by $d(a, b)$, and so

$$d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

(taking always the positive square root). This is called the *Euclidean distance function in* $\mathbb{R}^2$.

The concept of distance generalizes readily. Given any two points $a, b \in \mathbb{R}^n$, $n \geq 1$, we have

Definition 2.14 Given points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$, $n \geq 1$, the **Euclidean distance** between them is

$$d(a, b) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

Example 2.12 Find the Euclidean distances between the following points:

(i) 2 and 3 in $\mathbb{R}$
(ii) (2, 3) and (4, 1) in $\mathbb{R}^2$
(iii) (2, 3, 4) and (4, 1, −5) in $\mathbb{R}^3$
(iv) (2, 3, 4, 5) and (−2, 4, 1, −5) in $\mathbb{R}^4$

Solution

(i) $d(2, 3) = \sqrt{(2-3)^2} = 1$
(ii) $d[(2, 3), (4, 1)] = \sqrt{(2-4)^2 + (3-1)^2} = \sqrt{8} = 2.83$
(iii)

$$\begin{aligned} d[(2, 3, 4), (4, 1, -5)] &= \sqrt{(2-4)^2 + (3-1)^2 + (4-(-5))^2} \\ &= \sqrt{89} = 9.43 \end{aligned}$$

(iv)

$$\begin{aligned} &d[(2,\ 3,\ 4, 5), (-2, 4, 1, -5)] \\ &\quad = \sqrt{(2-(-2))^2 + (3-4)^2 + (4-1)^2 + (5-(-5))^2} \\ &\quad = \sqrt{126} = 11.22 \end{aligned}$$

■

Given this definition of distance, we can now generalize the ideas of openness, closedness, boundedness, and convexity to points sets in general. First we define an ϵ-neighborhood.

Definition 2.15 An **ϵ-neighborhood** of a point $x_0 \in \mathbb{R}^n$ is given by the set $N_\epsilon(x_0) = \{x \in \mathbb{R}^n : d(x_0, x) < \epsilon\}$. Simply, $N_\epsilon(x_0)$ is the set of points lying within a distance ϵ of x_0.

Example 2.13 Describe the following ϵ-neighborhoods:

(i) $N_\epsilon(2)$
(ii) $N_\epsilon[(2, 3)]$
(iii) $N_\epsilon[(2, 3, 1)]$

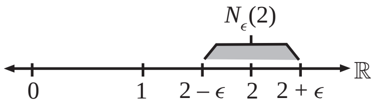


Figure 2.12 $N_\epsilon(2)$

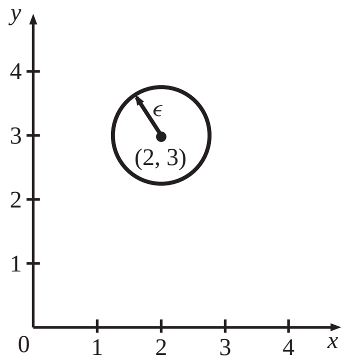


Figure 2.13 $N_\epsilon[(2, 3)]$

Solution

(i) $N_\epsilon(2) = \{x \in \mathbb{R} : \sqrt{(x-2)^2} < \epsilon\}$. This is the open interval $(2-\epsilon, 2+\epsilon)$, as figure 2.12 illustrates.

(ii) $N_\epsilon[(2, 3)] = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x-2)^2 + (y-3)^2} < \epsilon\}$. This is the set of points in $\mathbb{R}^2$ lying inside a circle centered on (2, 3), and with radius ϵ. See figure 2.13.

(iii) $N_\epsilon[(2, 3, 1)] = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2} < \epsilon\}$. This is the set of points lying within a sphere centered at (2, 3, 1) and with radius ϵ. Figure 2.14 illustrates this neighborhood. ■

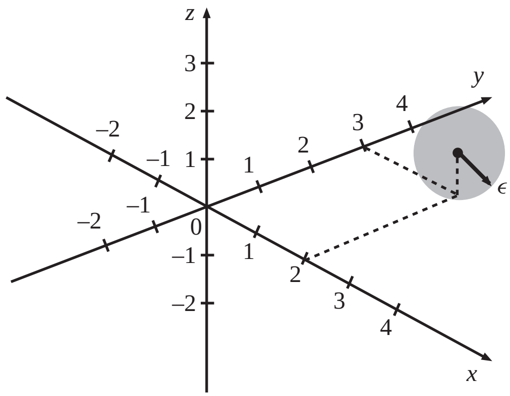


Figure 2.14 $N_\epsilon[(2, 3, 1)]$

Definition 2.16 A set $X \subset \mathbb{R}^n$ is **open** if, for every $x \in X$, there exists an ϵ such that $N_\epsilon(x) \subset X$.

Thus a set is open if it is possible at every point within it to find an ϵ-neighborhood of that point that lies entirely within the set.

Definition 2.17 A **boundary point** of a set $X \subset \mathbb{R}^n$ is a point x_0 such that *every* ϵ-neighborhood $N_\epsilon(x_0)$ contains points that are in and points that are not in X.

Given these two definitions, it is clear that an open set does not contain any boundary points it may have.

Definition 2.18 A set $X \subset \mathbb{R}^n$ is **closed** if its complement $\bar{X} \subset \mathbb{R}^n$ is an open set.

Quite simply, a closed set is one that contains its boundary points.

Definition 2.19 A set $X \subset \mathbb{R}^n$ is **bounded** if, for every $x_0 \in X$, there exists an $\epsilon < \infty$ such that $X \subset N_\epsilon(x_0)$.

In other words, a set is bounded if it can be enclosed in a (sufficiently large) ϵ-neighborhood of any of its points.

To formalize the definition of convexity of a set in $\mathbb{R}^n$, we need first to generalize the idea of a convex combination.

Definition 2.20 Given two points $x, x' \in \mathbb{R}^n$, their **convex combination** is the set of points $\bar{x} \in \mathbb{R}^n$ for some $\lambda \in [0, 1]$, given by

$$\begin{aligned} \bar{x} &= \lambda x + (1-\lambda)x' \\ &= [\lambda x_1 + (1-\lambda)x'_1, \ldots, \lambda x_n + (1-\lambda)x'_n] \end{aligned}$$

Example 2.14 Find the convex combinations of the points

(i) (2, 1) and (−3, 2)
(ii) (2, 1, 0) and (−3, 2, 1)
(iii) (2, 1, 0, −2) and (−3, 2, 1, 5)

Solution

(i) $\bar{x} = \lambda(2, 1) + (1-\lambda)(-3, 2) = [2\lambda - 3(1-\lambda), \lambda + 2(1-\lambda)]$. For example, if $\lambda = 1/2$, $\bar{x} = (-1/2, 3/2)$.
(ii) $\bar{x} = \lambda(2, 1, 0) + (1-\lambda)(-3, 2, 1) = [2\lambda - 3(1-\lambda), \lambda + 2(1-\lambda), 0 + (1-\lambda)]$. For example, if $\lambda = 1/4$, $\bar{x} = (-7/4, 7/4, 3/4)$.
(iii) $\bar{x} = \lambda(2, 1, 0, -2) + (1-\lambda)(-3, 2, 1, 5)$ Or $\bar{x} = [2\lambda - 3(1-\lambda), \lambda + 2(1-\lambda), 0 + (1-\lambda), -2\lambda + 5(1-\lambda)]$. For example, if $\lambda = 2/3$, $\bar{x} = (1/3, 4/3, 1/3, 1/3)$. ■

In definition 2.20, we have implicitly introduced the idea of *adding* points in $\mathbb{R}^n$ by adding their corresponding coordinates. This is a first step in developing linear algebra, which we will take up in detail in chapters 7 to 10.

Intuitively in $\mathbb{R}^2$ and $\mathbb{R}^3$ the convex combination of two points consists of the set of points lying on the line segment between those points, as figures 2.15 and 2.16 illustrate.

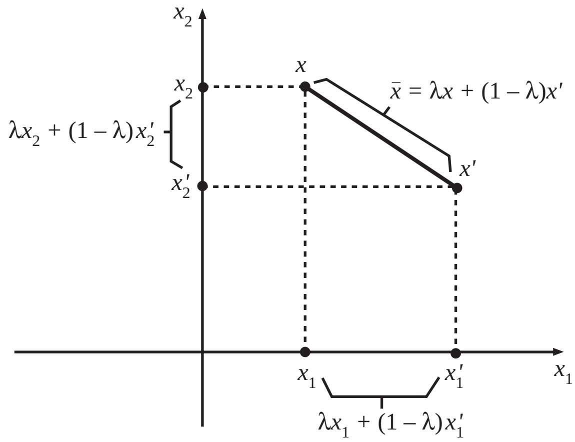


Figure 2.15 Convex combination in $\mathbb{R}^2$

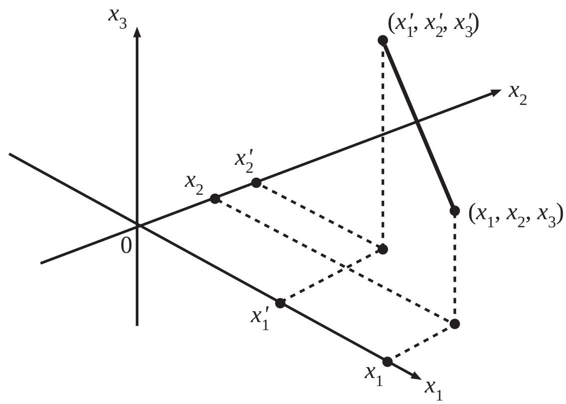


Figure 2.16 Convex combination in $\mathbb{R}^3$

Definition 2.21 A set $X \subset \mathbb{R}^n$ is **convex** if for every pair of points $x, x' \in X$, and any $\lambda \in [0, 1]$, the point

$$\bar{x} = \lambda x + (1 - \lambda)x'$$

also belongs to the set X.

In words, a set is convex if every point on the line segment between every pair of points in the set is also in the set. In (a) of figure 2.17 we show some convex sets in $\mathbb{R}^2$, and in (b) some nonconvex sets. Although the idea of convexity is a very simple one geometrically, it is extremely important.

Definition 2.22 An **interior point** of a set $X \subset \mathbb{R}^n$ is a point $x_0 \in X$ for which there exists an ϵ such that $N_\epsilon(x_0) \subset X$.

Thus we can always find an ϵ-neighborhood of an interior point that lies entirely within the set. Then we have

Definition 2.23 A set $X \subset \mathbb{R}^n$ is **strictly convex,** if for every pair of points $x, x' \in X$, and every $\lambda \in (0, 1)$, we have that $\bar{x}$ is an interior point of X, where

$$\bar{x} = \lambda x + (1 - \lambda)x'$$

In figure 2.17 (a), only the sets A and B are strictly convex. Note that in the definition we exclude the cases $\lambda = 0$ and $\lambda = 1$ because x or x' could be a boundary point of a set.

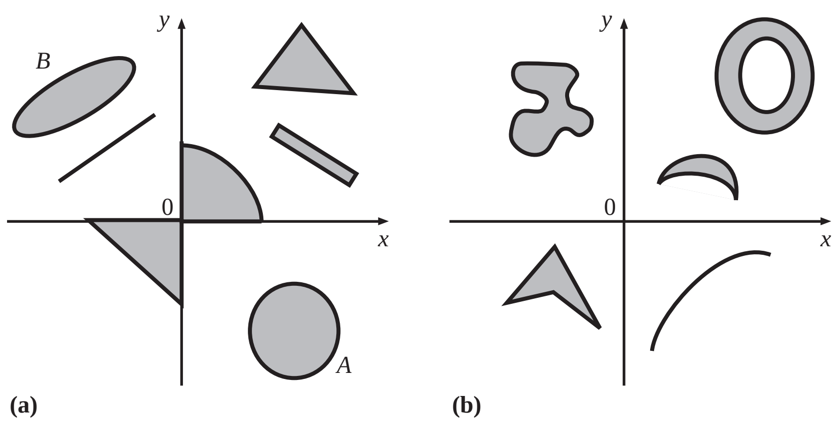


Figure 2.17 Convex and nonconvex sets in $\mathbb{R}^2$

EXERCISES

1. Form the Cartesian products of the following sets:

(a) $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8, 9\}$

(b) Z_+ and Z_+

(c) The set of even elements of Z_+ and the set of odd elements of Z_+

Illustrate these product sets in $\mathbb{R}^2$.

2. The consumption set of a consumer is

$$C = \{(x, y) \in \mathbb{R}^2_+ : x \geq x' > 0, y \geq y' > 0\}$$

Illustrate this set. Is it closed? bounded? convex? How would you interpret x' and y'?

3. A consumer's budget set is

$$B = \{(x, y) \in \mathbb{R}^2_+ : p_1 x + p_2 y \leq m\}$$

where $p_1, p_2 > 0$ are prices and $m > 0$ is income. Illustrate this set. Is it closed? bounded? convex?

Consider the set $X = B \cap C$ where C is defined in exercise 2. Sketch this set. How would you interpret the case $X = \emptyset$? Is X closed? bounded? convex?

4. A consumer's preferences over bundles of two goods (x, y) are represented by the smooth, convex-to-the-origin indifference curves of standard economics textbooks. Take the consumption bundle (x', y') and define the *better set*

$$B(x', y') = \{(x, y) \in \mathbb{R}^2_+ : \ (x, y) \text{ is preferred or indifferent to } (x', y')\}$$

Is this set closed? bounded? convex?

5. Find the Euclidean distance between the following pairs of points:

(a) 4 and -5 in $\mathbb{R}$

(b) $(-6, 2)$ and $(8, -1)$ in $\mathbb{R}^2$

(c) $(5, -3, 0, 8)$ and $(12, -6, 3, 1)$ in $\mathbb{R}^4$

6. Prove that

(a) an ϵ-neighborhood is an open set

(b) an open set does not contain its boundary points

(c) the intersection of two closed sets is closed

(d) the Cartesian product of two closed sets is closed

(e) the intersection of two convex sets is convex

(f) (from definitions 2.15, 2.16, and 2.17) a closed set contains its boundary points

(g) the union and intersection of two bounded sets is bounded

7. For $\epsilon = 0.1$ and $\epsilon = 10$, describe the ϵ-neighborhoods:

(a) $N_\epsilon(-1)$

(b) $N_\epsilon(-1, 1)$

(c) $N_\epsilon(-1, 1, -1)$

8. Prove that the set

$$X = [1, 2] \cup [3, 4] \subset \mathbb{R}$$

is not convex.

2.4 Functions

We define a function as follows:

Definition 2.24 Given two sets X and Y, a **function** from X to Y is a rule that associates with each element of X, one and only one element of Y.

The set X is called the **domain** of the function, Y is called the **codomain,** and the set of elements in Y (which may or may not be the whole of Y) associated with the elements of X by the function is called the **range** of the function. Denoting the rule for associating the elements of the two sets by f, we can write the function as

$$f : X \mapsto Y$$

or as

$$y = f(x), \qquad x \in X$$

where y is often referred to as the **image** of x or the **value** of the function f at x. In this book, we will be mainly concerned with cases in which Y is $\mathbb{R}$, the set of real numbers, and $X \subseteq \mathbb{R}^n$, $n \geq 1$, while the rule f is a standard algebraic expression. For example, we have the functions

$$y = a + bx, \qquad x \in \mathbb{R}$$

$$y = ax_1^{\alpha} x_2^{\beta}, \qquad (x_1, x_2) \in \mathbb{R}_+^2$$

$$y = \log x, \qquad x \in \mathbb{R}_+$$

$$y = ae^x, \qquad x \in \mathbb{R}$$

each of which we will be examining in some detail below. However, definition 2.27 is more general than these examples, since it applies to sets X and Y and rules of association f of any kind. To emphasise this generality, we also use the term **mapping** as a synonym for function. We refer to the above examples, then, as **real-valued functions** or mappings.

The range of a function can be written as the **image set**

$$f(X) = \{y \in Y : y = f(x),\ x \in X\}$$

that is, as the entire set of y's that we obtain when we substitute into the function the entire set of x's. If $f(X) \subset Y$, we say that f maps X *into* Y, while if $f(X) = Y$, we say that f maps X *onto* Y. In the first case not every $y \in Y$ is an image of an $x \in X$; in the second case it is.

It may be the case that a given y may be the image of more than one x. An extreme case of this would be the constant mapping $f(X) = y^0$, a single element of Y. On the other hand, it may be that each x has as its image a different element of Y, in which case the mapping is said to be one-to-one. A mapping that is both one-to-one and onto is called a **one-to-one correspondence.** Corresponding to each x there is a distinct element of Y and all the elements of Y are images of points x in X.

Given $y = f(x)$, we may often want to invert this function and write x as a function of y, written as $x = f^{-1}(y)$. Clearly, this can only be done if f is one-to-one (into or onto), since otherwise for some y we would have more than one x as the image, and this condition violates the definition of a function. If f is one-to-one onto, then the domain of the inverse function will be Y, and $(f^{-1})^{-1} = f$.

Example 2.15 Find the inverse of $y = x^2$, $x \in \mathbb{R}$.

Solution

As figure 2.18 shows, this function does not possess an inverse, since, for every nonzero value of y, there are two x's—the positive and the negative square roots.

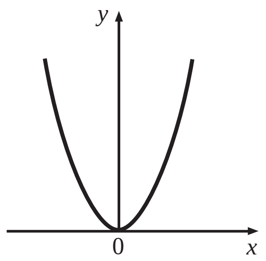


Figure 2.18 Graph of $y = x^2$

However, if we had two functions

$$y = x^2, \qquad x \in \mathbb{R}_+$$

$$y = x^2, \qquad x \in \bar{\mathbb{R}}_{++}$$

then each of these is a one-to-one correspondence and their inverses exist. ■

If the mapping f is one-to-one into (i.e., not onto), then we can still define an inverse function, but we must take care with its domain. Let $Y' = f(X) \subset Y$ denote the image set of f. Then we can define Y' as the domain of the inverse function.

Finally, we can define the **composite mapping** of two mappings $f : X \to Y$ and $g : Y \to Z$ as

$$g \circ f : X \to Z$$

or

$$z = g[f(x)]$$

We substitute each $x \in X$ into f and then substitute the resulting image y into g to obtain an element $z \in Z$ so that overall we have a mapping from X to Z. Note that, for a composition of mappings to be possible, the *range* of the first mapping (f) must be a subset of the *domain* of the second mapping (g), as figure 2.19 makes clear.

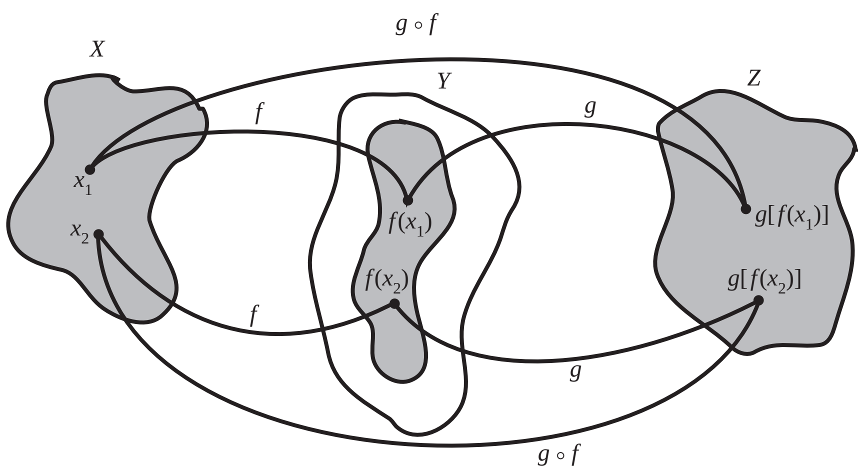


Figure 2.19 Composition of mappings

We will be concerned exclusively with real-valued functions, and we now describe some common types of functions encountered in economics along with some of their properties.

Linear Functions

The function

$$y = ax, \qquad x \in \mathbb{R} \tag{2.7}$$

where a is some real number, is graphed in figure 2.20. In (a) of the figure the parameter a is positive, and in (b) it is negative.

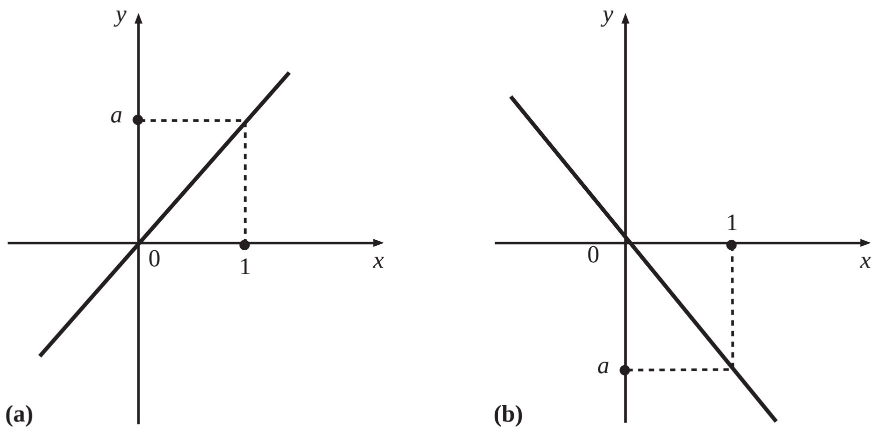


Figure 2.20 Linear functions

The reason for calling this function **linear** is obviously that its graph is a straight line. The steepness of the line is determined by the absolute value of a. Taking two x-values, we can write

$$y_1 = ax_1, \quad y_2 = ax_2$$

implying that

$$y_2 - y_1 = a(x_2 - x_1)$$

or

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

where Δy is read "the change in y," and likewise for Δx. The ratio $\Delta y/\Delta x$ is called the **slope** of the line and so a is the **slope coefficient.** Note that the line $y = ax$ is fully determined once a is chosen.

In equation (2.7) y is often referred to as the **dependent variable** and x as the **independent variable.** This terminology suggests that often we will have a particular causality in mind when writing down the relationship between variables, such as "a change in x *causes* a change in y." When both x and y can serve as dependent or independent variables (i.e., where the causation is unknown or unimportant), we can write the equation as an **implicit function:**

$$f(x, y) = 0$$

The lines so far have passed through the origin. We can displace them, so that their position changes but not their slope, by adding the **intercept term** $b \in \mathbb{R}$. Thus, if we write

$$y = ax + b, \qquad x \in \mathbb{R}$$

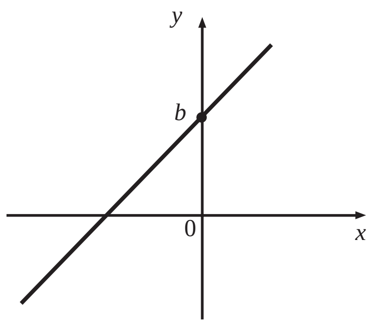


Figure 2.21 Linear function with a positive intercept

then, since $y = b$ when $x = 0$, the line cuts the y-axis at b, as figure 2.21 shows. Varying b generates a whole family of parallel lines.

We can express the same idea in the implicit form as

$$a_1x + a_2y - c = 0, \qquad x, y \in \mathbb{R}$$

where we now have $b = c/a_2$. Varying c with a_1 and a_2 fixed again generates a family of parallel lines. Setting $y = 0$ allows us to obtain the intercept of the line on the x-axis simply as

$$\frac{c}{a_1} = \frac{ba_2}{a_1} = \frac{-b}{a}$$

Note that if (x_1, y_1) and (x_2, y_2) are any two points on a line, then we have $a = (y_2 - y_1)/(x_2 - x_1)$. Now, since for all x,

$$y = ax + b$$

we have

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

as a general equation for the line. This is the algebraic form of the obvious geometric fact that any two points fully determine a line.

Finally, we consider further the important concept, already introduced in section 2.3, of a convex combination of points. To start, suppose that we have two numbers $x', x'' \in \mathbb{R}$, with $x' < x''$. Then we can define the set of numbers

$$\bar{x} = \lambda x' + (1 - \lambda)x'', \qquad \text{for } \lambda \in [0, 1]$$

Clearly, $x' \leq \bar{x} \leq x''$, and writing

$$\bar{x} = x'' - \lambda(x'' - x')$$

we have

$$\frac{(x'' - \bar{x})}{(x'' - x')} = \lambda$$

Thus λ gives us the difference between x'' and $\bar{x}$ as a proportion of the entire interval $[x', x'']$.

This idea generalizes to $\mathbb{R}^2$. Take any two distinct points (x', y') and (x'', y''), and define their convex combination $(\bar{x}, \bar{y})$ as

$$\begin{aligned}(\bar{x}, \bar{y}) &= \lambda(x', y') + (1 - \lambda)(x'', y'') \\ &= [\lambda x' + (1 - \lambda)x'', \lambda y' + (1 - \lambda)y''], \qquad \lambda \in [0, 1]\end{aligned}$$

Then, as figure 2.22 illustrates, the points $(\bar{x}, \bar{y})$ all lie on the line joining (x', y') and (x'', y''). In fact we can think of the convex combination as giving an expression

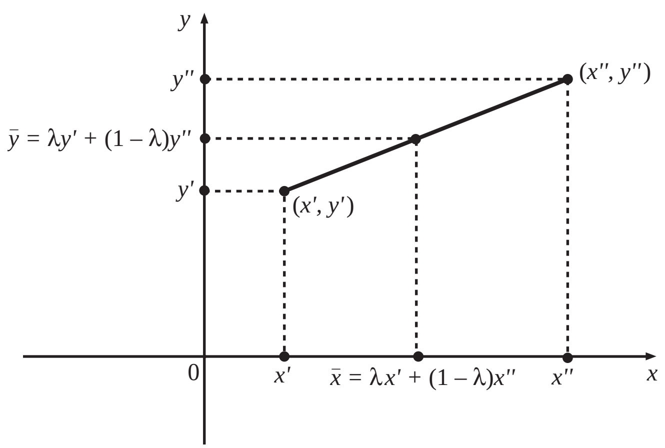


Figure 2.22 A convex combination

for the line joining two points. To see this, we can show that if (x', y') and (x'', y'') lie on a line, then $\bar{y}$ is the y-value on this line corresponding to an x-value of $\bar{x}$. Suppose that the line is $y = ax + b$; then we must have

$$y' = ax' + b$$

and

$$y'' = ax'' + b$$

Multiplying the first expression by λ and the second by $(1 - \lambda)$ gives

$$\begin{aligned}\bar{y} &= \lambda y' + (1-\lambda)y'' = \lambda(ax' + b) + (1-\lambda)(ax'' + b)\\ &= a[\lambda x' + (1-\lambda)x''] + b\\ &= a\bar{x} + b\end{aligned}$$

as required. This is called a convex combination because the set of points $\{(\bar{x}, \bar{y})\}$ is a convex set, as can be seen in figure 2.22. If we now lift the restriction that $\lambda \in [0, 1]$ and allow any $\lambda \in \mathbb{R}$, then the expression

$$(x, y) = \lambda(x', y') + (1-\lambda)(x'', y'')$$

defines the entire line through (x', y') and (x'', y''). Thus we have

$$x = \lambda x' + (1-\lambda)x'' = x'' + \lambda(x' - x'')$$

$$y = \lambda y' + (1-\lambda)y'' = y'' + \lambda(y' - y'')$$

From the first equation we obtain

$$\lambda = \frac{(x - x'')}{(x' - x'')}$$

Substituting this expression into the second equation and rearranging, we have

$$y - y'' = \frac{(y' - y'')}{(x' - x'')}(x - x'')$$

which is exactly the two-point characterization of a line we obtained earlier.

Quadratic Functions

We can write a quadratic function in explicit form as

$$y = ax^2 + bx + c, \qquad x \in \mathbb{R}, \quad a \neq 0$$

As figure 2.23 shows, this is a useful function in economics because in its convex form, with $a > 0$, it could be used to depict a typical U-shaped average or marginal-cost curve, while in its concave form, with $a < 0$, it could depict a typical total-revenue or total-profit curve. (Note that in these examples the domain of the function must be restricted to $\mathbb{R}_+$, since negative outputs are not allowed.) The unique minimum (in the convex case) or maximum (in the concave case) always occurs at the point $x^* = -b/2a$. Thus, if we want a function to have a maximum at a positive value of x, we must choose $b > 0$, while if we want a function to have a minimum at a positive value of x, we must choose $b < 0$. Finally, the value of c will determine whether y is positive, negative, or zero at this maximum or minimum.

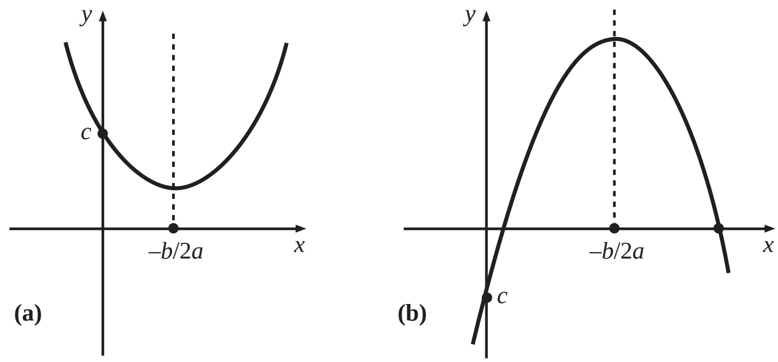


Figure 2.23 Quadratic functions

Rectangular Hyperbola

A **rectangular hyperbola** may be written

$$xy = \alpha \quad \text{or} \quad y = \frac{\alpha}{x}, \qquad x \in \mathbb{R} - \{0\}$$

for some positive constant α. The name stems from the fact that every rectangle drawn to the curve has the same area α. Note that the graph of the function in figure 2.24 has two parts, one entirely in the positive quadrant and the other entirely in the negative quadrant. In economics we often restrict x to $\mathbb{R}_+$, so only the upper curve is relevant. As x tends to zero, the curve approaches the y-axis asymptotically, and as x tends to infinity, it approaches the x-axis asymptotically. Increasing α shifts the curve outward, while retaining the general shape.

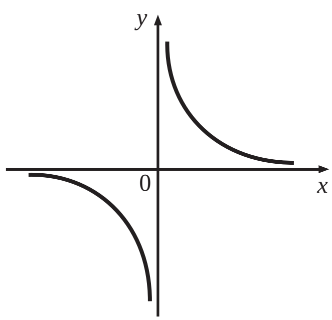


Figure 2.24 Rectangular hyperbola

Power, Exponential, and Logarithmic Functions

When a number a is multiplied by itself n times, we write a^n, where n is called the **exponent.** This leads to the rules of exponents:

$$a^n a^m = a^{n+m}$$

$$(a^n)^m = a^{nm}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$\frac{a^n}{a^n} = a^0 = 1$$

Intuitively, we may think of n as an integer, but in fact n could be any real number. The **power function** takes the form

$$y = ax^b, \qquad x \in \mathbb{R},\ a > 0$$

(Note that the rectangular hyperbola is a special form of the power function with $b = -1$. The linear function is also a special case with $b = 1$. The quadratic may be thought of as the sum of two power functions.)

Figure 2.25 shows two power functions for $x \in \mathbb{R}_+$ where $b > 1$ and $b < -1$.

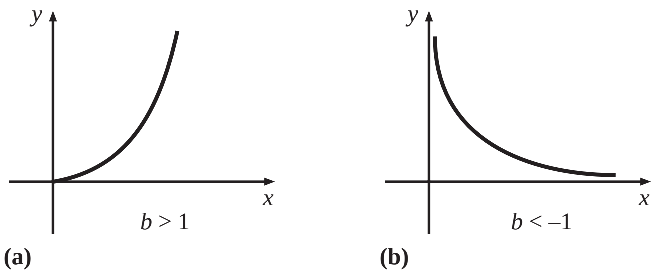


Figure 2.25 Power functions $ax^b : \mathbb{R}_+ \to \mathbb{R}_+$

If we take the *exponent* as the variable in the function, we then obtain the **exponential function**

$$y = ab^x$$

where b is called the **base** of the function. In many applications this base is taken to be the number $e \doteq 2.718$. For $a > 0$ the general shape of the exponential function is shown in figure 2.26.

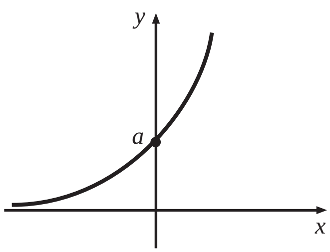


Figure 2.26 Exponential function $ae^x : \mathbb{R} \to \mathbb{R}$

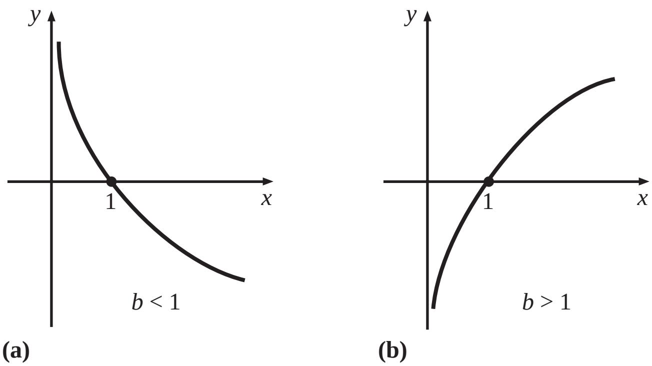


Figure 2.27 The logarithmic function $\log_b x : \mathbb{R}_+ \to \mathbb{R}$

If $x = b^y$, then we say that y is the *logarithm* of x to base b, and we write

$$y = \log_b x$$

This expression defines y as a **logarithmic function** of x. We define the domain of this function to be $\mathbb{R}_+$, and we choose $b > 0$. Moreover, since 1 raised to any power is still 1, we exclude $b = 1$. As figure 2.27 shows, if $b < 1$, the function is decreasing and convex, while if $b > 1$, it is increasing and concave. Very often the base, b, is chosen to be e so that $x = e^y$. The corresponding logarithm is called the **natural logarithm,** and we write

$$y = \ln x$$

(i.e., $\ln x = \log_e x$). By combining the definition of the logarithm with the rules of exponents, we have

$$x = b^{\log_b x} \quad \text{and} \quad z = b^{\log_b z}$$

which implies that

$$xz = b^{\log_b x} b^{\log_b z} = b^{\log_b x + \log_b z}$$

so

$$\log_b(xz) = \log_b x + \log_b z$$

Similarly

$$\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z$$

Moreover we have

$$x^a = (b^{\log_b x})^a$$

which implies that

$$\log_b(x^a) = a \log_b x$$

Concavity, Convexity, Quasiconcavity, Quasiconvexity

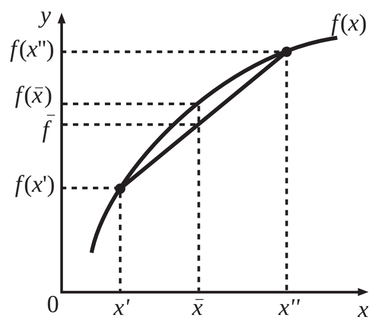


Figure 2.28 Strict concavity

In our description of some specific functions we used the terms “convexity” and “concavity.” Visually the meaning should be clear, but we now present a formal definition. Figure 2.28 shows how we proceed in the case of a concave function. First we must assume that the domain of the function is a convex set, because we want convex combinations of points in the domain to be in the domain. Take any two points x' and x'' in the domain of the function and the corresponding function values $f(x')$ and $f(x'')$. The key characteristic of a concave function is that it “arches above” the line joining these two function values. That is, the value of the function at an x between x' and x'' is higher than the point on the line immediately above that x value. Thus we have

$$\bar{x} = \lambda x' + (1-\lambda)x'', \qquad \lambda \in [0, 1]$$

Since the point on a line joining x' and x'' has the y-coordinate value

$$\bar{f} = \lambda f(x') + (1-\lambda) f(x''), \qquad \lambda \in [0, 1]$$

at $\bar{x}$, strict concavity can be expressed as the property $f(\bar{x}) > \bar{d}$. This is apparent from figure 2.28 and is summarized in

Definition 2.25 The function f is **concave** if

$$f(\bar{x}) \geq \lambda f(x') + (1-\lambda) f(x'')$$

where $\bar{x} = \lambda x' + (1-\lambda)x''$ and $\lambda \in [0, 1]$. It is **strictly concave** if the strict inequality holds when $\lambda \in (0, 1)$.

Note that the curve in figure 2.28 is strictly concave while the linear function is concave but not strictly concave.

By a similar argument we obtain

Definition 2.26 The function f is **convex** if

$$f(\bar{x}) \leq \lambda f(x') + (1-\lambda) f(x'')$$

where $\bar{x} = \lambda x' + (1-\lambda)x''$ and $\lambda \in [0, 1]$. It is **strictly convex** if the strict inequality holds when $\lambda \in (0, 1)$.

A convex function "bends below" a line joining any two function values. This is illustrated in figure 2.29. Clearly, f is (strictly) convex if $-f$ is (strictly) concave. From these definitions it follows that a linear function is both convex and concave but strictly neither.

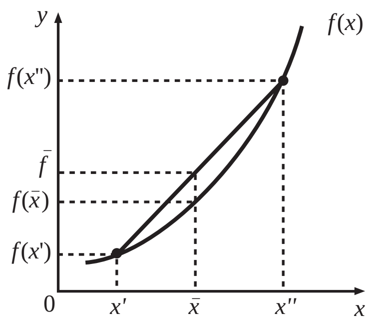


Figure 2.29 Strict convexity

Example 2.16 The point $\bar{x}$ in definitions 2.25 and 2.26 is a convex combination of the two points x' and x''. Suppose that the function is $f(x) = x^2$. Then, if we choose $\lambda = 0.4$, $x' = 2$, and $x'' = 5$, we have

$$\bar{x} = (0.4)2 + (0.6)5 = 3.8$$

so

$$f(\bar{x}) = f(3.8) = (3.8)^2 = 14.44$$

is the height of the function at the point $\bar{x}$. This is shown in figure 2.30. Now, from the convex combination, we can obtain a straight line connecting the two function values $f(x')$ and $f(x'')$:

$$\lambda f(x') + (1-\lambda) f(x'')$$

or

$$0.4(2^2) + 0.6(5^2) = 16.6$$

which is the height of a straight line connecting the points (2, 4) and (5, 25) at $x = 3.8$ in figure 2.30. Clearly, $f(x) = x^2$ is strictly convex between these two points.

To show generally that $f(x) = x^2$ is strictly convex, we need to show that

$$f(\bar{x}) < \lambda f(x') + (1-\lambda) f(x'')$$

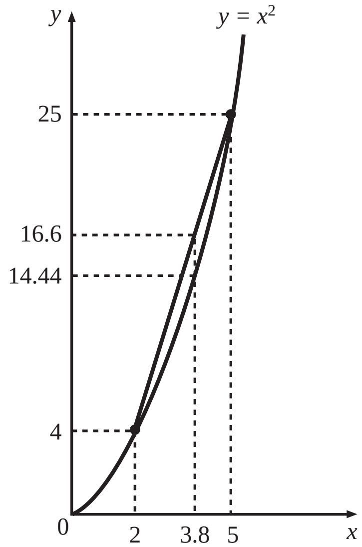


Figure 2.30 The function $f(x) = x^2$ is strictly convex

for any pair of x' and x'' with $\bar{x} = \lambda x' + (1-\lambda)x''$ and $\lambda \in (0, 1)$. The left-hand side of the inequality is

$$f(\bar{x}) = [\lambda x' + (1-\lambda)x'']^2 = \lambda^2(x')^2 + 2(1-\lambda)\lambda x'x'' + (1-\lambda)^2(x'')^2$$

The right-hand side of the inequality is

$$\lambda(x')^2 + (1-\lambda)(x'')^2$$

Hence we have

$$\begin{aligned} \lambda^2(x')^2 + 2(1-\lambda)\lambda x'x'' + (1-\lambda)^2(x'')^2 &< \lambda(x')^2 + (1-\lambda)(x'')^2 \\ (\lambda^2 - \lambda)(x')^2 + 2(1-\lambda)\lambda x'x'' + [(1-\lambda)^2 - (1-\lambda)](x'')^2 &< 0 \\ -\lambda(1-\lambda)(x')^2 + 2(1-\lambda)\lambda x'x'' - \lambda(1-\lambda)(x'')^2 &< 0 \\ (x')^2 - 2x'x'' + (x'')^2 = (x' - x'')^2 &> 0 \end{aligned}$$ ■

To develop the idea of **quasiconcavity,** we first define the notion of a **level set** of a function of n variables.

Definition 2.27

A **level set** of the function $y = f(x_1, x_2, \ldots, x_n)$ is the set

$$L = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : f(x_1, x_2, \ldots x_n) = c\}$$

for some given number $c \in \mathbb{R}$.

In other words, the level set shows the set of points in the domain of the function that gives equal values of the function. As special cases we take two functions of two variables, the linear function

$$y = a_1x_1 + a_2x_2, \qquad (x_1, x_2) \in \mathbb{R}^2_+, \; a_1, a_2 > 0$$

and an example of a power function, called in economics the **Cobb-Douglas function,**

$$y = x_1^a x_2^b, \qquad (x_1, x_2) \in \mathbb{R}^2_+, \; a, b > 0, \; a + b > 1$$

The three-dimensional graphs of these functions are shown in figures 2.31 and 2.32 for particular parameter values.

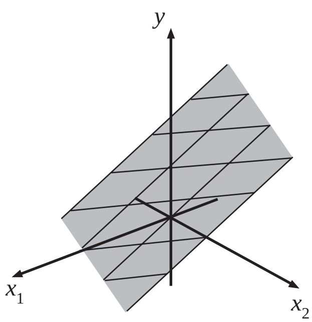


Figure 2.31 The function $y = 2x_1 + 3x_2$

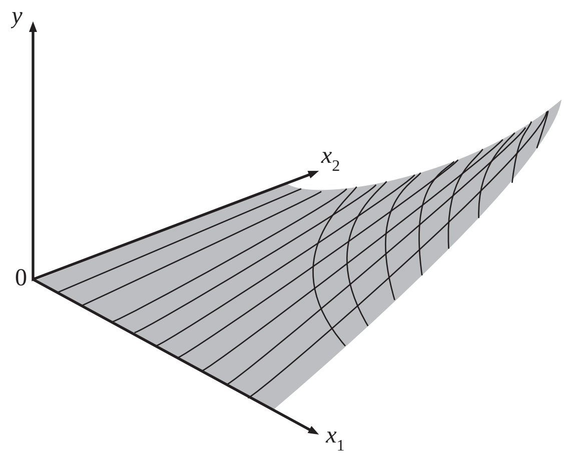


Figure 2.32 The function $y = x_1^2 x_2^2$

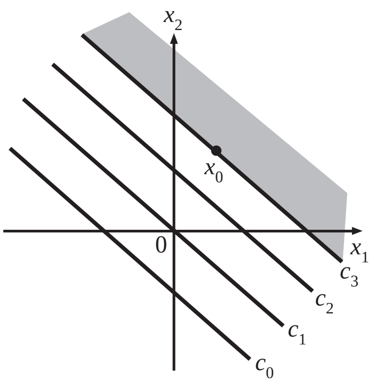


Figure 2.33 Level sets of the function $y = 2x_1 + 3x_2$

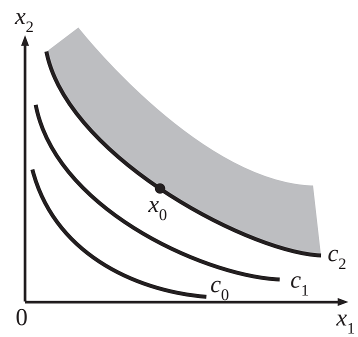


Figure 2.34 Level sets of the function $y = x_1^2 x_2^2$

Applying the definition of a level set gives

$$a_1x_1 + a_2x_2 = c \quad \text{or} \quad x_2 = \frac{c}{a_2} - \left(\frac{a_1}{a_2}\right)x_1$$

and

$$x_1^a x_2^b = c \quad \text{or} \quad x_2 = \left(cx_1^{-a}\right)^{1/b}$$

The level sets can be graphed in two dimensions, with each curve representing a different value of c. Examples from figures 2.31 and 2.32 are shown in figures 2.33 and 2.34. In economics, level sets are encountered in consumer theory (where they are called **indifference curves**), producer theory (where they are called **isoquants**), and a large range of other applications.

Quasiconcavity is essentially concerned with the shapes of the level sets of a function. First, define the **better set** of a point $x_1^0, x_2^0, \ldots, x_n^0$ in the domain $X \subseteq R^n$ of the function $f(x_1, x_2, \ldots, x_n)$.

Definition 2.28 The **better set** of the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is

$$B\left(x_1^0, x_2^0, \ldots, x_n^0\right) = \left\{(x_1, x_2, \ldots, x_n) \in X : f(x_1, x_2, \ldots x_n) \geq f\left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\}$$

That is, the better set of a point is simply the set of points in the domain that yields at least as large a function value. In figures 2.33 and 2.34 the better sets of the point x_0 are shaded.

Definition 2.29 A function f with domain $X \subseteq \mathbb{R}^n$ is **quasiconcave** if, for every point in X, the better set B of that point is a convex set. It is **strictly quasiconcave** if B is strictly convex.

Thus we see that the linear function shown in figure 2.33 is quasiconcave but not strictly quasiconcave, while the Cobb-Douglas function shown in figure 2.34 is strictly quasiconcave.

The precise shapes of the level sets of quasiconcave functions will depend on the direction in which the function increases. In the examples shown in figures 2.33 and 2.34, the functions are increasing in both variables, so the level sets must have negative slopes and the convexity of the better sets implies the shapes shown. However, if the function were increasing in one variable and decreasing in the other, or decreasing in both variables, then quasiconcavity would imply quite different shapes for the level sets. Exercise 8 of this section asks you to explore this situation further.

The terminology in use here may be confusing. Why the term quasi*concavity* when the relevant set must be *convex?* The reason becomes clear if we recall the definition of a concave function: $f(x)$ is concave if, given x' and x'' in its (convex) domain, we have

$$f(\bar{x}) \geq \lambda f(x') + (1 - \lambda) f(x'')$$

where $\bar{x} = \lambda x' + (1 - \lambda)x''$ and $\lambda \in [0, 1]$. Since this holds for *any* points x' and x'', it must hold for a point $x'' \in B(x')$. This implies that $B(x')$ is a convex set and therefore that *any concave function is also quasiconcave*. The converse, however, is not true. For example, the Cobb-Douglas function

$$y = x_1^a x_2^b, \qquad a, b > 0,\ a + b > 1$$

(see figure 2.32), is quasiconcave but not concave.

We can proceed in a similar way with the property of quasiconvexity. Given a function $f(x_1, x_2, \ldots, x_n)$ with domain $X \subseteq \mathbb{R}^n$, we can define the **worse set** of a point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ in the domain:

Definition 2.30 The **worse set** of the point $\left(x_1^0, x_2^0, \ldots, x_n^0\right)$ is

$$W\left(x_1^0, x_2^0, \ldots, x_n^0\right) = \left\{(x_1, x_2, \ldots, x_n) \in X : f(x_1, x_2, \ldots x_n) \leq f\left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\}$$

Then we have

Definition 2.31 A function $f(x_1, x_2, \ldots, x_n)$ with domain $X \subseteq \mathbb{R}^n$ is **quasiconvex** if, for every $(x_1^0, x_2^0, \ldots, x_n^0) \in X$, the worse set $W(x_1^0, x_2^0, \ldots, x_n^0)$ is a convex set. It is **strictly quasiconvex** if W is strictly convex.

Again, it is possible to show that any convex function is quasiconvex, and not vice versa. This topic is further explored in the exercises.

EXERCISES

1. Give equations and sketch graphs of the lines

(a) passing through (0, 1) and having slope −2

(b) passing through (−2, 2) and parallel to $y = 2 - 5x$

(c) passing through (−1, 1) and parallel to

$$\frac{x}{-2} + \frac{y}{-3} = 1$$

2. In a class of 120 students, everyone would take two hamburgers if the price were zero, and no one would buy hamburgers if the price were \$4 or more. Assume that the class demand curve for hamburgers is linear and give its equation. Explain what this implies about the demand for hamburgers when the price is \$3.99.

3. Find the convex combinations of the following pairs of points and, where possible, show them graphically:

(a) −2 and 4

(b) (−1, 1) and (3, 4)

(c) (−2, 0, 1) and (1, −2, 2)

4. Total revenue is price × quantity sold. Show that the total revenue curve corresponding to the demand curve found in exercise 2 is a quadratic. Is it convex or concave? At what value of x does its maximum or minimum occur?

5. A firm's *average-cost function* is given by the quadratic

$$y = x^2 - 20x + 120$$

where y is average cost in dollars per unit of output. The output price is \$10 per unit, and is the same at all levels of output. Find the output levels at which the firm just breaks even (i.e., price = average cost).

Sketch the average-cost function and show the solution. Over what range of prices does the firm make a loss at all output levels?

6. Simplify the following expressions:

(a) $\sqrt{a^5}/a^3$

(b) a^2b^3/a^2b

(c) $bx_1^{b-1}x_2^c/cx_1^bx_2^{c-1}$

(d) $(x_1^{1/b}x_2^c)^{b/c}$

(e) $6x^{0.2} = 5y^{0.4}$ (solve for y in terms of x)

(f) $x = (2^{-1/2})^{-1/2}$ (solve for x)

(g) $y = ax_1^{b_1}x_2^{b_2}x_3^{b_3}$ (What is $\log y$?)

(h) $\log_b(b^x)$

(i) $b^{-\log_b(1/x)}$

(j) $\log_b[b(\log_a a^2)]$

7. Sketch typical level sets of the following functions and state whether they are (strictly) quasiconcave or (strictly) quasiconvex. Then say whether the functions are concave, convex, or neither.

(a) $y = 2x_1^2 - x_1x_2 + 2x_2^2$

(b) $y = (0.5x_1^2 + 0.5x_2^2)^{1/2}$

(c) $y = 2x_1^{1/2}x_2^{1/2}$

8. **(a)** Given the strictly quasiconcave function $y = f(x_1, x_2)$, sketch a typical level set in each of the following cases:

(i) The function is increasing in x_1 and decreasing in x_2.

(ii) The function is decreasing in x_1 and increasing in x_2.

(iii) The function is decreasing in both variables.

(Hint: First determine which way the curve of the level set must slope, then identify the area that gives the better set, and then find how the curvature must look to make the better set convex.)

(b) Repeat part (a), assuming that the function is strictly quasiconvex, and illustrate the level set in each case.

9. Construct an example of a strictly quasiconcave function that is not a concave function.

10. Using the points $x' = 1$, $x'' = 9$, and $\lambda = 5/8$, illustrate definition 2.25 for the concave function $y = x^{1/2}$, $x > 0$. Use a graph in your answer.

11. Show that the function $y = x^{1/2}$, $x > 0$, is strictly concave according to definition 2.25.

12. Show that the function $f(x_1, x_2) = x_1^2 + x_2^2$, is strictly convex according to definition 2.26.

CHAPTER REVIEW

Key Concepts

base
better set
boundary point
bounded
cardinality
Cartesian product
Cobb-Douglas function
closed interval
codomain
compact
complement
completeness property
composite mapping
concavity
convex combination
convexity
coordinate system
dependent variable
dimensions
disjoint
distance
domain
exponent
exponential function
elements
empty set
Euclidean distance
function
image
image set
implicit function
independent variable
indifference curves
infimum
integers
intercept term
interior numbers
interior point
intersection
intervals
irrational numbers
isoquants
level set
linear function
logarithmic function
mapping

natural logarithm
natural numbers
neighborhood
nonnegative numbers
one-to-one correspondence
ordered pair
partition
point sets
power function
power set
proper subset
pure number
quasiconcavity
quasiconvexity
rational numbers
range
rectangular hyperbola
real line
real numbers
real-valued functions
relative difference
set
singleton
slope
slope coefficient
subset
supremum
union
universal set
Venn diagram
worse set

Review Questions

1. How does a Venn diagram help to illustrate the possible relationships between sets and subsets?
2. What is meant by “the real line”?
3. What is a supremum? What is an infimum?
4. What is a point set? What is a convex set?
5. Distinguish between closedness and boundedness of a point set.
6. Distinguish between concavity and convexity of a function.
7. Distinguish between quasiconcavity and concavity.
8. Distinguish between quasiconvexity and convexity.
9. What is the difference between a necessary condition and a sufficient condition?

Review Exercises

1. Write out the convex combinations of the following pairs of points:
   - **(a)** -2 and 2 in $\mathbb{R}$
   - **(b)** $(-2, 2)$ and $(-3, 3)$ in $\mathbb{R}^2$
   - **(c)** $(0, 0)$ and (x_1, x_2) in $\mathbb{R}^2$
   - **(d)** $(-2, 2, 5)$ and $(-3, 3, 8)$ in $\mathbb{R}^3$

In cases (b) and (c), draw a graph to show that the convex combination lies on the line segment between two points in $\mathbb{R}^2$.

2. A firm's production set is

$$Y = \{(x, y) \in \mathbb{R}^2_+ : y \leq \sqrt{x}\}$$

Sketch this set in $\mathbb{R}^2$. Is it closed? bounded? convex? Explain why we would interpret a boundary point of the set as "efficient" and an interior point as "inefficient."

3. Give the equation and sketch the graph of the line

(a) passing through $(-1, 20)$ and having slope 2

(b) passing through $(-2, 1)$ and parallel to $3x - 4y = 2$

4. Find the convex combinations of the pair of points $(0, -2, 1, -1)$ and $(-1, 3, 1, -2)$.

5. Simplify the following expressions:

(a) $(ab)^3/a^2b$

(b) $a(b/a)^q$

(c) $10x^{0.25} = 2y^{1/8}$ (solve for y in terms of x)

(d) $\log_b(b^x)^3$

6. Sketch typical level sets of the function

$$y = 10x_1^{1/4}x_2^{1/2}$$

and state whether it is (strictly) quasiconcave or (strictly) quasiconvex. Is the function concave, convex, or neither?

7. Show that the function $y = 10 - x^2$ is strictly concave according to definition 2.28.

8. By using the points $x' = 2$, $x'' = 6$, and $\lambda = 1/2$, illustrate definition 2.25 for the concave function $y = 10 - x^2$. Use a graph to demonstrate this.

9. Show that the function $f(x_1, x_2) = (x_1 + x_2)^{1/2}$, $x_1, x_2 > 0$ is concave according to definition 2.25.

Chapter 3 Sequences, Series, and Limits

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- More Applications of Series
- The Keynesian Multiplier
- St. Petersburg Paradox
- Practice Exercises

Studying sequences and series is the best way to gain intuition about the rather perplexing notions of arbitrarily large numbers (infinity) and infinitesimally small (but nonzero) numbers. We gain such understanding by using the idea of the limit of a sequence of numbers. Thus, from a mathematical perspective, this chapter provides very useful background to the important property of continuity of a function, which we will explore fully in chapter 4. There are also some interesting economic applications of series and sequences, in particular the notion of discounting a future stream of payments or receipts, which is a critical aspect of judging the value of an investment by a business or a government.

3.1 Definition of a Sequence

A **sequence** is simply a succession of numbers. For example, the sequence of numbers 1, 4, 9, 16, ... appears to consist of the squares of the natural numbers (i.e., $1^2, 2^2, 3^2, 4^2, \ldots$). It is common to see questions on IQ or mathematical aptitude tests asking one to fill in the next number in a sequence, which involves figuring out or guessing the formula that generates the numbers of the given terms of the sequence. We could write that formula for the sequence above as

$$f(n) = n^2, \qquad n = 1, 2, 3, 4, \ldots$$

Then the "next" term in the given sequence of numbers is $5^2 = 25$.

Example 3.1 Find a function or formula that corresponds to the first three terms of the sequence

$$1, 4, 7, \ldots$$

Solution

One function that works is

$$f(n) = 3n - 2, \qquad n = 1, 2, \ldots \tag{3.1}$$

Another is

$$f(n) = (n-2)^3 + 2n, \qquad n = 1, 2, \ldots \tag{3.2}$$

■

Example 3.1 illustrates the general result that for any n numbers intended to describe a sequence, there is more than one function that can be used to generate the sequence of numbers given. In fact there are an infinite number of possibilities. Therefore such questions on IQ tests requesting one to fill in the next number of a sequence are poorly designed. If the respondent considering example 3.1 had in mind the equation (3.1), then she would offer 10 as the "next number" while if she had in mind the equation (3.2), she would offer 16 as the "next number." Both responses are correct.

A formal definition of a sequence follows, along with several examples.

Definition 3.1 A **sequence** is a function whose domain is the positive integers.

The following are examples of sequences:

1. $f(n) = 2n$	or	$2, 4, 6, 8, 10, \ldots$	(figure 3.1)
2. $f(n) = 1/n$	or	$1, 1/2, 1/3, 1/4, 1/5, \ldots$	(figure 3.2)
3. $f(n) = -1/n$	or	$-1, -1/2, -1/3, -1/4, -1/5, \ldots$	(figure 3.3)
4. $f(n) = (-1)^n$	or	$-1, 1, -1, 1, -1, \ldots$	(figure 3.4)
5. $f(n) = -n^2$	or	$-1, -4, -9, -16, -25, \ldots$	(figure 3.5)
6. $f(n) = (-2)^n$	or	$-2, 4, -8, 16, -32, \ldots$	(figure 3.6)

It is clear that the terms in the sequences for examples 2 and 3 above are getting smaller in absolute value and closer and closer to zero. We say that such sequences have a **limit,** and in these two cases the limit is the same (zero) even though no two elements of the sequences equal each other and no term in either sequence ever actually takes on the value zero. The terms in the other sequences do not tend to approach a single, finite value, and so they do not have a limit. In examples 1 and 5 we can see that the values of the sequence become either arbitrarily large or arbitrarily small, and so we say the sequences are not bounded. The sequence in example 4 is bounded, but still the elements don't become arbitrarily close to any single value, and so we also say it has no limit. The idea of limits is presented more formally in the following section.

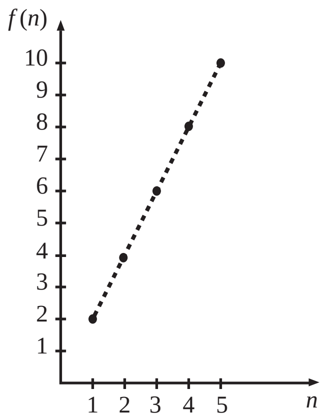


Figure 3.1 The first five terms of the sequence $f(n) = 2n$

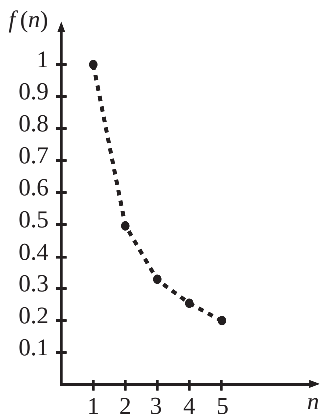


Figure 3.2 The first five terms of the sequence $f(n) = 1/n$

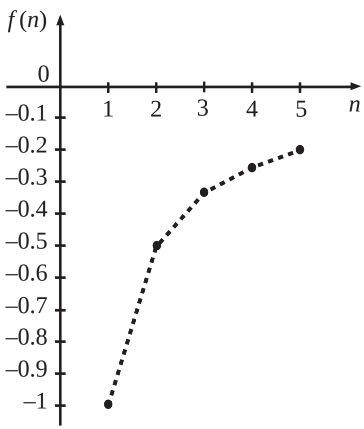


Figure 3.3 The first five terms of the sequence $f(n) = -1/n$

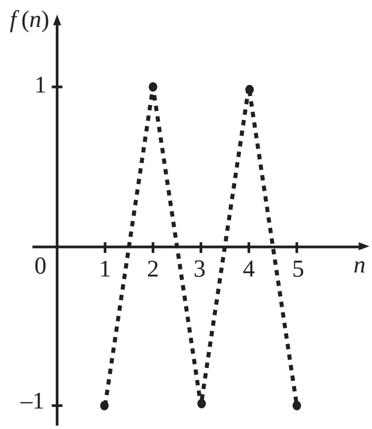


Figure 3.4 The first five terms of the sequence $f(n) = (-1)^n$

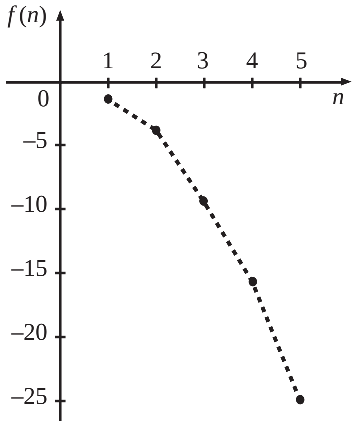


Figure 3.5 The first five terms of the sequence $f(n) = -n^2$

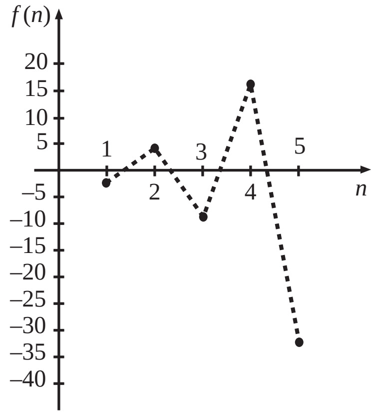


Figure 3.6 The first five terms of the sequence $f(n) = (-2)^n$

EXERCISES

1. Determine the first 10 terms of each of the following sequences. In each case, draw a graph such as those in figures 3.1 to 3.6.

(a) $f(n) = 5 + 1/n$

(b) $f(n) = 5n/(2^n)$

(c) $f(n) = (n^2 + 2n)/n$

2. Determine the first 10 terms of each of the following sequences. In each case, draw a graph such as those in figures 3.1 to 3.6.

(a) $f(n) = 5 - 1/n$

(b) $f(n) = n/(n + 1)$

(c) $f(n) = c + [(-1)^n(1/n)]$ for c constant

3. Show how the sequence of terms $f(n) = 2n$, $n = 0, 1, 2, \ldots$ can be written using the domain $n = 1, 2, 3, \ldots$.

4. Show how the sequence of terms $f(n) = n^2$, $n = 5, 6, 7, \ldots$ can be written using the domain $n = 1, 2, 3, \ldots$.

5. Show how to write all terms beginning with the 26th term of the sequence $f(n) = (1 + r)^n$, $n = 1, 2, 3, \ldots$ using the same domain, $n = 1, 2, 3, \ldots$.

3.2 Limit of a Sequence

It is convenient to refer to each term of a sequence by a_n where $a_n = f(n)$, $n = 1, 2, 3, \ldots$, corresponding to definition 3.1. The intuitive notion of a limit for a sequence is that the terms a_n “get close" to some unique and finite value as n gets “large.” If a sequence does have a limit, we say it is **convergent,** while if it does not, we say it is **divergent.**

Definition 3.2

A sequence is said to have the **limit** L if, for any $\epsilon > 0$, however small, there is some value N such that $|a_n - L| < \epsilon$ whenever $n > N$. Such a sequence is said to be **convergent,** and we write its limit as $\lim_{n\to\infty} a_n = L$.

In less formal language, the definition above states that a sequence has a limit L provided that all values of the sequence “beyond some term” can be made as close to L as one wishes (i.e., the condition $|a_n - L| < \epsilon$ can be met for as small a positive number ϵ as one likes by choosing a sufficiently large value of N). For example, consider the sequence $a_n = 1/n$ (figure 3.2), which has the limit $L = 0$. We see that $|a_n - 0| < 0.01$ for any choice of $N > 100$, while $|a_n - 0| < 0.002$ for any choice of $N > 500$. More generally, $|1/n - 0| < \epsilon$ requires a choice of $N > 1/\epsilon$. One can think of N as formally being a function of ϵ and so write $N(\epsilon)$.

As is also the case for functions in general (see chapter 2), a sequence may be **bounded** or **unbounded.** In particular, we say that a sequence is bounded if there is some finite value $K > 0$ such that for some N it follows that

$$a_n < K \quad \text{for all } n > N \quad \text{bounded above}$$

and

$$a_n > -K \quad \text{for all } n > N \quad \text{bounded below}$$

and is not bounded if one or both of these conditions fails to hold. For example, the sequence

$$f(n) = 2n, \qquad n = 1, 2, 3, \ldots$$

illustrated in figure 3.1 is unbounded because it is *not* bounded above, while the sequence

$$f(n) = -n^2, \qquad n = 1, 2, 3, \ldots$$

illustrated in figure 3.5 is unbounded because it is *not* bounded below. A sequence may also be unbounded because it is neither bounded above nor below, as in the sequence

$$f(n) = (-2)^n, \qquad n = 1, 2, 3, \ldots$$

which is illustrated in figure 3.6.

It is quite easy to see that a sequence that grows without bound cannot have a limit and so doesn't satisfy definition 3.2 for convergence. The sequence $a_n = (-1)^n$ (figure 3.4) illustrates another type of sequence that is not convergent. The terms do not grow without bound, yet there is no limit because *all* terms of the sequence beyond the Nth term must satisfy definition 3.2 if the sequence is to have a limit. It is clear that whatever choice of L is made, terms of a_n for n greater than any N will include the values $+1$ and -1, and so all terms beyond a_N cannot be made arbitrarily close to any single value L. Thus a bounded sequence is not necessarily convergent.

Definition 3.3 If a sequence has no limit, it is **divergent.**

We can classify divergent sequences in two ways, either as definitely divergent or simply as divergent. For example, the sequence $a_n = 2n$ (figure 3.1) is divergent, since there is no (finite) number L which the terms of the sequence approach. However, the terms grow without bound in a positive direction, and so we say the sequence approaches positive infinity ($+\infty$). This is an example of what is called a *definitely divergent sequence*.

Definition 3.4 A divergent sequence is said to be **definitely divergent** if either one of the following conditions holds:

(i) If for any (arbitrarily large) value of K there is an N sufficiently large that $a_n > K$ for all $n > N$, then we say the sequence is definitely divergent and $\lim_{n\to\infty} a_n = \infty$.

(ii) If for any (arbitrarily large) value of K there is an N sufficiently large that $a_n < -K$ for all $n > N$, then we say the sequence is definitely divergent and $\lim_{n\to\infty} a_n = -\infty$.

Notice that the sequences in figures 3.1, 3.4, 3.5, and 3.6 are all divergent but only the ones in figure 3.1 and 3.5 are definitely divergent. Also note that the sequence in figure 3.6 is not bounded yet its terms do not approach either $+\infty$ or $-\infty$.

The following set of examples illustrates how one can show formally that a sequence either converges or diverges, as the case may be.

Example 3.2 Show that the sequence $f(n) = 1/n$, $n = 1, 2, 3, \ldots$, has the limit zero; that is, $\lim_{n\to\infty} 1/n = 0$, and so converges to zero.

Solution

According to definition 3.2 we need to show that for any $\epsilon > 0$ there must be some value N such that

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

for every $n > N$. In other words, we need to find an N such that $1/n < \epsilon$ for every $n > N$.

By multiplying both sides of this inequality by n, we have that the condition becomes

$$1 < \epsilon n$$

or

$$\epsilon n > 1$$

or

$$n > \frac{1}{\epsilon}$$

Thus, if we choose N to be the next integer greater than $1/\epsilon$, we will satisfy the condition. ■

Example 3.3 Show that the sequence $f(n) = (-1)^n$, $n = 1, 2, 3, \ldots$ is divergent.

Solution

According to definition 3.3 a sequence is divergent if it has no limit, L. It is easy to illustrate diagrammatically that there is no value L such that all the terms in the sequence $f(n) = (-1)^n$, $n > N$, lie within distance ϵ of L for every $\epsilon > 0$, no matter how large we choose N to be. The reason is that no matter how large we choose N, the terms of the sequence $f(n) = (-1)^n$, $n > N$ will include both the

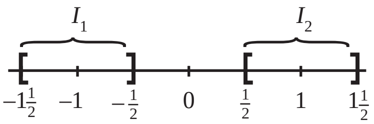


Figure 3.7 Illustration that $f(n) = (-1)^n$ has no limit (example 3.3)

values -1 and $+1$. So, if we take $\epsilon = 1/2$, for example, there is no number L which lies within $1/2$ of a unit of both -1 and $+1$. We see this in figure 3.7.

For the limit L to be within $\epsilon = 1/2$ of the number -1, it must lie in the interval marked I_1 while for the limit L to be within $\epsilon = 1/2$ of the number $+1$, it must lie in the interval I_2. Since these two intervals do not intersect, there is no value L that satisfies both conditions (i.e., no number lies in *both* of these intervals). Algebraically, for L to be a limit of this sequence, it must satisfy both of the following conditions for every $\epsilon > 0$:

Condition A, $|-1 - L| < \epsilon$
Condition B, $|+1 - L| < \epsilon$

For $\epsilon = 1/2$, for example, condition A requires that

$$|-1 - L| < \frac{1}{2}$$

which will not hold if $L > 0$, since in that case $|-1 - L| = 1 + L$ which is greater than $1/2$, while condition B requires that

$$|+1 - L| < \frac{1}{2}$$

which will not hold if $L < 0$, since in that case $|+1 - L| = 1 + |L|$ which is also greater than $1/2$.

Therefore, since L cannot be both negative and positive, there can be no limit to this sequence. ■

Example 3.4 Show that the sequence $a_n = 2n$, $n = 1, 2, 3, \ldots$ is definitely divergent.

Solution

By definition 3.4, a sequence is definitely divergent if either $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$. The first of these two cases holds for the sequence $a_n = 2n$. To see this formally, we need to show that for any (arbitrarily large) value K, one can always find an N large enough that $a_n > K$ for every $n > N$. This is clearly the case for $a_n = 2n$, since for any K we have

$$2n > K$$

provided $n > K/2$. Therefore, we need only choose our value for N to be the next integer greater than $K/2$ in order to satisfy the condition. ■

EXERCISES

1. Using definition 3.2, show that each of the following sequences has the limit as specified:

(a) $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$

(b) $\lim_{n\to\infty} 5 + \dfrac{1}{n} = 5$

(c) $\lim_{n\to\infty} \left(-\dfrac{1}{2}\right)^n = 0$

2. Determine the limit for each of the following sequences, and prove that your choice is correct according to definition 3.2:

(a) $\lim_{n\to\infty} \dfrac{1}{n^2+2}$

(b) $\lim_{n\to\infty} \dfrac{n}{(n+1)^2}$

3. Show that each of the following sequences is divergent. If the sequence is definitely divergent, show this to be the case according to definition 3.4.

(a) $\lim_{n\to\infty} n^2$

(b) $\lim_{n\to\infty} (-n)^3$

(c) $\lim_{n\to\infty} (-c)^n$ (c a constant and $c \neq 0$)

3.3 Present-Value Calculations

An important economic application of sequences is the determination of the **present value** of a sum of money to be received at some point in the future. This computation is the inverse of determining how much money one would have in the future upon investing a certain amount now. Suppose, for example, that one had \$90.91 to invest currently at an annual interest rate of 10%. Then the amount of money received at the end of one year would be $\$90.91(1 + 0.1) = \$100$. In general, investing $\$X$ today at an annual rate of return r will generate $V = X(1+r)$ at the end of one year. Therefore it is equivalent to say that the *present value* of amount V to be received in one year's time is $X = V/(1 + r)$, where r is the rate of return (or rate of interest). For the example above the present value of \$100 to be received at the end of one year is \$90.91 if the annual interest rate is 10%. The reason is that \$90.91 is precisely the amount required to be invested now in order to generate \$100 in one year's time given the interest rate of 10%.

The same rationale can be used to value an amount received any number of periods into the future. If one received \$$Y$ now and could invest at 10% per year, with compounding (i.e., the accumulation and reinvestment of interest payments) at the end of each year, then the \$$Y$ would be worth $Y(1+0.1)$ at the end of the first year and, after reinvesting (including interest payments), would be worth $[Y(1+0.1)](1+0.1) = Y(1+0.1)^2$ at the end of the second year. Therefore the present value of \$100 to be received in two years' time is Y, where $Y(1+0.1)^2 = 100$ or $Y = \$100/(1+0.1)^2 = \$82.64$ (approximately). Following this line of argument leads to the following formula, which determines the present value PV_t of amount V to be received t periods from now when the interest rate is r per period and compounding occurs at the end of each period:

$$PV_t = \frac{V}{(1+r)^t} \tag{3.3}$$

Notice that for $r > 0$ the denominator of $(1+r)^t$ becomes larger as t becomes larger, and thus PV_t gets smaller. In other words, receiving a certain sum in the future has a lower present value the longer one has to wait for the payment. This is natural since the further in the future one receives the fixed amount V, the less one would need to invest now to replicate that future payment. For this reason economists refer to the *discounting* of future benefits and the value $1/(1+r)$ is referred to as the **discount rate,** or **discount factor.** Moreover $(1+r)^t$ grows without bound as $t \to \infty$, and so $PV_t \to 0$ as $t \to \infty$. The proof of this statement is left as an exercise at the end of this section.

Example 3.5 Compute the present value of \$500 to be received in one year's time given the interest rate of 8%.

Solution

According to equation (3.3) we have

$$PV_1 = \frac{V}{(1+r)^1} = \frac{V}{1+r} = \frac{500}{1+0.08} = \frac{500}{1.08} = \$462.96$$

■

One can readily check that this is correct since if one had \$462.96 presently to invest at an interest rate of 8% for one year, then at the end of the year one would have $\$462.96(1+0.08) = \$500$.

Example 3.6 Compute the present value of receiving \$1 million at the end of each of the next three years given the interest rate of 12%.

Solution

$$PV \text{ of \$1 million at the end of the first year } = \frac{\$1{,}000{,}000}{1.12} = \$892{,}857.14$$

$$PV \text{ of \$1 million at the end of the second year } = \frac{\$1{,}000{,}000}{(1.12)^2} = \$797{,}193.88$$

$$PV \text{ of \$1 million at the end of the third year } = \frac{\$1{,}000{,}000}{(1.12)^3} = \$711{,}780.25$$

Total: \$2,401,831.27

So the present value of the sum of these three annual payments of \$1 million is \$2,401,831.27. ■

Example 3.7 Given an interest rate of 12%, guaranteed for the next three years, how much money would one need to be given presently as a lump sum in order to finance expenditures of \$1 million to occur at the end of each of the next three years?

Solution

From example 3.6 we can see immediately that the answer to this question is \$2,401,831.27.

To see that this is so, suppose that we start with \$2,401,831.27, invested at 12% for one year, thus generating

$$\$2{,}401{,}831.27 \times 1.12 = \$2{,}690{,}051.02$$

at the end of the first year. After spending \$1 million, we have \$1,690,051.02 left over to invest for the second year, thus generating

$$\$1{,}690{,}051.02 \times 1.12 = \$1{,}892{,}857.14$$

at the end of the second year. After spending \$1 million, we have \$892,857.14 to invest for the third and final year, thus generating

$$\$892{,}857.14 \times 1.12 = \$1{,}000{,}000.00$$

as required for expenditures at the end of that year. ■

Continuous Compounding

Although it is conventional to quote interest rates on the basis of annual payments, it is not unusual to see a different length of time used to determine the compounding of interest earned or owed. For example, it is not unusual to have interest earned in a savings account accrue and be automatically reinvested (compounded) on a monthly basis. The result of doing this is that more money is earned than in the case of only a single instance of coumpounding at the end of the full year. This result is developed below through the use of an example.

Suppose that the annual rate of interest on money in a savings account is 12% ($r = 0.12$) and an individual places \$1,000 into her account. No withdrawals are made until the end of the year. If interest is compounded on an annual basis, the amount in her account at the end of one year will be $\$1{,}000 \times (1 + 0.12) = \$1{,}120$. Alternatively, suppose that the bank computes its interest payments on a half-yearly basis and deposits the appropriate sum into their customers' savings accounts accordingly. This means that interest is compounded semiannually. Since an annual interest rate of 12% implies a semiannual interest rate of 6%, this means that the \$1,000 on deposit earns $\$1{,}000 \times 0.06 = \$60$ interest in the first half-year. When deposited into the account, this means that the principal for the second half-year is \$1,060 which earns $\$1{,}060 \times 0.06 = \$63.60$ and so the total value of the deposit at the end of the year is $\$1{,}000 + \$60 + \$63.60 = \$1{,}123.60$. A more mathematically convenient way to express this is to note that after the first half-year there is $\$1{,}000 \times (1 + 0.06)$ in the account and this amount earns 6% interest in the second half-year, implying that at the end of the year

$$[\$1{,}000 \times (1 + 0.06)] \times (1 + 0.06) = \$1{,}000(1.06)^2 = \$1{,}123.60$$

is the amount in the account.

Suppose that the bank instead decides to offer compounding every three months (i.e., each quarter of a year). The interest rate for a quarter of a year is 3% and compounding quarterly means the value of \$1,000 at the end of a year will be

$$\$1{,}000(1 + 0.03)(1 + 0.03)(1 + 0.03)(1 + 1.03) = \$1{,}000(1.03)^4 = \$1{,}125.51$$

If the bank offered compounding monthly, the relevant interest rate is 1% (per month) and the value of \$1,000 at the end of a year becomes

$$\$1{,}000\,\underbrace{(1+0.01)(1+0.01)\ldots(1+0.01)}_{12\text{ times}} = \$1{,}000(1.01)^{12} = \$1{,}126.82$$

Not surprisingly, the more frequently interest payments are compounded, the greater is the value of the savings at the end of the year.

If we let r represent the annual interest rate and n the number of times per year that interest is compounded, then the relevant *per period* interest rate is r/n, and so in each period we need to apply the factor $(1 + r/n)$ to determine the value of each dollar at the end of that period. It follows that the value of $\$P$ invested at an annual rate of interest r compounded n times per year is worth

$$P\,\underbrace{\left(1+\frac{r}{n}\right)\left(1+\frac{r}{n}\right)\ldots\left(1+\frac{r}{n}\right)}_{n\text{ times}} = P\left(1+\frac{r}{n}\right)^{n}$$

at the end of a year. (Try this formula for the example developed above where $n = 1, 4,$ and 12 for annual, quarterly, and monthly compounding respectively.)

By **continuous compounding** we mean that interest is compounded instantaneously or, in effect, $n \to \infty$. Thus we can treat the term $(1 + r/n)^n$ in the formula above as a sequence with its limit $\lim_{n\to\infty}(1 + r/n)^n$, giving us the factor to be applied to the principal P in order to determine the value of $\$P$ invested for a year with continuous compounding. It turns out that for the special case of $r = 1$ (100% interest rate), we get

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

where e is the so-called natural number first introduced in chapter 2. Recall that it has approximate value $e \doteq 2.71828$.

Although interesting, this formula in itself has little application, since interest rates are rarely 100% per year. If the interest rate is r, then the relevant calculation for continuous compounding becomes

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n}$$

In order to develop a usable formula for this expression, we first introduce the new variable s where $s \equiv n/r$. Then $r/n = 1/s$ and $n = sr$. Upon substituting these values for r/n and n into the formula above, and noting that as $n \to \infty$ so does $s \to \infty$, we get

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \lim_{s\to\infty}\left(1+\frac{1}{s}\right)^{sr} = \lim_{s\to\infty}\left[\left(1+\frac{1}{s}\right)^s\right]^r$$

Ignoring the term r for a moment, we see that

$$\lim_{s\to\infty}\left[\left(1+\frac{1}{s}\right)^s\right] = e$$

since it is exactly the same expression as before except for a change in variable names. It follows that

$$\lim_{s\to\infty}\left[\left(1+\frac{1}{s}\right)^s\right]^r = e^r$$

Thus $\$P$ invested for one year at an interest rate of r with continuous compounding takes on the value

$$P\lim_{s\to\infty}\left[\left(1+\frac{1}{s}\right)^s\right]^r = Pe^r$$

at the end of one year. For our example with $P = \$1{,}000$ and $r = 0.12$, this value is

$$Pe^r = \$1{,}000e^{0.12} = \$1{,}127.50$$

All that remains to be able to apply generally the idea of compounding interest payments is to consider what happens when the investment period differs from one year. In the case of compounding n times per year, where n is finite and r is the annual rate of interest, the following formula indicates the value of $\$P$ invested for t years:

$$V = P\left[\left(1+\frac{r}{n}\right)^n\right]^t = P\left(1+\frac{r}{n}\right)^{nt}$$

This formula is derived by noting that the factor $(1 + r/n)$ must be applied n times each year for each of t years, which implies that the exponent in this expression is nt. For our own example this means that if the \$1,000 were left in the savings account for two years, it would be worth

$V = \$1{,}000(1 + 0.12)^2 = \$1{,}254.40$ with annual compounding
$V = \$1{,}000(1 + 0.06)^4 = \$1{,}262.48$ with semiannual compounding
$V = \$1{,}000(1 + 0.01)^{24} = \$1{,}269.73$ with monthly compounding

An analogous result applies to the case of continuous compounding with the value of $\$P$ invested at interest rate r for t periods being worth

$$V = P(e^r)^t = Pe^{rt}$$

at the end of t periods. For our example, the value of \$1,000 invested for two years at interest rate 12%, compounded continuously will be

$$V = \$1{,}000e^{0.12\times 2} = \$1{,}000e^{0.24} = \$1{,}271.25$$

It is worth noting that even if the process of continuous compounding seems implausible in a real-world context, it may nonetheless be a sufficiently accurate and useful approximation. Notice that the difference in the value of \$1,000 invested for two years with interest compounded monthly, compared to the case of interest being compounded continuously, is minimal. Since the formula for continuous compounding is much easier to work with, economists often assume that they can apply it even if it is not *exactly* correct.

Since computing the present value of a payment to be received in the future means determining how much money one would need now to generate this future amount, it follows that the frequency with which money invested now could be compounded must be factored into present-value calculations. For example, \$1,000 received one year from now has present value \$909.09 if the interest rate is 10% and interest is compounded annually; that is, since

$$\$909.09(1.10) = \$1{,}000 \text{ or } \frac{\$1{,}000}{(1.10)} = \$909.09$$

However, if interest were compounded twice annually, the \$909.09 would generate *more than* \$1,000 after one year. It would generate

$$\$909.09(1 + 0.05)^2 = \$1{,}002.27$$

and so the present value of \$1,000 received in one year's time is worth less than \$909.09 in this scenario. The correct computation is

$$\frac{\$1{,}000}{(1.05)^2} = \$907.03$$

To check the intuition, note that \$907.03 invested at interest rate 10% (per year) with compounding on a semiannual basis generates $\$907.03(1.05)^2 = \$1{,}000$.

In general terms, if $\$V$ is to be received t years from now and the interest rate is r (per year) with compounding n times per year, then one would need $\$X$ now to generate this future value, where

$$X\left(1+\frac{r}{n}\right)^{nt} = V \quad \text{or} \quad X = \frac{V}{[1+(r/n)]^{nt}}$$

Thus the appropriate discounting formula for finding the present value PV_t of a future amount V when discounting occurs n times per year is

$$PV_t = \frac{V}{[1+(r/n)]^{nt}}$$

Similarly, if interest is compounded continuously, the amount ($\$X$) needed now to generate $\$V$ in t years' time is determined by the relationship

$$Xe^{rt} = V \quad \text{or} \quad X = \frac{V}{e^{rt}} = Ve^{-rt}$$

and so the appropriate discounting formula under continuous discounting is

$$PV_t = Ve^{-rt}$$

All of these results are illustrated in the following two examples. Note that a case where compounding occurs a finite number of times per year is referred to as **discrete compounding,** while if $n \to \infty$, it is referred to as **continuous compounding.**

Example 3.8 Determine how much money an investment of \$10,000 will generate in the following situations. In each case assume that the annual interest rate is 3%.

(i) At the end of one year given semiannual compounding
(ii) At the end of five years given semiannual compounding
(iii) At the end of one year given monthly compounding
(iv) At the end of five years given monthly compounding
(v) At the end of one year given continuous compounding
(vi) At the end of five years given continuous compounding

Solution

Using the formulas

$$V = P\left(1+\frac{r}{n}\right)^{nt} \qquad \text{for discrete compounding}$$

$$V = Pe^{rt} \qquad \text{for continuous compounding}$$

we get

(i) $V = \$10{,}000\left(1 + \frac{0.03}{2}\right)^2 = \$10{,}000(1.015)^2 = \$10{,}302.25$

(ii) $V = \$10{,}000\left(1 + \frac{0.03}{2}\right)^{10} = \$10{,}000(1.015)^{10} = \$11{,}605.41$

(iii) $V = \$10{,}000\left(1 + \frac{0.03}{12}\right)^{12} = \$10{,}000(1.0025)^{12} = \$10{,}304.16$

(iv) $V = \$10{,}000\left(1 + \frac{0.03}{12}\right)^{60} = \$10{,}000(1.0025)^{60} = \$11{,}616.17$

(v) $V = \$10{,}000e^{0.03} = \$10{,}000(1.030454) = \$10{,}304.54$

(vi) $V = \$10{,}000e^{0.15} = \$10{,}000(1.161834) = \$11{,}618.34$ ■

Example 3.9 Determine the present value of $25,000 to be received in the future in the following situations. In each case, assume the interest rate is 8%.

(i) Payment is received at the end of one year's time given annual compounding
(ii) Payment is received at the end of 20 years' time given annual compounding
(iii) Payment is received at the end of one year's time given quarterly compounding (i.e., every three months)
(iv) Payment is received at the end of 20 years' time given quarterly compounding
(v) Payment is received at the end of one year's time given continuous compounding
(vi) Payment is received at the end of 20 years' time given continuous compounding

Solution

Using the formulas

$$PV_t = \frac{V}{[1 + (r/n)]^{nt}} \qquad \text{for discrete compounding}$$

$$PV_t = Ve^{-rt} \qquad \text{for continuous compounding}$$

we get

(i) $\frac{\$25{,}000}{1 + 0.08} = \frac{\$25{,}000}{1.08} = \$23{,}148.15$

(ii) $\frac{\$25{,}000}{(1 + 0.08)^{20}} = \frac{\$25{,}000}{(1.08)^{20}} = \$5{,}363.70$

(iii) $\dfrac{\$25{,}000}{[1+(0.08/4)]^4} = \dfrac{\$25{,}000}{(1.02)^4} = \$23{,}096.14$

(iv) $\dfrac{\$25{,}000}{[1+(0.08/4)]^{80}} = \dfrac{\$25{,}000}{(1.02)^{80}} = \$5{,}127.74$

(v) $\$25{,}000e^{-0.08} = \$25{,}000(0.9231163) = \$23{,}077.91$

(vi) $\$25{,}000e^{-1.6} = \$25{,}000(0.2018965) = \$5{,}047.41$ ■

As the applications above indicate, computing the present value of streams of payments or periodic payments can be quite tedious. These computations, however, are essential to banks and other financial institutions that need to determine the equivalence of a stream of payments to a fixed current amount. Some further techniques and formulas for this purpose are developed in section 3.5.

EXERCISES

1. Find the present value of \$100 to be received three years from now, assuming annual compounding of interest, given an interest rate of 12%.

2. If the interest rate is 10% how much money would one need to receive now to be equivalent to \$1 million received two years from now if:

(a) Interest is compounded annually?

(b) Interest is compounded semiannually?

(c) Interest is compounded monthly?

(d) Interest is compounded continuously?

3. Suppose that the interest rate (r) is such that the present value of receiving $\$V_2$ in t_2 years from now is the same as the present value of receiving $\$V_1$ in t_1 years from now, $t_2 > t_1$. Assume that interest is compounded annually.

(a) Show that $V_2 > V_1$.

(b) Show that the present value of receiving $\$V_2$, $(t_2 + k)$ years from now is also equal to the present value of receiving $\$V_1$, $(t_1 + k)$ years from now for any value of k. (That is, it is the absolute difference between time periods that matter.)

4. Prove, according to definition 3.2, that the present value of an amount of money, V, received t periods from now and evaluated at an interest rate $r > 0$ approaches zero as $t \to \infty$.

5. Suppose a country with a current population of 100 million is expected to experience a population growth rate of 2% per year for the next 50 years. Assuming continuous compounding, what is the expected size of the population:

(a) 5 years from now

(b) 10 years from now

(c) 20 years from now

6. Making use of the same situation and assumptions as in question 5 above, find the length of time required for the population in this country to double.

3.4 Properties of Sequences

The following results concerning convergent sequences are straightforward and so are given without proof.

Theorem 3.1 Suppose that a_n and b_n are convergent sequences with limits L^a and L^b respectively. It follows that:

(i) $\lim_{n\to\infty} ca_n = cL^a$ for c any constant
(ii) $\lim_{n\to\infty}(a_n \pm b_n) = L^a \pm L^b$
(iii) $\lim_{n\to\infty}(a_n)(b_n) = L^a L^b$
(iv) $\lim_{n\to\infty}(a_n/b_n) = L^a/L^b$ provided that $L^b \neq 0$

The usefulness of theorem 3.1 lies in the fact that it is sometimes easier to determine independently the limits of certain *parts* of the terms of a sequence. Consider, for example, the sequence formed by the terms

$$\left(\frac{\alpha + \dfrac{2}{n^2+3}}{\beta + \dfrac{1}{n}}\right)$$

To expand this and determine the limit as $n \to \infty$ would be a tedious exercise. However, it is clear that the limit of the numerator is α and the limit of the denominator is β. Thus one can apply result (iv) of theorem 3.1 to see that the limit of the expression is α/β. The following two examples illustrate the first two results of theorem 3.1.

Example 3.10 Use the result that $\lim_{n\to\infty} 1/n = 0$ and theorem 3.1 (i) to find the limit of the sequence $f(n) = 2/n$, $n = 1, 2, 3, \ldots$.

Solution

Since

$$\frac{2}{n} = 2\left(\frac{1}{n}\right)$$

we can write

$$\lim_{n\to\infty} \frac{2}{n} = \lim_{n\to\infty} 2\frac{1}{n} = 2\left(\lim_{n\to\infty} \frac{1}{n}\right) = 2(0) = 0$$

■

Example 3.11 Use the results that $\lim_{n\to\infty} 1/n = 0$ and, $\lim_{n\to\infty} 1/(n+1) = 0$, and theorem 3.1 (ii) to find the limit of the sequence

$$f(n) = \frac{2n+1}{n^2+n}, \qquad n = 1, 2, 3, \ldots$$

Solution

Since

$$\frac{2n+1}{n^2+n} = \frac{n+n+1}{n(n+1)} = \frac{n}{n(n+1)} + \frac{n+1}{n(n+1)} = \frac{1}{n+1} + \frac{1}{n}$$

we can write

$$\lim_{n\to\infty} \frac{2n+1}{n^2+n} = \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n}\right] = \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{n} = 0 + 0 = 0$$

■

Theorem 3.2 provides some simple and useful results when we are faced with algebraic combinations of definitely divergent sequences in conjunction with convergent sequences. An analogous set of results applies for the case where b_n has limit $-\infty$.

Theorem 3.2 Suppose that a_n is a convergent sequence with limit L^a, b_n is a definitely divergent sequence with limit $+\infty$, and c is a constant. It follows that:

(i) $\lim_{n\to\infty} cb_n = +\infty$ for $c > 0$ and $-\infty$ for $c < 0$
(ii) $\lim_{n\to\infty}(a_n + b_n) = +\infty$
(iii) $\lim_{n\to\infty}(a_n - b_n) = -\infty$
(iv) $\lim_{n\to\infty}(a_n)(b_n) = +\infty$ for $L^a > 0$ and $-\infty$ for $L^a < 0$
(v) $\lim_{n\to\infty}(a_n/b_n) = 0$

and, as a special case,

$$\lim_{n\to\infty} \frac{c}{b_n} = 0 \quad \text{for } c \text{ constant}$$

An analogous set of results applies for the case where b_n has limit $-\infty$.

The following examples illustrate results (i) and (iv) of theorem 3.2.

Example 3.12 Use the result that $\lim_{n\to\infty} n = +\infty$ and result (i) of theorem 3.2 to find $\lim_{n\to\infty} n/2$.

Solution

$$\lim_{n\to\infty} \frac{n}{2} = \lim_{n\to\infty} \frac{1}{2}n = \frac{1}{2}\left[\lim_{n\to\infty} n\right]$$

and so $\lim_{n\to\infty} n/2 = +\infty$. ■

Example 3.13 Use the results that

$$\lim_{n\to\infty} \frac{n+3}{n} = 1, \qquad \lim_{n\to\infty} \frac{n^2-1}{n} = +\infty$$

and result (iv) of theorem 3.2 to find

$$\lim_{n\to\infty} \frac{(n+3)(n^2-1)}{n^2}$$

Solution

Since

$$\frac{(n+3)(n^2-1)}{n^2} = \left(\frac{n+3}{n}\right)\left(\frac{n^2-1}{n}\right)$$

we can write

$$\lim_{n\to\infty} \frac{(n+3)(n^2-1)}{n^2} = \lim_{n\to\infty} \frac{n+3}{n} \lim_{n\to\infty} \frac{n^2-1}{n} = +\infty \quad \blacksquare$$

Another useful application of this theorem concerns the present-value formula developed in section 3.3:

$$PV_t = \frac{V}{(1+r)^t}$$

Theorem 3.2 provides a proof of the claim that, if $r > 0$, then $PV_t \to 0$ as $t \to \infty$. Since the denominator is a definitely divergent sequence (if $r > 0$) and the numerator is a constant, then part (v) of theorem 3.2 establishes the result.

Since sequences are functions with their domains being the set of positive integers, one can define characteristics of monotonicity and boundedness in an analogous manner as was done for general functions in chapter 2. This is done formally below. (The property of boundedness was addressed informally earlier in this chapter.)

Definition 3.5 A sequence is **monotonically increasing** if $a_1 < a_2 < a_3 < \dots$ and is **monotonically decreasing** if $a_1 > a_2 > a_3 > \dots$. In either case the sequence is said to be monotonic.

Definition 3.6 A sequence is **bounded** if and only if it has a lower bound and an upper bound.

The following theorem is of obvious use in determining whether certain sequences are convergent.

Theorem 3.3 A monotonic sequence is convergent if and only if it is bounded.

Example 3.14 Use theorem 3.3 to show that the sequence $a_n = 1/2^n$, $n = 1, 2, 3, \dots$, is convergent.

Solution

The sequence $a_n = 1/2^n$ is monotonically decreasing. To see this, note that

$$a_{n+1} = \frac{1}{2^{n+1}} = \frac{1}{2^n}\frac{1}{2}, \quad a_n = \frac{1}{2^n}, \qquad \text{and so } a_{n+1} < a_n$$

Moreover this sequence is bounded, since

$$0 < \frac{1}{2^n} < 1 \qquad \text{for every } n = 1, 2, 3, \ldots$$

Therefore the sequence is convergent. ■

Example 3.15 Use theorem 3.3 to show that the sequence $a_n = -2n$, $n = 1, 2, 3, \ldots$, is divergent.

Solution

The sequence is monotonically decreasing. To see this, note that

$$a_{n+1} = -2(n+1) = -2n - 2, \qquad a_n = -2n$$

and so $a_{n+1} < a_n$. Moreover this sequence is unbounded since for every $K > 0$, no matter how large,

$$-2n < -K \qquad \text{for every } n > \frac{K}{2}$$

That is, this sequence is not bounded below. Therefore the sequence is divergent. ■

We can see from these two examples that, for the case of a monotonic sequence, it can be much easier to check for convergence or divergence using theorem 3.3 than by using definitions 3.2, 3.3, and 3.4. If a sequence is not monotonic, then theorem 3.3 is not of any use. The following two sequences illustrate this fact. Both the sequence

$$a_n = (-1)^n, \qquad n = 1, 2, 3, \ldots$$

and the sequence

$$a_n = \left(-\frac{1}{2}\right)^n, \qquad n = 1, 2, 3, \ldots$$

are bounded and neither is monotonic; yet the first is not convergent while the second one is.

EXERCISES

1. Use theorem 3.3 to show that the sequence $PV_t = V/(1+r)^t$ is convergent when $r \geq 0$ and divergent when $-1 < r < 0$.
2. Prove result (i) of theorem 3.2.
3. Prove result (iii) of theorem 3.2.

3.5 Series

A **series** is a special *type* of sequence. Consider, for example, the sequence $a_t = 1/(t^2+t)$, $t = 1, 2, 3, \ldots$. By summing the first n terms of this sequence, we generate another sequence:

$$s_n = \sum_{t=1}^{n} a_t = \sum_{t=1}^{n} \frac{1}{(t^2+t)} = 1 - \frac{1}{(n+1)} = \frac{n}{(n+1)}$$

To see how to derive this result, note that

$$a_t = \frac{1}{(t^2+t)} = \frac{1}{t} - \frac{1}{(t+1)}$$

By taking the right-hand side of this equality to the common denominator $t(t+1)$, we get

$$\frac{1}{t} - \frac{1}{(t+1)} = \frac{(t+1)-t}{t(t+1)} = \frac{1}{t^2+t}$$

Therefore we can rewrite s_n as

$$\begin{aligned} s_n &= \sum_{t=1}^{n} \left(\frac{1}{t} - \frac{1}{t+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ &\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \end{aligned}$$

The second term of every pair combined with the first term of the subsequent pair is always zero. So we are left with

$$s_n = 1 - \frac{1}{n+1}$$

The sum s_n is itself a sequence. Since the sequence is the sum of the first n terms of some other sequence, it is called a series.

A simpler example is that $s_n = n$ is the series associated with the constant sequence $a_t = 1$. Thus we have the following definition:

Definition 3.7 If $a_t,\ t = 1, 2, 3, \ldots$ is a sequence, then $s_n = \sum_{t=1}^{n} a_t,\ n = 1, 2, 3, \ldots,$ is called a **series.**

Since a series is just a specific type of sequence, any results derived for sequences also apply to series. For example, if a series is monotonic and bounded, then it has a limit (theorem 3.3). However, due to the particular property of series, as specified in definition 3.7, there are some further useful results that can be generated. The following theorem, for example, is very useful in determining whether a series converges:

Theorem 3.4 If $s_n = \sum_{t=1}^{n} a_t$ is the series associated with sequence a_t and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

it follows that:

(i) if $L < 1$, then the series s_n converges
(ii) if $L > 1$, then the series s_n diverges
(iii) if $L = 1$, then the series s_n may converge or diverge

Theorem 3.4 offers a simple test for determining whether a series converges or diverges. It is based on the ratio of the absolute value of successive terms of the underlying sequence that is summed in order to obtain the series. If this ratio eventually exceeds 1, that is,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$$

then this means that the series is generated by adding numbers of increasing size, and so it is not surprising that the series diverges. Alternatively, if this ratio is less than 1, then the series is generated by adding successively smaller numbers, and so it converges. For a case where this ratio equals 1, the test is not informative, and so more effort is required to determine whether the series in question converges or diverges.

Example 3.16 **The Geometric Series**

One of the most important series in mathematics and in economics is the **geometric series.** It provides us with an excellent example of the usefulness of the test for convergence given in theorem 3.4. To see this, consider the sequence $a_t = a\rho^{t-1}$ for a and ρ constants. The series formed from this sequence,

$$s_n = \sum_{t=1}^{n} a\rho^{t-1} = a + a\rho + a\rho^2 + a\rho^3 + \cdots + a\rho^{n-1}$$

is the geometric series. Upon forming the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a\rho^n}{a\rho^{n-1}}\right| = |\rho|$$

we see from theorem 3.4 that this series converges if $|\rho| < 1$ and diverges if $|\rho| > 1$. It is also easy to see that if $|\rho| = 1$ we get $a_t = a$ and $s_n = na$, which diverges for any $a \neq 0$. The following example provides another case in which Theorem 3.4 fails to determine whether a series converges.

Example 3.17 The series constructed from the sequence $a_n = 1/n$, called the **harmonic series,** does not converge even though $\lim_{n\to\infty} a_n = 0$. (This is left as an exercise.) The intuition is that if a series is to converge, the nth term (a_n) of its associated sequence must approach zero "quickly enough" that the sum of the terms is finite even as $n \to \infty$. This is the rationale of the condition stated in theorem 3.4, that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

which ensures that a series is convergent. Note that for the harmonic sequence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = 1$$

■

For the case of $|\rho| < 1$, it is fairly easy to derive the limit of the geometric series,

$$\lim_{n\to\infty} s_n = \frac{a}{1-\rho}$$

This is done by noting that upon multiplying each term in the expression for s_n by ρ, we get

$$\rho s_n = a\rho + a\rho^2 + a\rho^3 + \cdots + a\rho^{n-1} + a\rho^n$$

where

$$s_n = a + a\rho + a\rho^2 + a\rho^3 + \cdots + a\rho^{n-1}$$

We can see from the two expressions above that

$$s_n - \rho s_n = a - a\rho^n$$

This allows us to write the sum s_n as

$$s_n = a\left(\frac{1-\rho^n}{1-\rho}\right) \tag{3.4}$$

Since for $|\rho| < 1$ we get $\rho^n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} s_n = \frac{a}{1-\rho} \tag{3.5}$$

Both equation (3.4), the formula for the sum of a finite geometric series, and equation (3.5), the formula for the sum of an infinite geometric series, have many uses in economics. In this chapter we focus on applications of series that involve evaluating streams of money payments or receipts, such as in an investment project or a mortgage. See the Web page http://mitpress.mit.edu/math_econ3 for some additional applications as well as a discussion of the classic problem referred to as the Achilles paradox.

Present Value of a Stream of Payments

Earlier we saw how $PV_t = V/(1+r)^t$ (see equation 3.3) represents the present value of an amount of money V received t periods into the future. In many economic settings we need to compute the equivalent present value of a series (i.e., the sum total) of such amounts. For example, a mortgage or other long-term loan represents a current sum of money loaned to an individual or institution in return for a stream of future payments. Thus, if an individual makes annual payments at the end of each year in amount V for T years, with the interest rate being r, then the present value of this stream of payments is

$$P_T = \sum_{t=1}^{T} \frac{V}{(1+r)^t} = \frac{V}{(1+r)^1} + \frac{V}{(1+r)^2} + \cdots + \frac{V}{(1+r)^T} \tag{3.6}$$

Note that expression (3.6) is simply a geometric series as described earlier in this section in example 3.16. Replace the first term (a) from that series with $V/(1+r)$ and the multiplicative factor ρ with $1/(1+r)$, where ρ is referred to as the discount factor. For finite valued T, equation (3.4) can be used to recast equation (3.6) as

$$P_T = \frac{V}{(1+r)} \left\{ \frac{1-[1/(1+r)]^T}{1-[1/(1+r)]} \right\} \tag{3.7a}$$

In the limit as $T \to \infty$, equation (3.5) would be recasted as

$$P_\infty = \lim_{T\to\infty} P_T = \lim_{T\to\infty} P_T \frac{V}{(1+r)} \left(\frac{1-\left(\frac{1}{1+r}\right)^T}{1-\left(\frac{1}{1+r}\right)} \right)$$

Although this expression is a little messy, it can easily be simplified by noting that $1 - 1/(1+r) = r/(1+r)$ and by applying a little algebra to get

$$P_\infty = \frac{V}{r} \tag{3.7b}$$

So, for example, the present value of a stream of payments or receipts of \$8,000 occurring at the end of each of 50 years over the future at an interest rate of 8% (i.e., $V = \$8{,}000$, $r = 0.08$, and $T = 50$) would be computed using equation (3.7a) to obtain the value $P_{50} = \$97{,}867.88$. If this stream is continued forever $(T \to \infty)$, it will be worth $P_\infty = \$100{,}000$. This second result is straightforward as \$100,000 invested at an annual interest rate of 8% will sustain a flow of income of \$8,000 forever.

What may be surprising is that the flow of \$8,000 a year forever is worth only slightly more than if the flow stops after the 50th year. The reason is that all the income received after the end of the 50th year in the case of $T \to \infty$ must be discounted heavily. In fact, the value of \$8,000 a year received forever from (end of) year 50 onward is also worth, **at the beginning of the 50th year**, \$100,000. To find what this is worth currently, one must discount this value by the rate $1/(1+r)^{50}$

to obtain its present value. This gives the amount of $\$100{,}000/(1+0.08)^{50} =$ \$2,132.12. This is precisely the difference between P_∞ and P_T (with $T = 50$) in this example (i.e. $\$100{,}000 - \$97{,}867.88 = \$2{,}132.12$).

In many circumstances we want to do the reverse of the exercise above. For example, a bank lending \$100,000 for a mortgage at an annual interest rate of 8% to a homeowner who will make the same payment value V at the end of each year needs to compute this value V. If the payments were to be made forever (in perpetuity) it would be a trivial exercise to compute V by inverting equation (3.7b); that is,

$$V = P_\infty \cdot r = \$100{,}000 \cdot (0.08) = \$8{,}000$$

If, however, the payments are expected to be made for only 50 years, one must invert equation (3.7a) to obtain

$$V = \frac{P_T \cdot (1+r)}{\left(\frac{1-[1/(1+r)]^T}{1-[1/(1+r)]}\right)} = \frac{\$100{,}000 \cdot (1.08)}{\left(\frac{1-[1/1.08]^{50}}{1-[1/1.08]}\right)} = \$8{,}174.29$$

Note that the payment level required for a 50-year period is not mush different than that for an infinite period (i.e., \$8,174.29 vs. \$8,000.00). This is because payments made beyond 50 years into the future are discounted heavily in present-value calculations at least for a relatively high interest rate of 8% per year.

Example 3.18 Suppose that a stream of equal payments of amount \$10,000 per year at the end of each year is to continue in perpetuity. At the interest rate of 6% compute

(i) the present value of this entire stream of benefits
(ii) the present value of the benefits beginning at the end of the 51st year
(iii) the present value of the first 50 years of benefits

Solution

(i)

$$P_\infty = \lim_{T\to\infty} P_T = \sum_{t=1}^{\infty} \frac{\$10{,}000}{(1.06)^t} = \frac{\$10{,}000}{0.06} = \$166{,}666.67$$

(ii) as of the end of the 50th year the present value of \$10,000 per year in perpetuity is \$166,666.67, as computed in part (i). Since this is in effect received at the end of the 51st year, its *current* (i.e., as of now) present value must be discounted so that it becomes

$$\frac{\$166{,}666.67}{(1.06)^{50}} = \$9{,}048.06$$

(iii) The present value of the first 50 payments is simply the answer in (i) less that in (ii):

$$\$166{,}666.67 - \$9{,}048.06 = \$157{,}618.61 \qquad \blacksquare$$

Note that we can also compute the value in (iii) using equation (3.7a); that is,

$$P_{50} = \frac{10{,}000}{1.06}\left(\frac{1-(1/1.06)^{50}}{1-(1/1.06)}\right) = 157{,}618.61$$

The examples above all deal with the problem of determining the present value of a series of equal payments. In general, however, one can evaluate the present value of any pattern of payments. Suppose, for example, that a business firm is considering the possibility of making a current (and immediate) investment of $\$C$, the payoff of which will be the sales revenue of a product whose sales will increase over time. Let us assume that the production process will begin at the end of one year and that net profit from sales of the product is $\$\pi(1+g)$ the first year (as measured assuming it accrues at the end of the year) and will grow at a rate of g each subsequent year. Thus the profit for period t will be $a_t = \pi(1+g)^t$, and the (undiscounted) value of the stream of benefits (gross benefits) will be

$$GB = \lim_{T\to\infty}\sum_{t=1}^{T} a_t = \lim_{T\to\infty}\sum_{t=1}^{T}\pi(1+g)^t$$

which is a divergent series if π and g are positive. The *discounted* or present value of the stream of benefits is

$$PVB = \lim_{T\to\infty}\sum_{t=1}^{T}\frac{\pi(1+g)^t}{(1+r)^t} \tag{3.8}$$

which is just a geometric series, with $a = \pi(1+g)/(1+r)$ and $\rho = (1+g)/(1+r)$, so that $PVB = \pi(1+g)/(r-g)$ and is finite valued if and only if $g < r$ (i.e., $|\rho| < 1$). To decide whether the investment is profitable, one merely needs to determine whether $PVB > C$ or $PVB < C$. This illustrates how discounting is used in assessing the net benefits of a project. In most instances the costs are heavily concentrated in the early periods with the benefits spread out over a longer time horizon.

Example 3.19 Suppose that a stream of payments arising from some business venture begins with an amount of \$20,000 immediately and grows at the rate of 4% per annum thereafter, forever. Given an interest rate of 8%, find the present value of this stream of payments.

Solution

From the formula given in equation (3.8), we have that the present value of the stream of payments, beginning at the end of one year, is

$$PVB = \lim_{T\to\infty} \sum_{t=1}^{T} \frac{\$20{,}000(1+0.04)^t}{(1+0.08)^t}$$
$$= \lim_{T\to\infty} \sum_{t=1}^{T} \$20{,}000 \left(\frac{1.04}{1.08}\right)^t$$

This is an example of a geometric series, with $a = \$20{,}000(1.04/1.08)$ and $\rho = (1.04/1.08)$. Using the formula given in equation (3.5) for finding the sum of an infinite geometric series, we get

$$PVB = \frac{a}{1-\rho} = \$520{,}000$$

Upon adding the initial payment of \$20,000 the final answer is \$540,000. ■

This illustration leads us into a more general and very powerful application of present-value calculations in the area of *project evaluation*. Many investment plans or projects involve costs and benefits that occur disproportionately over time. It is often the case that a larger fraction of the costs are incurred early on (up front) while benefits are more evenly spread over a long interval of time. Construction of a hydroelectric project is a stark example of this phenomenon. In order to compare many of the front-end costs to the stream of benefits that will occur later, it is standard procedure to compare all costs and benefits evaluated in present-value terms. The following example shows how one can creatively use present-value calculations to ascertain whether a project is worthwhile.

Example 3.20 Suppose that a profit-minded public utility is trying to decide whether to expand its hydroelectric capacity by building a new dam. The project has the following costs and benefits:

Building costs	\$200 million	Immediately
	\$100 million	At the end of each of the next three years

Operating costs	\$5 million	Beginning at the end of the fourth year and continuing thereafter forever
Revenue	\$30 million	Beginning at the end of the fourth year and continuing thereafter forever

The interest rate is 6%. Should the utility proceed with this expansion?

Solution

$$\begin{aligned} PV \text{ of building costs} &= 200\text{m} + \frac{100\text{m}}{(1+0.06)} + \frac{100\text{m}}{(1+0.06)^2} + \frac{100\text{m}}{(1+0.06)^3} \\ &= 200\text{m} + \frac{100\text{m}}{1.06} + \frac{100\text{m}}{1.1236} + \frac{100\text{m}}{1.1910} \\ &= 200\text{m} + 94{,}339{,}623 + 88{,}999{,}644 + 83{,}961{,}928 \\ &= 467{,}301{,}195 \end{aligned}$$

Revenue and operating costs occur over the same time period, and so the value of the stream of net operating revenue (i.e., revenue net of operating costs but not net of building costs) is

$$\sum_{t=1}^{\infty} \frac{25{,}000{,}000}{(1+0.06)^t} = \frac{25{,}000{,}000}{0.06} = 416{,}666{,}667$$

from the perspective of the end of the third year. It follows that we must discount this figure accordingly to compare it to the present value of building costs evaluated from *today's* perspective. Therefore

$$\begin{aligned} PV \text{ of net operating revenue} &= \frac{416{,}666{,}667}{(1+0.06)^3} \\ &= \frac{416{,}666{,}667}{1.191016} \\ &= 349{,}841{,}368 \end{aligned}$$

Since the present value of building costs (\$467,301,195) exceeds that of net operating revenues \$349,841,368.21, the utility should not proceed with the project. ■

The following simple example explains clearly the economic rationale for discounting revenues and costs using net present-value formulas in order to decide on the economic viability of an investment. As the example illustrates, the present-value approach is an appropriate framework regardless of whether the funds needed to finance the costs must be borrowed.

Example 3.21 An investment project that requires immediate costs of \$200,000 generates net revenues of \$15,000 per year beginning at the end of the first year and continuing forever.

(i) Find the net present value of this investment if the interest rate is 9%.
(ii) Illustrate why this investment is not worthwhile if the investor must borrow the money at an interest rate of 9%.
(iii) Illustrate why this investment is not worthwhile if the investor doesn't need to borrow the money but can invest any financial capital at a rate of 9%.

Solution

(i) The present value of the revenue stream is

$$P_\infty = \frac{15{,}000}{0.09} = 166{,}666.67$$

while the present value of costs is \$200,000. Since the present value of costs exceeds the present value of revenues, this is not a worthwhile scheme.
(ii) If the investor must borrow the funds at $r = 9\%$, then the interest cost will be $\$200{,}000 \times 0.09 = \$18{,}000$ per year, which exceeds the annual revenues.
(iii) If the investor has the financial capital available, she could simply invest that money at 9% and generate more money (\$18,000 per year) than by investing in the project (\$15,000 per year). ■

In situations as in the preceding example, it is useful to compute the **internal rate of return** of the investment. This is the implicit rate of return earned on the funds invested. If it exceeds the market rate of interest, then it is a financially worthwhile project and not otherwise. For the example above, since an immediate investment of \$200,000 generates a stream of payments of \$15,000 per year forever, it follows that the internal rate of return is

$$\frac{15{,}000}{200{,}000} = 0.075 \quad \text{or} \quad 7.5\%$$

This concept is defined more generally below.

Definition 3.8 The **internal rate of return** of a project or investment is the rate of interest that equates the present value of benefits and costs.

The single calculation of the internal rate of return is useful in that it implicitly determines the set of interest rates under which the investment is economically viable. Notice that for the investment described in example 3.21, if the market interest rate exceeds 7.5% (the internal rate of return of the project), then it is not an economically viable project. As the next example illustrates, this rate of return is not always found so easily, and often computer simulation is required to calculate it.

Example 3.22 Find the internal rate of return for the following investment project:

Initial costs (immediate)	\$100,000
Subsequent costs (end of one year and continuing forever)	\$50,000 annually
Revenues (from end of year 6 and continuing forever)	\$70,000 annually

Solution

The internal rate of return is that value of the interest rate, $\hat{r}$, which when used in the net present-value calculations will exactly balance costs against revenues (i.e., lead to a net present value of zero). Therefore this value for $\hat{r}$ is the solution to the following equation:

$$\underbrace{\$100{,}000 + \sum_{t=1}^{\infty} \frac{\$50{,}000}{(1+\hat{r})^t}}_{PV \text{ of costs}} = \underbrace{\sum_{t=6}^{\infty} \frac{\$70{,}000}{(1+\hat{r})^t}}_{PV \text{ of revenues}}$$

The right side of this equation represents the amount \$70,000 every year in perpetuity beginning six years from now. As of the beginning of year 6 (i.e., end of year 5), this has a present value of $\$70{,}000/\hat{r}$, and so discounting this number means that it has a current value of

$$\frac{(\$70{,}000/\hat{r})}{(1+\hat{r})^5}$$

It is not easy to solve analytically for $\hat{r}$ in this relationship, which can be written

$$\underbrace{\$100{,}000 + \frac{\$50{,}000}{\hat{r}}}_{PV \text{ of costs}} = \underbrace{\frac{(\$70{,}000/\hat{r})}{(1+\hat{r})^5}}_{PV \text{ of benefits}}$$

By using a computer to try several values for r, we find the solution to be $\hat{r} = 0.05$. Values less than $\hat{r} = 0.05$ give a higher present value for benefits than costs, while values greater than $\hat{r} = 0.05$ give a higher present value for costs than for benefits.

The reason for this is that the benefits of this project accrue relatively further in the future and so are reduced to a greater extent than are costs as the interest rate rises. Table 3.1 illustrates this point.

Table 3.1

Interest rate (r)	PV of costs	PV of benefits
0.04	\$1,350,000	\$1,438,000
0.05	\$1,100,000	\$1,100,000
0.06	\$933,300	\$871,800

■

Further examples of using the formulas to understand various economic models and concepts are provided on the Web page http://mitpress.mit.edu/math_econ3.

EXERCISES

1. Consider the trivial sequence $a_n = c$, $c > 0$ a constant. Show that this is an example of a sequence for which

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

and the series generated by it diverges (see theorem 3.4).

2. Show that the harmonic series

$$s_n = \sum_{i=1}^{n} a_i, \qquad a_i = \frac{1}{i}$$

diverges. [Hint: Group terms from $i = 2^k + 1$ to 2^{k+1}, $k = 1, 2, 3, \ldots,$ and note that the sum in each group is greater than $1/2$.] Show that this is an example of a sequence for which $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$ and the series generated by it diverges.

3. By writing out and expanding expressions for s_n and $\rho^2 s_n$, prove that for $|\rho| < 1$,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{t=1}^{n} a\rho^{2t-1} = \frac{a\rho}{1-\rho^2}$$

4. Evaluate the net present value of the following streams of income:

(a) $V = \$1{,}000$ per year at an interest rate of 5% in perpetuity

(b) $V = \$1{,}000$ per year at an interest rate of 12% in perpetuity

(c) $V = \$1$ million per year at an interest rate of 10% in perpetuity

(d) $V_t = \$1$ million per year in perpetuity, but not begining until year $t = 5$ at an interest rate of 15%

5. Suppose that a project has an immediate cost of \$10 million and running costs of \$1 million per year beginning at the end of a one-year construction period. At the end of this year, annual gross revenue from the project of \$1.5 million per year is generated in perpetuity. (You may assume that the running costs and revenues accrue at the end of each year.)

(a) Is the project profitable in a net present-value sense if the interest rate is 8%?

(b) For what range of interest rates (nonnegative) is the present value of net revenues (including the immediate cost of \$10 million) positive?

6. A power company can develop a hydroelectric project at one of two capacity levels, one megawatt or two megawatts, at the cost of \$1 billion or \$1.75 billion respectively. Construction in either case takes one year's time with the cost being incurred immediately. If the smaller capacity is chosen, it is not possible to increase capacity at a later date. The company can sell the capacity from one megawatt on a contractual basis for a net revenue of \$200 million per year (in perpetuity) with payments beginning in two years' time (i.e., one year after the construction period.) No customer will currently purchase the second megawatt of capacity, but a second party (e.g., another state or country) will contract to purchase the second megawatt of power for \$300 million per year in perpetuity beginning 10 years from now (with first payment at the end of that year).

(a) Find the net present value of building for each capacity level if the interest rate is 5%. Should the power company invest in 0, 1, or 2 megawatts of capacity?

(b) Do the same as in part (a) for interest rates of 10% and 15%.

CHAPTER REVIEW

Key Concepts

bounded above
bounded below
bounded sequence
continuous compounding
convergent sequence
definite divergence
divergent sequence
discrete compounding
geometric series
harmonic series
internal rate of return
limit
monotonically decreasing
monotonically increasing
present value
sequence
series
unbounded sequence

Review Questions

1. Explain and provide the notation describing a sequence.
2. What does it mean for a sequence to be bounded?
3. What does it mean for a sequence to have a limit?
4. What does it mean for a sequence to be monotonically decreasing or increasing?
5. Why is boundedness not sufficient for a sequence to have a limit unless the sequence is monotonically increasing or decreasing?
6. What is the difference between an unbounded sequence that is definitely divergent and one that is not? Give an example.
7. Explain the relationship between a sequence and a series.
8. Write out the formula for a finite geometric series.
9. Write out a formula for an infinite geometric series.
10. Why is it that when computing the present value of a project yielding fixed payments, received periodically over a long period of time, it is often assumed that the payments continue in perpetuity?
11. What does continuous compounding mean?

Review Exercises

1. Determine the first five terms of each of the following sequences. In each case, draw a graph such as those in figures 3.1 to 3.6.

   (a) $f(n) = 1/n^2$

   (b) $f(n) = 2 + [(-1)^n\,(1/n)]$

(c) $f(n) = n/(3n + 2)$

(d) $f(n) = -n^2$

(e) $f(n) = (n^2 + 2n + 1)/(n + 1)$

(f) $f(n) = 5 + 1/n$

(g) $f(n) = 5 - 1/n$

2. Determine the limit, if one exists, for each sequence given in question 1. If a sequence is divergent, determine whether it is definitely divergent. [Hint: Use definitions 3.2 and 3.4.]

3. **(a)** Compute the present value of the following amounts of money, given an interest rate of 8%.

(i) \$100 received one year from now

(ii) \$150 received five years from now

(b) Suppose an individual can earn a rate of return of 8% or can borrow money at this same rate. Explain intuitively why the resulting ranking of the alternatives given in part (a) should correspond to the relative present values.

4. *Prove result (iv) of theorem 3.2.

5. Evaluate the net present values of the following streams of income:

(a) $V =$ \$100 per year in perpetuity at an interest rate of 10%

(b) $V =$ \$100 per year at an interest rate of 10% for 25 years

(c) $V =$ \$100 per year beginning after 25 years at an interest rate of 10% for 25 years

6. Suppose that an investment project has an immediate cost of \$100 million followed by costs of \$50 million at the end of one year and a further \$25 million at the end of two years. Net revenues (i.e., revenues in excess of operating costs) accrue in the amount of \$16 million at the end of each year beginning in three years' time. Find the net present value of this investment project given an interest rate of 9%.

7. A student in economics has completed her undergraduate degree and has been accepted into a one-year postgraduate business program. Upon completion of this degree she will earn an additional \$2,000 a year for each of the next 40 years. She will give up \$18,000 income that she would otherwise earn and sustain additional costs of \$2,000 for educational materials such as books. She cares only about the financial implications of her decisions.

(a) If the interest rate for borrowing and investing is the same, at amount 8%, should she accept the offer into the business program? Explain.

(b) What is the internal rate of return of attending this business program? How does your answer indicate conditions for which it is worthwhile to attend this program?

(c) Do your answers to parts (a) and (b) depend on whether this individual would need to borrow money or use any savings she has? Explain.

(d) How does this example explain the phenomenon that attendance in post-secondary programs tends to rise when the unemployment rate rises?

Part II Univariate Calculus and Optimization

Chapter 4 Continuity of Functions

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Revenue Function, Cost Function, and Profit Function for a Perfectly Competitive Firm
- Hotelling's Location Model
- Intermediate-Value Theorem
- Practice Exercises

The idea of continuity of a function is extremely important in mathematics. Many convenient techniques of analysis can be used if a function is continuous but not if it is discontinuous. In modeling economic problems, we often assume that we can represent various economic concepts by continuous functions (e.g., the relationship between the quantity of some commodity produced by the firm and its profit level). Thus it is important to know precisely what is the content of this assumption, especially since in many instances there is a natural reason to believe that the function will *not* be continuous everywhere, and in some cases this turns out to be an important consideration from an economic standpoint. As we will see in this and the subsequent chapter, both continuity and differentiability can be conveniently defined in terms of the existence of certain limits.

4.1 Continuity of a Function of One Variable

The intuitive notion of continuity can be explained easily with the aid of a graph. A function is continuous if the graph of the function has no *breaks* or *jumps* in it. On the following pages figures 4.1 and 4.2 present graphs of continuous functions, while figures 4.3 to 4.6 present graphs of discontinuous functions. We first define the continuity of a function $y = f(x)$, $x \in \mathbb{R}$, at a specific point, say $x = a$. This approach is, not surprisingly, referred to as *pointwise* continuity. We then say that a function is continuous on a set of points if it is continuous at each point of that set. If the function is continuous at each point in the domain, we simply say that the function is continuous. Before giving this definition of continuity formally, it is useful to give the definitions for left- and right-hand limits of a function at a given point.

Definition 4.1 The **left-hand limit** of a function $f(x)$, which is defined to the left of a, at the point $x = a$, exists and is equal to L^L, written

$$\lim_{x \to a^-} f(x) = L^L$$

if for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L^L| < \epsilon$, $\forall\, x$, satisfying $a - \delta < x < a$.

This definition states that if it is always possible to find some (possibly very small) range of values of x to the left of a (i.e., strictly less than a) for which the function values $f(x)$ can be made to be *arbitrarily* close to some value L^L, then we say that the left-hand limit of $f(x)$ at $x = a$ exists and is equal to L^L.

Example 4.1 Provide a specific example of a sequence of x-values that approaches the point $x = a$ from the left ($x \to a^-$).

Solution

An example of a sequence of x-values that approaches $x = a$ from the left is $x_n = a - (1/n)$, $n = 1, 2, 3, \ldots$. Choosing a value of δ *sufficiently small* to satisfy the condition $|f(x) - L^L| < \epsilon$ is equivalent to choosing n *sufficiently large* (i.e., $a - \delta < x_n < a$ for $n > 1/\delta$). In fact, if the left-hand limit is L^L, then the sequence of values $f_n = f(x_n)$ must converge to the value L^L for $x_n = a - (1/n)$. ■

This example of a sequence of values x_n that approaches $x = a$ from the left illustrates the relationship between limits of functions and limits of sequences (see chapter 3) and also explains why the notation $x \to a^-$ is used to denote a left-hand limit ($x = a$ minus some *small* positive value). One must realize, however, that for the left-hand limit of the function to exist and equal L^L, it must be the case that *any* sequence of x_n values approaching a from the left must induce a sequence of function values $f_n = f(x_n)$, which converges to the limit L^L.

A similar definition applies for the right-hand limit, namely that if there is always some (possibly very small) range of values of x to the right of a (i.e., strictly greater than a) for which the function values $f(x)$ can be made arbitrarily close to some value L^R, then we say that the right-hand limit of $f(x)$ at

$x = a$ exists and is equal to L^R. An example of a sequence of x-values that approaches $x = a$ from the right is $x_n = a + 1/n$, $n = 1, 2, 3, \ldots$. Hence we have the notation

$$\lim_{x \to a^+} f(x)$$

for the right-hand limit.

Definition 4.2

The **right-hand limit** of a function $f(x)$ which is defined to the right of $x = a$, at the point $x = a$, exists and is equal to L^R, written

$$\lim_{x \to a^+} f(x) = L^R$$

if for any $\epsilon > 0$, however small, there exists some $\delta > 0$, such that $|f(x) - L^R| < \epsilon$, $\forall x$, satisfying $a < x < a + \delta$.

Suppose that a function is defined on some open interval including the point a. We say that the $\lim_{x \to a} f(x)$ exists at a if the left- and right-hand limits exist and are equal to each other. We can now offer two equivalent definitions of pointwise continuity.

Definition 4.3

A function $f(x)$ defined on an open interval including the point $x = a$ is **continuous** at that point if

(i)

$$\lim_{x \to a} f(x)$$

exists; that is,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

and

(ii)

$$\lim_{x \to a} f(x) = f(a)$$

Definition 4.4 A function $f(x)$, defined on an open interval including the point $x = a$ is continuous at that point if there is some $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$, whenever $|x - a| < \delta$ for any $\epsilon > 0$.

A function that is *not* continuous is said to be **discontinuous.** Definition 4.3 is the more useful definition when trying to show that a function is discontinuous at a certain point. This will become clear when discussing why the examples of the functions illustrated in figures 4.3 to 4.6 are discontinuous. Each case demonstrates a distinct property that is the cause of the discontinuity. Together the cases exhaust the reasons why a function may be discontinuous.

It is often simpler, however, to use definition 4.4 to show that a function is continuous, as will be seen for the examples of continuous functions illustrated in figures 4.1 and 4.2. Definition 4.4 illustrates the similarity between the definition of continuity of a function and that of the limit of a sequence (definition 3.2). According to definition 4.4, a function $f(x)$ is continuous at the point $x = a$ if $f(x)$ is *arbitrarily close to* $f(a)$ for all values of x *close to* $x = a$ (i.e., for $|x - a| < \delta$), while a sequence $f(n)$ has limit value L if $f(n)$ is *arbitrarily close to* L for all values n beyond some value that is sufficiently large (i.e., $n > N$).

Example 4.2 Show that the linear function $f(x) = 2x$ is continuous.

Solution

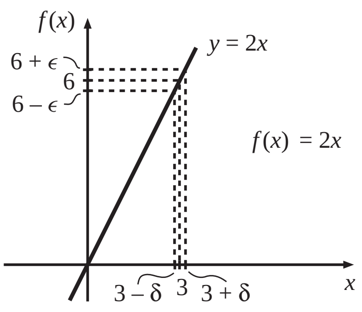


Figure 4.1 Function $y = 2x$ (with demonstration that it is continuous at $x = 3$)

Take the function $f(x) = 2x$. This function is continuous at every point $x \in \mathbb{R}$. To see this, consider the point $x = 3$ in figure 4.1. The function value at $x = 3$ is $f(3) = 6$. According to definition 4.4, this function is continuous at this point if, no matter how small a number $\epsilon > 0$ we choose, there is some value $\delta > 0$ (possibly very small) such that all the function values defined on the set of x values $(3-\delta, 3+\delta)$ lie within the set $(6-\epsilon, 6+\epsilon)$. Intuitively this means that the function values defined at points *near* $x = 3$ are all *close to* $f(3)$. For example, if we choose $\epsilon = 0.01$, then by choosing $\delta = 0.002$, we find that all function values $f(x)$ defined on the set $x \in (3 - 0.002, 3 + 0.002)$ lie within distance ϵ of $f(3)$. That is, $f(x) \in (f(3 - 0.002), f(3 + 0.002))$, which implies that $f(x) \in (6 - 0.004, 6 + 0.004)$. All of these values are within distance $\epsilon = 0.01$ of $f(3) = 6$. This would be true for any choice of $\delta \leq 0.005$. More generally, we see that at any value $x = a$ we can be assured that $f(x)$ will lie within distance ϵ of $f(a)$ as long as we choose $\delta < \epsilon/2$. That is, for $x \in (a - \epsilon/2, a + \epsilon/2)$, we get $f(x) \in (2a - \epsilon, 2a + \epsilon)$. ■

Informally then, a function is continuous at a point $x = a$ if the function is defined at that point (i.e., $f(a)$ exists) and you can meet the following challenge. No matter how small a value $\epsilon > 0$ is chosen, it is always possible to find a sufficiently small but positive δ such that for all the x-values within distance δ of the point $x = a$, the function values fall within distance ϵ of $f(a)$.

Example 4.3 Show that the polynomial function $f(x) = x^3 - 2x^2 + 1$ is continuous at the point $x = 1/2$.

Solution

The graph for the polynomial function $f(x) = x^3 - 2x^2 + 1$ is drawn in figure 4.2. At $x = 1/2$, we see that the function takes on the value $f(1/2) = 5/8$. It is clear from the diagram that if for any $\epsilon > 0$ we can find a value δ such that for $x \in (1/2 - \delta, 1/2 + \delta)$, the function values will lie within the set $f(x) \in (5/8 - \epsilon, 5/8 + \epsilon)$. In this case the actual choice for δ for any given ϵ is not so easy to ascertain. Some reasonably straightforward cases are given in the exercises following this section.

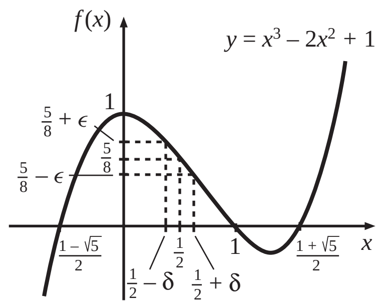


Figure 4.2 Function $y = x^3 - 2x^2 + 1$ (with demonstration that it is continuous at $x = 1/2$) ■

Example 4.4 Show that the function below is discontinuous:

$$f(x) = \begin{cases} +1, & x \leq 0 \\ -1, & x > 0 \end{cases}$$

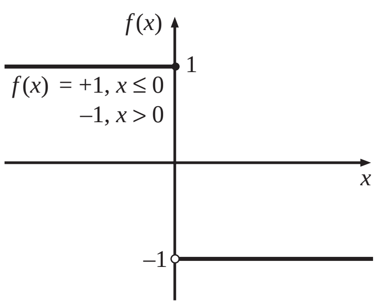


Figure 4.3 Function in example 4.4 (which is discontinuous at $x = 0$)

Solution

This function is graphed in figure 4.3. Notice that the function has an obvious "break" or "jump" at the point $x = 0$. As one approaches the point $x = 0$ from the left, the value of the function is $+1$, while as one approaches $x = 0$ from the right, the value of the function is -1. In other words, the left-hand limit, $\lim_{x \to 0^-} f(x) = +1$, is not equal to the right-hand limit, $\lim_{x \to 0^+} f(x) = -1$. Thus condition (i) of definition 4.3 is not satisfied at the point $x = 0$, and so the function is not continuous at $x = 0$.

We can also use definition 4.4 to see that this function is discontinuous at the point $x = 0$. Suppose that we try to find an interval (open set) of x-values including the point $x = 0$ such that for all x-values in this interval, the function value is within distance $\epsilon = 0.5$ of $f(0)$. We will fail. No matter how small a value $\delta > 0$ we choose, we will always find that $f(x)$ will take on both the values $+1$ and -1 for some $x \in (-\delta, \delta)$. ■

Before considering further examples of functions which are discontinuous, it is useful to introduce the concept of the asymptote of a function.

Definition 4.5

If the value of a function $f(x)$, $x \in \mathbb{R}$, becomes unbounded as x approaches some value $x = a$ either from the left or the right, then we say the line $x = a$ is an **asymptote** of the function.

In such cases the function is *not* continuous. If any one or more of the following possibilities holds, we say that the line $x = a$ is a **vertical asymptote.**

$$\lim_{x \to a^+} f(x) = +\infty, \quad \lim_{x \to a^+} f(x) = -\infty,$$

$$\lim_{x \to a^-} f(x) = +\infty, \quad \text{or} \quad \lim_{x \to a^-} f(x) = -\infty.$$

Example 4.5

Show that although the function $f(x) = 1/x^2$ has the same left-hand and right-hand asymptotic limits at the point $x = 0$, it is not continuous.

Solution

The line $x = 0$ is a vertical asymptote for the function $f(x) = 1/x^2$ because the left-hand and right-hand limits of $f(x)$ at $x = 0$ are both $+\infty$:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = +\infty$$

We say that the asymptotic limit of the function at $x = 0$ is $+\infty$. This possibility is similar to the idea of a sequence being *definitely divergent*, as discussed in chapter 3. Although the left- and right-hand limits of the function are the same, the function is not continuous because $f(0)$ is not defined. Thus condition (ii) of definition 4.3 is violated. The same holds true for definition 4.4 which requires that $f(x)$ must be defined on some open interval including the point $x = 0$ for it to be continuous at $x = 0$ (see figure 4.4). ■

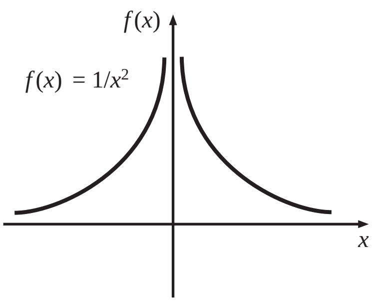


Figure 4.4 The function $f(x) = 1/x^2$ (which is discontinuous at $x = 0$ since $1/x^2$ is undefined there)

Example 4.6

Find an example of a function that has different left-hand and right-hand asymptotic limits at a point and so is not continuous at that point.

Solution

The function $f(x) = 1/(x-1)$ has different left-hand and right-hand asymptotic limits at the point $x = 1$. For x *slightly* less than 1, it follows that $x - 1$ is a small negative number, and so $1/(x-1)$ is negative but large in absolute value. For x *slightly* greater than 1, it follows that $(x-1)$ is a small positive number, and so $1/(x-1)$ is a large positive number. Thus $\lim_{x \to 1^-} f(x) = -\infty$, while $\lim_{x \to 1^+} f(x) = +\infty$, as illustrated in figure 4.5. As was the case for example 4.5, this function is not defined at the point in question ($x = 1$) and so is not continuous at $x = 1$. In this case there is the additional problem that the left- and right-hand limits are not the same.

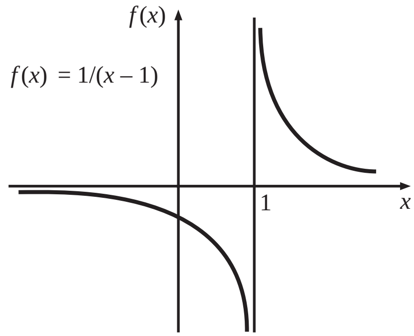


Figure 4.5 Function $f(x) = 1/(x-1)$ (which is discontinuous at $x = 1$ since $1/(x-1)$ is undefined there) ■

Example 4.7 Show that a function with a hole in it is not continuous.

Solution

Consider the function

$$f(x) = \begin{cases} 3x, & \forall\, x \in \mathbb{R}, x \neq 2 \\ 3, & x = 2 \end{cases}$$

which is graphed in figure 4.6. Despite the fact that this function is defined at $f(2)$ and its left- and right-hand limits are equal at $x = 2$, the function is not continuous here because condition (ii) of definition 4.3 is not satisfied.

We can also see formally that this function is not continuous by using definition 4.4. At the point $x = 2$ we have $f(x) = 3$. However, consider the value $\epsilon = 1$. It is not possible to find an interval of x-values, $(2 - \delta, 2 + \delta)$, for which the function values are within distance ϵ of $f(2)$ for every x in this interval. This is true no matter how small a value for $\delta (\delta > 0)$ is chosen. ■

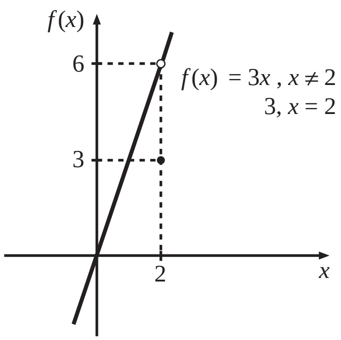


Figure 4.6 Function in example 4.7 (which is discontinuous at $x = 2$)

The following results, which summarize some useful properties of continuous functions, are intuitively straightforward. We provide a proof for only one of them, which should give the flavor of the rest of the results.

Theorem 4.1 Suppose that $f(x)$ and $g(x)$ are continuous functions and that $c \neq 0$ is a constant. The following are also continuous:

(i) $cf(x)$
(ii) $f(x) + c$
(iii) $f(x) \pm g(x)$
(iv) $f(x)g(x)$
(v) $f(x)/g(x)$ for $g(x) \neq 0$
(vi) $f^{-1}(x)$, if it exists

Proof of Part (i)

Since it is assumed that $f(x)$ is continuous, we know from definition 4.3 that for any $x = a$, $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$. That is, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$. Thus, letting $h(x) = cf(x)$, we get

$$\lim_{x\to a^-} h(x) = \lim_{x\to a^-} cf(x) = c \lim_{x\to a^-} f(x) = cf(a)$$

Similarly $\lim_{x\to a^+} h(x) = cf(a)$ and $h(a) = cf(a)$. Thus all of the conditions for continuity are satisfied for the function $h(x) = cf(x)$. ■

There are many instances in economics where the domain of a function is not the entire real line but is a proper subset of it. For example, in considering the amount of labor L for a firm to employ, it does not make sense to consider negative values. Thus the relevant domain of a production function $q = Q(L)$ involves a lower bound or boundary, and we would write $L \geq 0$ or $L \in [0, \infty)$. When considering continuity of the function $Q(L)$ at the point $L = 0$, it does not make sense to think about an open interval containing that point. In other instances the domain may have an upper bound or both a lower and upper bound. For example, in the short run a firm is constrained by its existing capital stock, say $K \leq \bar{K}$, and so the relevant set of values for capital to choose from would be $K \in [0, \bar{K}]$.

To deal with cases where the domain has boundary points, we make use of the concepts of right-hand and left-hand limits as given in definitions 4.1 and 4.2. Thus suppose that the function $f(x)$ is defined only on the closed interval $x \in [a, b]$, $a \leq x \leq b$. Since x cannot be less than a, we cannot define the left-hand limit of $f(x)$ as $x \to a^-$, and since x cannot be greater than b, we cannot define the right-hand limit of $f(x)$ as $x \to b^+$. However, as long as $f(x)$ is continuous at every point strictly within the interval $[a, b]$, meaning for $a < x < b$, and is continuous as $x \to a^+$ (from the right) and $x \to b^-$ (from the left), we say that the function $f(x)$ is continuous on the closed interval $[a, b]$.

Definition 4.6

Let $f(x)$ be defined on the closed interval $[a, b]$, $x \in \mathbb{R}$ and $a < b$. We say that

(i) $f(x)$ is **continuous from the right** at the point $x = a$ if $\lim_{x\to a^+} f(x)$ exists, $f(a)$ exists, and $\lim_{x\to a^+} f(x) = f(a)$.

(ii) $f(x)$ is **continuous from the left** at the point $x = b$ if $\lim_{x\to b^-} f(x)$ exists, $f(b)$ exists, and $\lim_{x\to b^-} f(x) = f(b)$.

(iii) $f(x)$ is continuous on the closed interval $[a, b]$ if it is continuous at every point x strictly within the interval (i.e., $a < x < b$), is continuous from the right at $x = a$ and is continuous from the left at $x = b$.

Examples of functions relevant to economics that are discontinuous, including some that are defined only on proper closed subsets of the real numbers, are provided in the next section.

EXERCISES

1. For each of the following functions, generate the sequence of function values associated with the sequence of x-values, $x_n = 2 - 1/n, n = 1, 2, 3, \ldots$. Show that the sequence of function values converges in each case and find the limit. In each case, what does this suggest about the left-hand limit of $f(x)$ at $x = 2$, namely $\lim_{x\to 2^-} f(x)$?

(a) $f(x) = 5x$ **(b)** $f(x) = -3x + 4$

(c) $f(x) = mx + b$ for m and b constant **(d)** $f(x) = x^2$

2. For the same functions as in question 1 above, generate the sequence of function values associated with the sequence of x-values, $x_n = 2 + 1/n$, $n = 1, 2, \ldots$. Show that the sequence of function values converges in each case and find the limit. In each case, what does this suggest about the right-hand limit of $f(x)$ at $x = 2$, that is, $\lim_{x\to 2^+} f(x)$?

3. For each of the following functions, indicate at which point(s) the function is discontinuous and explain which of the conditions of definition 4.3 is not satisfied. In each case, graph the function. (The domain is $\mathbb{R}$ in each case.)

(a) $f(x) = \begin{cases} 2x + 3, & x < 1 \\ x + 5, & x \geq 1 \end{cases}$

(b) $f(x) = 1/x$

(c) $f(x) = 1/(x - 3)^2$

(d) $f(x) = (x - 2)/(x^2 - x - 2)$ [Hint: Factor the denominator.]

4. For each of the following functions, indicate at which point(s) the function is discontinuous and explain which of the conditions of definition 4.3 is not satisfied. In each case, graph the function. (The domain is $\mathbb{R}$ in each case.)

(a) $f(x) = \begin{cases} -3x + 12, & x < 4 \\ -2x + 10, & x \geq 4 \end{cases}$

(b) $f(x) = 1/2x$

(c) $f(x) = 1/(x - b)^2$ for b constant

(d) $f(x) = (x - 1)/(x^2 + 2x - 3)$ [Hint: Factor the denominator.]

5. Prove that according to definition 4.4, the following functions are continuous at every point $x \in \mathbb{R}$:

(a) $f(x) = 4x + 3$

(b) $f(x) = mx + b$

6. Prove that according to definition 4.6, the function, $f(x) = 2x - 5$, defined on the interval $[0, 1]$ is continuous. Pay special attention to the points $x = 0$ and $x = 1$.

7. Suppose that we break up the interval $[0, 10]$ into subintervals, each of length 1, in the following way:

$$\{[0, 1), [1, 2), [2, 3), \ldots, [8, 9), [9, 10]\}$$

Define the function $f(x)$ as

$$f(x) = k \qquad \text{for } x \in [k - 1, k);\ k = 1, 2, 3, \ldots, 9$$
$$f(x) = 10 \qquad \text{for } x \in [9, 10]$$

Plot this function. Notice that the function values are a series of steps. For this reason it is called a *step function*. Discuss the continuity properties of this function in terms of definition 4.3. Also discuss the continuity properties of this function on each subinterval in terms of definition 4.6.

8. This question introduces a general definition of a step function. Suppose that we take the following set of points from the interval $[a, b]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

We can use these points to break up the interval $[a, b]$ into k subintervals:

$$\{[x_0, x_1), [x_1, x_2), [x_2, x_3), \ldots, [x_{n-2}, x_{n-1}), [x_{n-1}, x_n]\}$$

Suppose that the function $f(x)$ takes on one value, θ_i, in the subinterval $[x_{i-1}, x_i)$, $i = 1, 2, \dots, n-1$ and the value θ_n in the subinterval $[x_{n-1}, x_n]$). Discuss the continuity properties of this function in terms of definition 4.3. Also discuss the continuity properties of this function on each subinterval in terms of definition 4.6.

4.2 Economic Applications of Continuous and Discontinuous Functions

There are many natural examples of discontinuities from economics. In fact economists often adopt continuous functions to represent economic relationships when the use of *discontinuous* functions would be a more literal interpretation of reality. It is important to know when the simplifying assumption of continuity can be safely made for the sake of convenience and when it is likely to distort the true relationship between economic variables too much. Our first example illustrates a class of situations where it is usual to use a model with continuous functions even though this is a distortion of reality in a literal sense. In most such cases the assumption is not a harmful one. However, as some of the other examples illustrate, the idea of discontinuity may be inherent in an economic model, with the solution hinging entirely on the existence of some point of discontinuity of the relevant function.

Divisibility and the Production Function

The first step in modeling the decisions of a firm is usually an analysis of the available technology. This relationship between inputs used and outputs generated is generally presumed to be represented by some production function. In the case of one input, call it x, and one output, call it y, we can write $y = f(x)$.

What does it mean to say that this function is continuous on some range of values (usually $x \geq 0$, $x \in \mathbb{R}$)? In the first place, to assume that $f(x)$ is continuous at a point $x = a$ implies that $f(x)$ is defined on some open interval of real numbers containing the point a. This means that x must be *infinitely divisible*. That is, one can choose x to be a value that deviates even by infinitesimal amounts from $x = a$.

An example of an input (and an output) that would not be infinitely divisible would be bolts used in the production of an automobile. Since one would not use a fraction like a half of a bolt, it would only make literal sense to treat bolts as integer valued. Therefore it does not make sense to contemplate an open interval of points including some value $x = a$ bolts. However, if a manufacturer produces 20,000 vehicles per year using 1,050 bolts in each vehicle, it seems reasonable to simply treat bolts and vehicles as infinitely divisible and represent the relationship between them as $y = x/1{,}050$, where x is the number of bolts used and y is the number of vehicles produced. (Of course, we have ignored all the other inputs.)

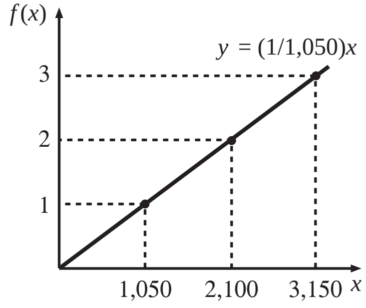


Figure 4.7 The function $y = (1/1{,}050)x$

Figure 4.7 illustrates this function, but the *true* relationship would include only the points (1,050, 1), (2,100, 2), (3,150, 3), and so forth. If one uses the continuous function $y = (1/1{,}050)x$ in the process of solving some decision problem for the firm and discovers the solution involves some value of x that is not a multiple of 1,050, then using the closest value that is a multiple of 1,050 would probably be reasonably accurate. Thus, even if a commodity is not infinitely divisible, we can often assume that it is, without distorting reality very much.

A Salary Schedule with a Bonus Payment

Suppose that a salesperson receives a salary according to a contract that establishes a relationship between pay and the level of sales made by the salesperson. In particular, suppose that the contract stipulates that the salesperson's monthly salary will be composed of three parts: (i) a basic amount of \$800, (ii) a commission of 10%, and (iii) a lump-sum bonus of \$500 if the salesperson's sales for the month reach or exceed \$20,000. From this description, one can see that her salary will jump by \$500 if the critical level of \$20,000 worth of sales is achieved. This implies a discontinuity in her salary schedule. Letting S represent sales per month and P represent the salesperson's pay for the month, it follows that the function describing her salary-sales relationship is

$$P = \begin{cases} \$800 + 0.1S, & S < \$20{,}000 \\ \$1{,}300 + 0.1S, & S \geq \$20{,}000 \end{cases}$$

which is illustrated in figure 4.8.

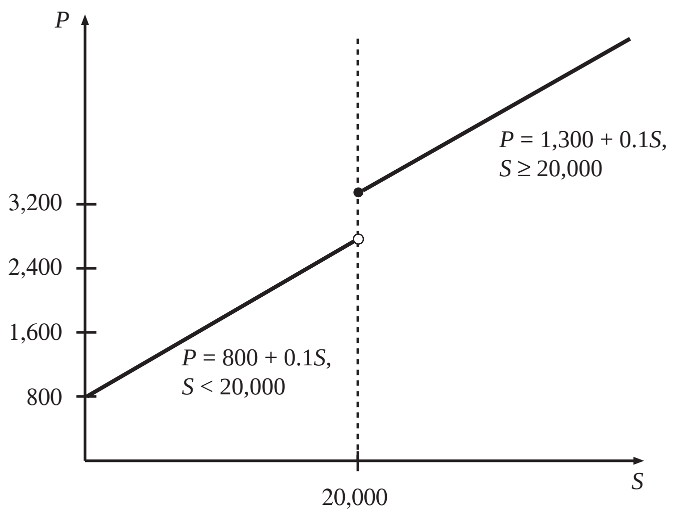


Figure 4.8 A salary schedule with a bonus payment

The fact that the bonus of $500 is achieved once S reaches the critical value of $20,000 leads to the result that the left-hand limit of the salary function at $S = \$20{,}000$ is $2,800 while the right-hand limit is $3,300. The existence of this discontinuity has interesting economic implications. Consider the following scenario. There are three salespersons, called A, B, and C. Their cumulative sales for the month, not including the last day, are $26,000 for A, $18,500 for B, and $6,000 for C. The 10% commission on sales will give each a similar incentive to make extra sales on the last day of the month. But will the $500 bonus possibility have a different effect on the three salespersons? Assuming that it is plausible to generate a few thousand dollars worth of sales in a day but virtually impossible to create more than $10,000 worth of sales, one would expect that salesperson B will try harder on the last day to increase sales than will the other two.

A Discontinuous Income Support Program

Many welfare programs or "income support programs" offer individuals who are not employed a fixed or lump-sum monthly payment that is made only if the individual does not earn any income. Once an individual earns any income whatsoever, the payment is stopped. Consider the following hypothetical example. Suppose that a single parent of two preschool-aged children can collect a monthly welfare payment of $750 provided she does not enter the labor market; once she earns *any* positive amount of income, the welfare payment stops. Assume that she can earn $15 per hour at some job for which the number of hours worked per month is entirely flexible. The income of this person, Y, as a function of hours worked, h, is given below:

$$Y(h) = \begin{cases} 750, & h = 0 \\ 15h, & h > 0 \end{cases}$$

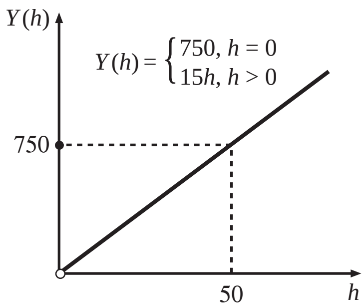


Figure 4.9 A discontinuous income support function

The graph depicting this person's income as a function of hours worked is provided in figure 4.9. It is clearly discontinuous at $h = 0$ hours worked.

This type of discontinuity, which is a property of many "all or nothing" income support programs, has been the subject of a great deal of debate. One can see that a person in such a program would have to work 50 hours per month just to match the income earned from the support payments. Since the person would face childcare and other costs of working, the "all or nothing" property of this program presents a serious deterrent to the incentive to work.

An alternative scheme would be to allow a person in this situation to keep a certain fraction of income earned in addition to the $750 monthly payment. Suppose, for example, that the person were allowed to retain 50% of any earnings, with the other 50% representing a *payback* of the income support up to the level where the entire $750 is paid back. A person facing a wage rate of $15 per hour will have paid back the full $750 only after working 100 hours or more per month ($0.5 \times 100 \times 15 = 750$). For $0 \leq h < 100$, the effective wage rate is 50% of $15 (or \$7.50), and so net income for this range of hours worked is $Y(h) = 7.5h$. After this amount of earnings, the individual would keep any excess. Therefore under this program the person's income schedule would be the following:

$$Y(h) = \begin{cases} 750 + 7.5h, & 0 \leq h \leq 100 \\ 15h, & h > 100 \end{cases}$$

The graph for this income schedule is provided in figure 4.10. Notice that it is continuous. [Check that $\lim_{h \to 100^-} Y(h) = \lim_{h \to 100^+} Y(h) = Y(100)$.]

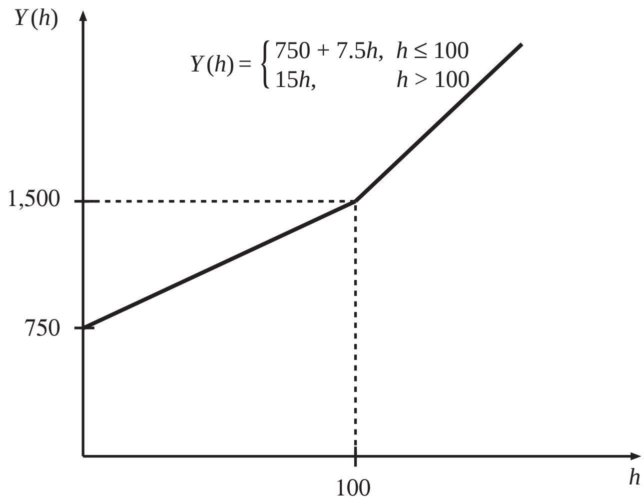


Figure 4.10 A continuous income-support function

Many economists prefer this second plan because it avoids the discontinuity of the first plan. In the first plan there is effectively a large penalty for working at all, since income drops from \$750 per month to almost zero if the individual chooses only a few hours of work. Under the second plan the person always earns more income by choosing to work more. The result is that the person will be more likely to choose some positive hours of work under the second plan making himself/herself better off and also reducing the cost of the program to the government.

Continuous Marginal-Product Functions

The **marginal product** of an input is the amount by which output increases as a result of an additional unit of that input being used, given fixed amounts of other inputs available. This concept is useful in economics when analyzing the decision-making problem of firms in the short run when the level of some inputs can be altered (variable inputs) but other input levels are fixed (fixed inputs).

A full treatment of the marginal product of an input will be taken up in the following chapter on derivatives. However, there are some interesting problems concerning the continuity of marginal-product functions that are useful to consider here. For example, suppose that the function $y = 10L$ relates the amount of output produced, y, to the amount of labor input employed, L, for given fixed levels of other inputs. One can then see that an increase in L of one unit always leads to an increase in output of 10 units. Thus the marginal product of labor function is the constant function $y = 10$ and so is continuous on the interval $[0, \infty)$.

Notice that this marginal-product function has the rather unrealistic property that more output is generated by using more labor even for very large values of labor. Since the amounts of all other inputs are fixed, one might anticipate that it makes more sense to imagine that as L increases, the added output generated begins to fall and may even become zero or negative. The following discussion shows that this phenomenon may occur in such a way that the marginal-product function should be modeled as a discontinuous function.

Marginal-Product Function with a Capacity Constraint

In many production processes the maximum output that can be produced by increasing the amount of a variable input depends on the amount of the other (fixed) inputs available. A good example is a coal-fired electricity generating station. There will always be some absolute maximum amount of power that can be generated from a single station. This maximum is generally referred to as the capacity of the station. For example, if a station has a 1,500-megawatt capacity, this means that no matter how much coal or other inputs are available, the maximum amount of energy that can be generated in a twenty-four hour period (per day) is 36,000 megawatt-hours (i.e., $24 \times 1{,}500$).

Suppose that we want to determine the marginal product of coal for a case in which there is enough of all inputs other than coal to keep a 1,500-megawatt power plant operating at capacity. Assume that it takes 250 pounds of coal to generate one megawatt-hour of energy, and so one ton of coal will generate eight megawatt-hours of energy. Therefore the marginal product (per day) of coal is eight megawatt-hours (per ton) as long as capacity has not been reached. Once capacity has been reached, however, the marginal product of coal drops to zero. Thus, once 4,500 tons of coal have been used in a day to generate electricity, the generating station will have reached capacity ($4{,}500 \times 8 = 36{,}000$). If we let x represent tons of coal used per day and y the marginal product of coal in megawatt-hours, then the marginal product of coal is given by the function

$$y = \begin{cases} 8, & 0 \leq x \leq 4{,}500 \\ 0, & x > 4{,}500 \end{cases}$$

This is illustrated in figure 4.11. The function is discontinuous at $x = 4{,}500$.

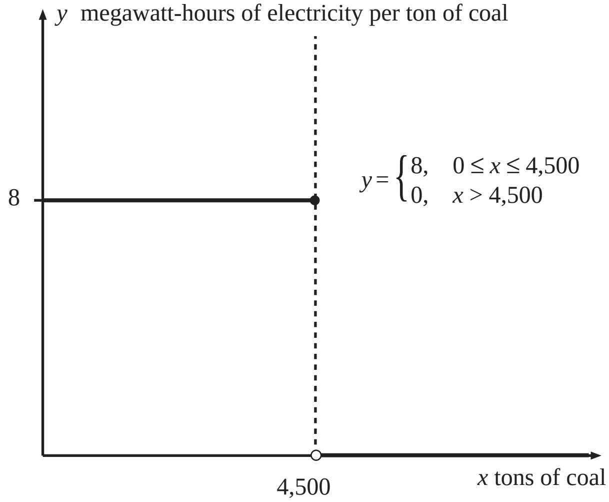


Figure 4.11 Marginal product of an input under a capacity constraint

The Bertrand Model of Price Competition

If there is more than one producer/seller in a market but not so many as to make the perfectly competitive model applicable, we say that the market structure is oligopolistic. The word *oligopoly* means "few sellers." One model that describes the possible behavior of firms in this setting is the so-called Bertrand model. To make matters simple, we assume that there are two firms in the market, although the qualitative nature of the outcome of the model is not altered if we assume that there are more than two firms.

In the Bertrand model the two firms are assumed to compete in prices. That is, each firm sets a price and then meets whatever demand exists for its product at that price. Assuming that the firms produce identical commodities, if one firm charges a lower price than the other, then all the consumers will purchase from that producer. If the two firms charge the same price, then we assume that consumers' purchases will be split evenly between the two producers. Thus we need to think about how revenue for each firm changes as prices are altered. To see how firms will behave in this situation, consider the following simple numerical example: Let the demand function be $y = 20 - 2p$, and let the marginal cost (i.e., the cost of producing one more unit of output) be the constant value $c = 4$ for each firm. That is, $C(y) = 4y$ is the cost function for each firm. Note that as long as the price exceeds 4, each firm can make an excess profit on each unit it sells. If the price equals 4, each firm can only earn normal (or zero economic) profit, and if the price falls below 4, either firm would incur a loss if it produces any output. Therefore each firm would produce zero output if the price falls below 4.

We begin the analysis by determining how firm 1's revenue changes for alternative prices given that a specific price has been set by firm 2, say $p_2 = 7$. Given that firm 2 is charging the price $p_2 = 7$, it follows that if firm 1 charges a price above 7, its sales will be zero and so will its revenue. If it charges a price equal to 7, it will share the market with firm 2. To find total market demand, note that at a market price of 7 we have

$$y = 20 - 2(7) = 6$$

Since the two firms share the market equally when they charge the same price, we have $y_1 = y_2 = 3$, and so firm 1's revenue is

$$R_1(p_1) = p_1 y_1 = 7(3) = 21$$

implying a profit level of

$$\pi_1 = R_1(p_1) - C_1(y_1) = p_1 y_1 - 4y_1 = 7(3) - 4(3) = 9$$

As noted, if firm 1 charges a price even slightly above 7, it loses all of its market share to firm 2 and so its revenue and profit drop to 0. If firm 1 charges a price even slightly less than 7, however, firm 1 will capture the entire market and so its revenue will jump accordingly. To see this, let $p_1 = 7 - \epsilon$, with ϵ positive but *small*. Since p_1 is less than the price charged by the other firm, firm 1's sales will be determined by the total market demand. Thus firm 1 sells output level

$$y_1 = 20 - 2p_1 = 20 - 2(7 - \epsilon) = 6 + 2\epsilon$$

and so earns revenue

$$R_1 = p_1 y_1 = (7 - \epsilon)(6 + 2\epsilon) = 42 + 8\epsilon - 2\epsilon^2$$

and profit

$$\pi_1 = R_1(p_1) - C_1(y_1) = (42 + 8\epsilon - 2\epsilon^2) - 4(6 + 2\epsilon) = 18 - 2\epsilon^2$$

For ϵ *small* (i.e., $\epsilon \to 0$), we find that firm 1 earns revenue $R_1 = 42$ and that profit $\pi_1 = 18$.

In fact, for any price $p_1 < 7$, firm 1 captures the entire market and so its revenue and profit functions become that of a simple monopolist. We can therefore write firm 1's revenue function as

$$R_1(p_1) = \begin{cases} p_1(20 - 2p_1), & p_1 < 7 \\ 21, & p_1 = 7 \\ 0, & p_1 > 7 \end{cases} \tag{4.1}$$

Firm 1's profit function is

$$\pi(p_1) = \begin{cases} p_1(20-2p_1) - 4(20-2p_1), & p_1 < 7 \\ 9, & p_1 = 7 \\ 0, & p_1 > 7 \end{cases} \tag{4.2}$$

The revenue and profit functions are illustrated in figure 4.12. Looking at the revenue function and returning to the conditions for continuity given in definition 4.3, we see that this is a particularly interesting case mathematically. At the point $p_1 = 7$, the left-hand limit of the revenue function is 42, the right-hand limit is 0, and the value of the function itself is 21. Thus we see that $\lim_{p_1 \to 7^-} R_1(p_1) = 42$, $\lim_{p_1 \to 7^+} R_1(p_1) = 0$, and $R_1(7) = 21$. Since these are all different, the function is not continuous at this point. A similar result holds for the profit function.

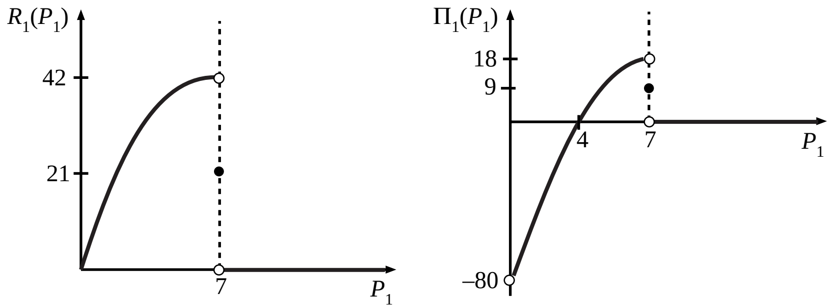


Figure 4.12 Revenue and profit functions of firm 1 for the model of Bertrand competition

From an economic perspective this discontinuity is also extremely important, since what happens at the point of discontinuity drives the model to its solution. To see this, consider any particular price, $\bar{p}_2$, that firm 2 may charge. As long as some profit can be made by charging the same price, firm 1 will never charge a higher price as that would mean no sales and zero profit. If firm 1 charges the same price, $p_1 = \bar{p}_2$, then the firms share the market. However, if firm 1 charges a slightly lower price than firm 2, it will capture the entire market and its profits will *jump* to a higher value. Thus, whatever price firm 2 charges, as long as it exceeds 4, firm 1 will always undercut it in order to capture the entire market. It follows that the revenue and profit functions, which are drawn in figure 4.13, are

$$R_1(p_1) = \begin{cases} p_1(20-2p_1), & p_1 < \bar{p}_2 \\ \left(\frac{1}{2}\right) p_1(20-2p_1), & p_1 = \bar{p}_2 \\ 0, & p_1 > \bar{p}_2 \end{cases} \tag{4.3}$$

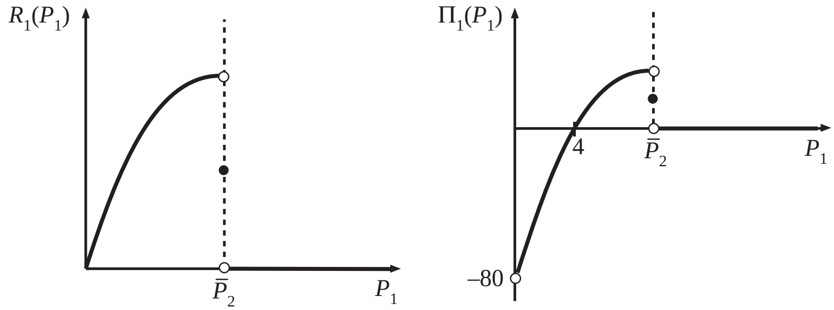


Figure 4.13 Revenue and profit functions of firm 1 for any price set by firm 2, $p_2 = \bar{p}_2 > 4$

The profit function is

$$\pi_1(p_1) = \begin{cases} p_1(20 - 2p_1) - 4(20 - 2p_1), & p_1 < \bar{p}_2 \\ \left(\frac{1}{2}\right)(p_1(20 - 2p_1) - 4(20 - 2p_1)), & p_1 = \bar{p}_2 \\ 0, & p_1 > \bar{p}_2 \end{cases} \tag{4.4}$$

If firm 2 sets its price at $\bar{p}_2 > 4$, firm 1 will always have an incentive to set its price slightly below firm 2's price in order to capture the entire market. Of course, the same applies to firm 2. It will always have an incentive to undercut firm 1's price slightly as long as firm 1 is charging a price above \$4. Beginning at any price above \$4 charged by either firm, we will expect the two firms to engage in a process of undercutting each other's price until the price both firms are charging is \$4 and they share the market equally. At this price both firms would earn zero economic profit and have no incentive to reduce the price further.

Thus, by concentrating on the discontinuity of the revenue or profit function, we can determine the final outcome of the model. Of course, one must realize that this type of competition will occur only under rather special assumptions. First, the result requires that consumers would immediately switch to buying from another firm if it offers a lower price. Thus consumers must always be fully aware of the prices the two firms are charging and must not care about any other characteristics of the firms, like convenience of location. It must also be the case that a firm can always supply whatever level of product is demanded by the market upon cutting the price below its rival's. This requires rather large inventories or a very quick response time for production. An example of when this outcome occurs would be if firms have excess capacity. This may explain why airline companies in the United States have frequently engaged in price wars since deregulation.

EXERCISES

1. Given the production function $Q(L) = bL$, $b > 0$, defined on $[0, +\infty)$, derive the cost function, $C(Q)$, and the profit function, $\pi(Q)$ for a perfectly competitive firm. Let fixed costs be c_0, and let w be the unit price of L. Prove that the production function is continuous (according to definition 4.4) and then use theorem 4.1 to show that the cost function and the profit function are continuous.

2. Given production function $Q(L) = L^2$, defined on $[0, +\infty)$, derive the cost function, $C(Q)$, and the profit function, $\pi(Q)$, for a perfectly competitive firm. Given that the production function is continuous (according to definition 4.4), use theorem 4.1 to show that the cost function and the profit function are continuous.

3. Suppose that a salesperson earns a basic monthly salary of \$800 plus a commission and possible bonuses based on her level of sales. Suppose that the commission rate is 15% and the possible bonuses are a lump-sum amount of \$1,000 if her monthly sales exceed \$10,000 and a further lump sum of \$2,500 if her monthly sales exceed \$15,000. Find the function that relates sales to earnings for this salesperson and graph it. At which points is the function discontinuous? By explicitly stating the left- and right-hand limits of the function at these points, show why these are points of discontinuity according to definition 4.3. What do these properties imply about the incentives created by this pay scheme? (Let S be monthly sales and P be the salesperson's pay.)

4. Suppose that a salesperson earns a basic monthly salary of \$500 plus a commission of 10% on sales if her monthly sales do not exceed \$20,000 for the month but receives a commission of 20% (on all sales) if her monthly sales do exceed \$20,000. Find the function that relates sales to earnings for this salesperson and graph it. Are there any points of discontinuity? If so, find them and indicate why, according to definition 4.3, each is a point of discontinuity. (Let S be monthly sales and P be the salesperson's pay.)

5. Suppose that the government has been taxing each person's income at a marginal rate of 0.25 for every dollar in excess of \$20,000. That is, the first \$20,000 earned is not taxed. The government decides to generate extra tax revenue but wishes to avoid increasing the tax burden on low- or middle-income earners. Therefore the government decides to impose a lump-sum surtax of \$1,000 on every person who earns \$60,000 or more. Write out and graph income after the tax, y, as a function of income before the tax, x. Indicate why, according to definition 4.3, the function has a point of discontinuity at $x = \$60,000$. Discuss any incentive effects on hours worked that may arise due to this discontinuity in the tax schedule.

6. In this question we consider two plans, each of which combines the effects of an income-support plan with an income-tax program. Let earned income (before tax) be x and income after taxes and any government transfers be y.

Plan A In this plan an individual who earns zero income ($x = 0$) receives a tax-free government transfer (income support) of \$6,000 annually. No income support is received by anyone with positive income ($x > 0$). Those who earn income in excess of \$6,000 pay income tax at the marginal rate of 20% on each dollar earned in excess of the \$6,000.
Plan B In this plan everyone receives a basic supplement of \$6,000. This is called a *demogrant*. Each person then pays income tax at the rate of 40% for every dollar of earned income (i.e., not including the \$6,000 demogrant).

(a) Define after-tax income, y, as a function of earned income, x, for both plans, and graph on the same diagram.

(b) Which tax system is more favorable from the point of view of the taxpayer? How does your answer depend on the individual's level of earned income?

(c) Which plan provides the greater incentive for an individual to earn income? Does the answer depend on the range of possible earnings of the individual? What role, if any, do discontinuities have in your answer?

7. A car production plant has a capacity to produce 100 cars per hour using 2,000 workers. The plant can operate around the clock, including weekends, using three shifts of workers. Let h represent the number of worker-hours used per week and MP the marginal product of workers measured by the number of cars produced per worker-hour.

(a) Draw a graph of the marginal-product function, $\text{MP} = \text{MP}(h)$. At what point is the marginal-product function discontinuous? Discuss in terms of definitions 4.3 and 4.4.

(b) Suppose that the automobile company earns a profit of \$1,000 per car not including labor costs. The wage for workers is \$30 per hour for any weekday shift but double that, \$60 per hour, for weekend shifts. The company will thus use weekend shifts only if it wants to produce more cars per week than it can using weekday shifts only. Let $\pi(y)$ be profit earned per week as a function of the number of cars produced per week, y. Find and graph this function. At what point is the function discontinuous? Explain intuitively.

8. A car production plant has a capacity to produce B cars per hour using N workers. The plant can operate around the clock, including weekends, using three shifts of workers. Let h represent the number of worker-hours used per week and MP the marginal product of workers measured by the number of cars produced per worker-hour.

(a) Draw a graph of the marginal-product function, $\text{MP} = \text{MP}(h)$. At what point is the marginal-product function discontinuous? Discuss in terms of definitions 4.3 and 4.4.

(b) Suppose that the automobile company earns a profit of $\$F$ per car, not including labor costs. The wage for workers is $\$w$ per hour for any weekday shift but double that for weekend shifts. The company will thus use weekend shifts only if it wants to produce more cars per week than it can using weekday shifts only. Let $\pi(y)$ be profit earned per week as a function of the number of cars produced per week, y. Find and graph this function. At what point is the function discontinuous? Explain intuitively. At what range of values for F (profit per car excluding labor costs) would it make sense to produce cars only during the week?

9. Consider the following example of the Bertrand model of price competition. The two firms, 1 and 2, set prices p_1 and p_2, respectively. The firm offering the lower price captures the entire market. If they charge the same price, then they share the market equally. Assume that market demand is determined by the function $y = 20 - p$, and that each firm faces cost function $C(y_i) = 2y_i$, which implies constant unit cost of \$2.

(a) If firm 2 sets a price $p_2 = 5$, find firm 1's revenue function, $R_1(p_1)$, and profit function, $\pi_1(p_1)$. Draw a graph of each of these functions, and explain why each is discontinuous at $p_1 = 5$.

(b) For a general price of $\bar{p}_2$ set by firm 2, where $\bar{p}_2 > 2$, do the same exercise as for part (a).

(c) Upon considering the above results, why is it the case that the outcome of each firm setting a price of \$2 is the equilibrium for this model? Explain.

CHAPTER REVIEW

Key Concepts

asymptote
continuous from the left
continuous from the right
continuous functions
discontinuous functions
left-hand limit
marginal product
right-hand limit
vertical asymptote

Review Questions

1. What is meant by the expression $\lim_{x\to a^-} f(x)$?

2. What is meant by the expression $\lim_{x\to a^+} f(x)$?

3. Under what condition does the limit of $f(x)$ as $x \to a$ exist?

4. Give two definitions of continuity of a function $f(x)$ at the point $x = a$.

5. Give two examples of functions that are not continuous.

6. What does it mean to say the line $x = a$ is a vertical asymptote of a function?

7. How do you describe conditions for continuity of a function defined on a closed interval?

Review Exercises

1. For each of the following functions, indicate at which point(s) the function is discontinuous and explain which of the conditions of definition 4.3 is not satisfied. In each case, graph the function. (The domain is $\mathbb{R}$ in each case.)

   (a) $f(x) = \begin{cases} 2x, & x < 5 \\ x + 6, & x \geq 5 \end{cases}$

   (b) $f(x) = (x + 1)/(x^2 - 1)$ [Hint: Factor the denominator.]

2. Prove that according to definition 4.4, the function, $f(x) = 3x$, is continuous at every point $x \in \mathbb{R}$.

3. Suppose that the government has been taxing each person's income at a marginal rate of 0.4 for every dollar in excess of $25,000 with the first $25,000 earned not taxed. In addition the government imposes a lump-sum surtax of $2,000 on every person who earns $100,000 or more. Write out and graph income after tax, y, as a function of income before tax, x. Indicate why, according to definition 4.3, the function has a point of discontinuity at $x = \$100{,}000$. Discuss any incentive effects on hours worked that may arise due to this discontinuity in the tax schedule.

4. Let $y = x^2$ be a production function relating input x to output y. Let $\bar{c}$ represent the unit cost of input x, and assume that total cost equals fixed costs, C_0, plus the cost of input x. Let $\bar{p}$ be the unit price of y. Find the revenue function, the cost function, and the profit function for the firm. Given that the function $f(x) = x^2$ is continuous, are these functions continuous? (Use theorem 4.1 to answer this question.) Discuss.

5. A railway company runs a train from A to B. The wages of the engineer and guard for one trip total $500. Each carriage on the train holds an absolute maximum of 50 passengers. The relationships between the cost of the energy required by the locomotive and the number of carriages it pulls are as follows:

Number	**Cost ($)**
1 carriage	1,000
2 carriages	1,800

Number	Cost ($)
3 carriages	2,400
4 carriages	2,600
5 carriages	3,400
6 carriages	4,600
7 carriages	8,000

It makes no difference to these costs whether the carriages are empty or full.

(a) What is the cost incurred to transport just one passenger from A to B?

(b) What is the *increase* in cost created by taking a second passenger?

(c) What is the *increase* in cost created by taking the 51st passenger?

(d) Draw the functions relating:

(i) total costs

(ii) average cost per person

to the total number of passengers, over the range 0 to 350 passengers, and comment on the continuity properties of these functions.

6. Consider the following example of the Bertrand model of price competition. Two firms, 1 and 2, set prices p_1 and p_2, respectively. The firm offering the lower price captures the entire market. If they charge the same price, then they share the market equally. Assume that market demand is determined by the function $y = 50 - p$, and each firm faces cost function $C(y_i) = 10y_i$, which implies constant unit cost of $10.

(a) If firm 2 charges a price $p_2 = 20$, find firm 1's revenue function, $R_1(p_1)$, and profit function, $\pi_1(p_1)$. Draw a graph of each of these functions and explain why each is discontinuous at $p_1 = 20$.

(b) For a general price charged by firm 2 of $\bar{p}_2$, where $\bar{p}_2 > 10$, do the same exercise as for part (a).

(c) Upon considering the results above, why is it the case that the outcome of each firm charging a price of $10 is the equilibrium for this model? Explain.

Chapter 5 The Derivative and Differential for Functions of One Variable

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Marginal Revenue Product of Labor for a Competitive Firm and for a Monopoly Firm
- Marginal Revenue Product of Labor: Example
- Further Details on the Elasticity Concept
- Finding Elasticities of Demand: Example
- Practice Exercises

The purpose of the derivative is to express in a convenient way how a change in the level of one variable (e.g., x) determines a change in the level of another variable (e.g., y). Much of economics is in fact concerned with just this sort of analysis. For example, we study how a change in a firm's output level affects its costs and how a change in a country's money supply affects the rate of inflation. Although expressing the relationship between x and y as a function $y = f(x)$ does capture this idea implicitly, it is much more convenient to relate explicitly how a change in x, denoted Δx, causes a change in y, Δy. Using the relationship between Δy and Δx allows one to perform what economists refer to as **marginal analysis,** which is of central importance in economics. For example, it allows us to derive important economic propositions such as: "A profit-maximizing firm should expand output so long as the added or marginal revenue (ΔR) exceeds the added or marginal cost (ΔC)." This and the subsequent chapter contain many economic applications of the usefulness of the derivative of a function of one variable.

5.1 Definition of a Tangent Line

A **tangent** to a curve is a straight line that just touches the curve at a given point. For example, the line l_P is the tangent to the curve $y = f(x)$ at point P in figure 5.1. Notice that this line just touches the curve at the point P without intersecting it at another point.

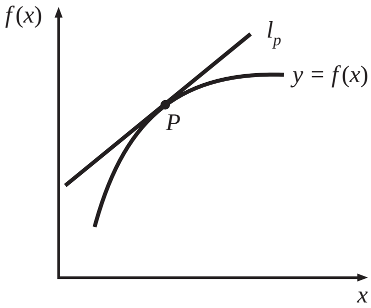


Figure 5.1 Tangent line

It is easy to see that there will be a different tangent line at each point on this curve. Letting $\Delta y/\Delta x$ represent the ratio of the change in y and some change in x, we say that the slope of a curve at a point is equal to this ratio as we take successively smaller values for Δx. For a *smooth* curve, such as the one drawn in figure 5.1, the slope of the tangent line is the same as the slope of the curve at the point where it touches. The slope of the tangent line at P is called the value of the derivative of the function $y = f(x)$ at point P. The derivative function gives the value of the slope of the tangent for different points along the function as determined by the value of x.

Before providing general definitions, it is useful to first consider a simple example that stresses the idea that the derivative has something to do with rates of change. Suppose that a person drives her car across a large city to work. Let y represent miles driven and t time spent driving with $y = f(t)$ indicating how far she has traveled after t minutes of driving. Suppose that the first 5 miles she drives takes her 20 minutes. Her average rate of speed over this period is then $\Delta y/\Delta t = 5/20 = 0.25$ miles per minute or 15 miles per hour. This is an average rate of speed and the rate at which she drives at any given point in this time interval may vary substantially. Suppose, for example, that we wish to consider her speed at the start of the tenth minute of the trip. Begin by denoting her rate of speed between the tenth and eleventh minute by $\Delta\hat{y}/\Delta\hat{t}$ where $\Delta\hat{y}$ is the distance traveled in that interval of time and $\Delta\hat{t} = 1$ minute is the length of time. This ratio, however, is still an average rate of speed. The process of defining an *instantaneous* speed or rate of change $\Delta y/\Delta t$ at the beginning of the tenth minute of the trip involves considering ratios of successively smaller values of Δt. We denote this by writing $\Delta t \to 0$ and saying the change in time approaches zero. The ratio $\Delta y/\Delta t$ as $\Delta t \to 0$ is the **instantaneous rate of change** of distance over time or, in other words, the speed of the vehicle. Although this ratio appears to involve a division by zero, this is not the case as Δt is never equal to zero and the associated Δy values also become successively smaller so that $\Delta y/\Delta t$ does not become arbitrarily large even though the denominator becomes arbitrarily small. The limit of the sequence of ratios $\Delta y/\Delta t$ as Δt approaches zero thus formed (if this limit exists) is the derivative of the function $y = f(t)$ at the point t. We now develop this idea formally.

To define the derivative of a function at a point, we first define a *secant*, which is the straight line connecting two points on the graph of a function. By taking a sequence of secants formed by connecting some point P to a sequence of points that comes arbitrarily close to, but is not coincident with, P we generate the tangent line at point P. In other words, the tangent line is the line through P that has the same slope as the limit of the sequence of secants thus formed, if this limit exists. The derivative of a function at point P is then defined as the value of the slope of this tangent line.

The easiest way to see this is to look at a graph. Consider a function $y = f(x)$ and two points on the graph of the function represented by $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$, as in figure 5.2. We call a line joining any such two points a

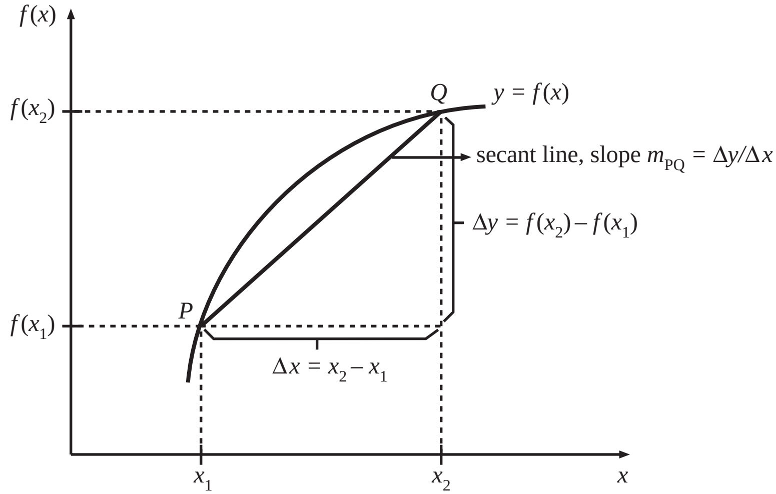


Figure 5.2 Secant line

secant line and define its slope as the change in y, written as $\Delta y = f(x_2) - f(x_1)$, divided by the change in x, written as $\Delta x = x_2 - x_1$; that is,

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Definition 5.1

Given two points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$, with $x_2 = x_1 + \Delta x$, on the graph of a function $y = f(x)$, we define the **secant line** as the straight line joining these points. The slope of the secant line is

$$m_{PQ} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

Suppose that we treat as fixed the point $P = (x_1, f(x_1))$ and consider a sequence of x_2 values such that x_2 becomes *arbitrarily close* to x_1. Since $x_2 = x_1 + \Delta x$, this can be seen to be equivalent to constructing a sequence of Δx values which become arbitrarily small ($\Delta x \to 0$). If for any sequence of Δx values with $\Delta x \to 0$, a sequence of values for m_{PQ} is generated that converges to some limit which we will call m^*, then the line which passes through P with slope m^* is called the tangent line through P. The sequence of Δx values, $\Delta x \to 0$, generates the sequence of points $Q_1, Q_2, Q_3, \ldots$ illustrated in figure 5.3.

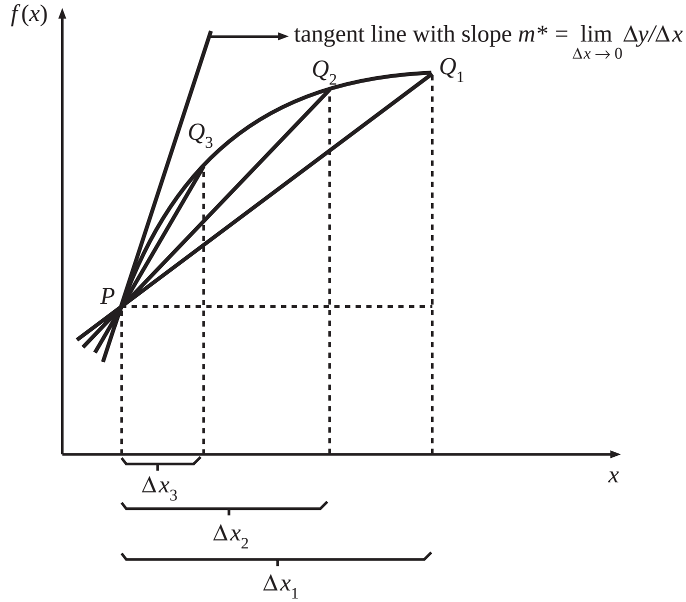


Figure 5.3 Sequence of secants

Definition 5.2 If the function $y = f(x)$ is defined on some open interval including the point $P = (x_1, f(x_1))$ and $\lim_{\Delta x \to 0} m_{PQ}$ exists, then the line passing through the point P with slope equal to $\lim_{\Delta x \to 0} m_{PQ}$ is the **tangent line** of the function $y = f(x)$ at P.

Consider, for example, the function $y = x^2$ illustrated in figure 5.4. If we compare the points $P = (2, 4)$ and $Q = (4, 16)$, we see that in moving from P to Q the change in x is $\Delta x = 2$ and the change in y is $\Delta y = 12$. Therefore the rate of change in y with respect to x between these two points is $\Delta y/\Delta x = 12/2 = 6$, the slope of the secant joining P and Q. If, however, we use the point $Q' = (5, 25)$ instead of Q, we get $\Delta y/\Delta x = 21/3 = 7$. Thus the rate of change in y with respect to x beginning at some point (P) depends on the amount by which x changes. In other words, the rate of change $\Delta y/\Delta x$ changes as one moves along the function. For this reason we refer to $\Delta y/\Delta x$ as the *average* rate of change between two points.

If for the function $y = x^2$ we form a sequence of changes in x, Δx, with $\Delta x \to 0$, we will find that the resulting sequence of values for the slope of the secant at a given point P converges (i.e., approaches a single value). For example, consider the sequence $\Delta x = 1/n$, $n = 1, 2, 3, \ldots$. As $n \to \infty$ we have $\Delta x \to 0$. Suppose that we begin at point $P = (2, 4)$ for the function $y = x^2$ and we use $\Delta x = 1/n$, which implies that $x + \Delta x = 2 + 1/n$ and $f(x + \Delta x) = (2 + 1/n)^2$. Then from definition 5.1, with x_1 written as x and x_2 as $x + \Delta x$, we get the following expression:

$$m_{PQ} = \frac{f(2+1/n) - f(2)}{(2+1/n) - 2} = \frac{(2+1/n)^2 - 2^2}{1/n} \tag{5.1}$$

$$= \frac{4 + 4/n + 1/n^2 - 4}{1/n} = 4 + \frac{1}{n} \tag{5.2}$$

As $n \to \infty$, we have $\Delta x \to 0$ and the sequence of values for m_{PQ} converges to the number 4. Therefore the slope of the tangent of the function $y = x^2$ at the point $x = 2$ is 4.

In general, the slope of the secant for the function $y = x^2$ between an arbitrary pair of points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$, where $x_2 = x_1 + \Delta x$, is

$$\begin{aligned} m_{PQ} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(x_1 + \Delta x)^2 - x_1^2}{\Delta x} \\ &= \frac{x_1^2 + 2(\Delta x)x_1 + \Delta x^2 - x_1^2}{\Delta x} \\ &= \frac{2(\Delta x)x_1 + \Delta x^2}{\Delta x} = 2x_1 + \Delta x \end{aligned}$$

Thus, for any sequence of values for Δx, as $\Delta x \to 0$, we have $\lim_{\Delta x \to 0} m_{PQ} = 2x_1$, which is the slope of the tangent at the point P. For example, at the point

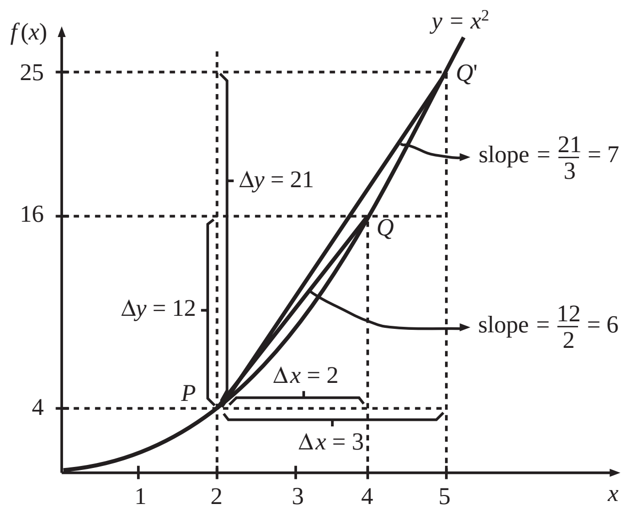


Figure 5.4 Slope of a secant depends on Δx

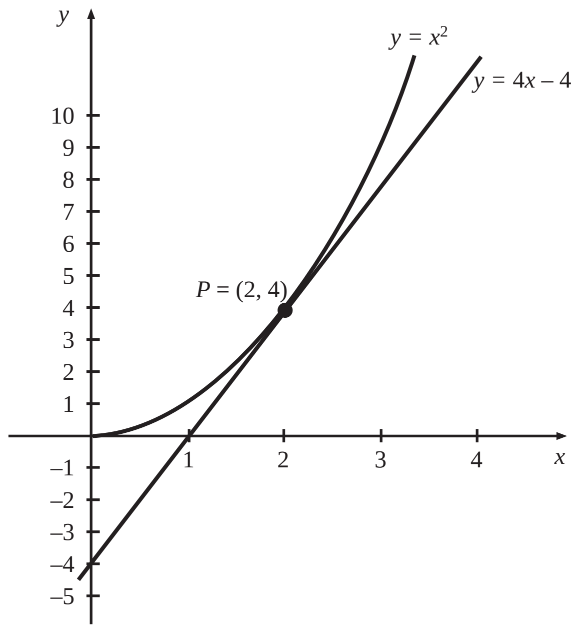


Figure 5.5 Tangent line for $y = x^2$ at value $x = 2$

$P = (2, 4)$, we have $x_1 = 2$ and so the tangent line has the slope 4 (i.e., $m_{PQ} = 2x_1 = 2(2) = 4$). By using the equation $y = 4x + b$ and the fact that the point $P = (2, 4)$ lies on this line, we can compute the intercept b from $4 = 4(2) + b$, which implies that $b = -4$. Thus the equation of the tangent line through P is $y = 4x - 4$. This is illustrated in figure 5.5. It should be obvious that a different tangent line is associated with each distinct point on the function.

It is generally the case that a function does not have the same tangent line at each point, although *by coincidence* the same line may be tangent to a function at more than one point, as illustrated in figure 5.6.

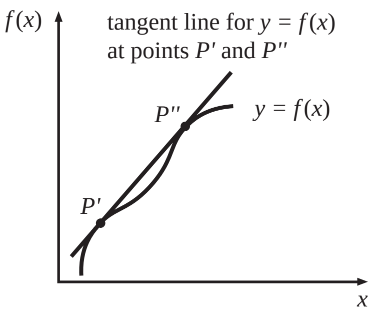


Figure 5.6 A function with the same tangent line at two points

A linear function is a very special case in that the secant line between any two points always coincides with the function itself. To see this, consider the general linear equation $y = mx + b$ where m and b are constants. The slope of the secant line between any two points $P = (x_1, f(x_1))$ and $Q = (x_2, f(x_2))$ is

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{where } x_2 = x_1 + \Delta x$$

$$= \frac{(m(x_1 + \Delta x) + b) - (mx_1 + b)}{(x_1 + \Delta x) - x_1}$$

$$= \frac{mx_1 + m\Delta x + b - mx_1 - b}{x_1 + \Delta x - x_1}$$

$$= \frac{m\Delta x}{\Delta x} = m$$

a constant. Thus the secant line clearly lies on the line itself, and its slope is independent of the size of Δx as well as the particular starting point, P, as illustrated in figure 5.7. Thus for a linear function we can write the relationship between Δx and Δy as $\Delta y/\Delta x = m$ or $\Delta y = m\Delta x$. In other words, a change in the x variable in amount Δx induces a change in the y variable in amount m times Δx. Moreover this relationship holds regardless of the size of the change Δx or the location of the particular point P. This is generally true only for a linear function.

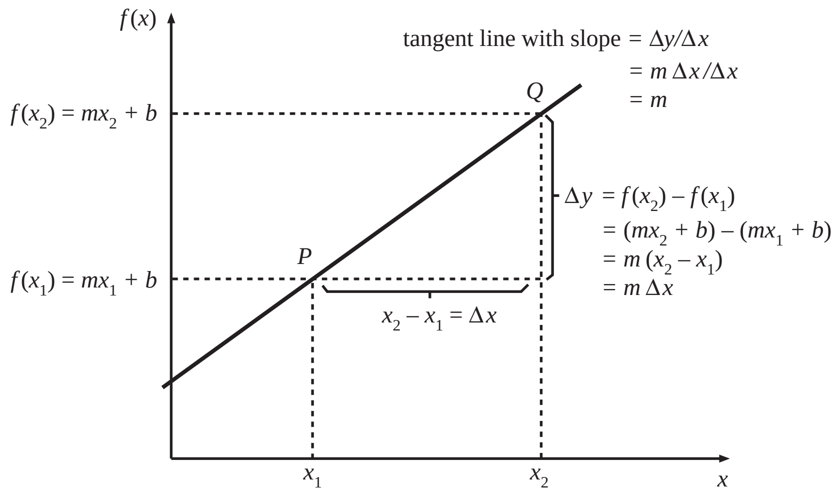


Figure 5.7 Tangent line on a linear function

EXERCISES

1. Suppose that we choose the point $P = (20, 400)$ on the function $y = x^2$. Find the ratio $\Delta y/\Delta x$ for each of the line segments (secants) found by connecting each of the points $Q_1 = (25, 625)$, $Q_2 = (24, 576)$, $Q_3 = (23, 529)$, $Q_4 = (22, 484)$, and $Q_5 = (21, 441)$, and arrange in a table as illustrated below.

Q_i	(25, 625)	(24, 576)	(23, 529)	(22, 484)	(21, 441)
Δx					
Δy					

Does the sequence of values look like it will converge as $\Delta x \to 0$? Illustrate with a graph.

2. As for question 1, compute a sequence of ratios $\Delta y/\Delta x$ for the function $y = x^2$ with respect to the fixed point $P = (20, 400)$. This time use $\Delta x_n = 1/n$, $n = 1, 2, 3, \ldots$ to generate a sequence of points $Q_n = ((20 + 1/n), (20 + 1/n)^2)$ and so a sequence of ratios

$$\frac{\Delta y_n}{\Delta x_n} = \frac{(20 + 1/n)^2 - (20)^2}{(20 + 1/n) - (20)}$$

Show that as $n \to \infty$ (i.e., $\Delta x_n \to 0$) this ratio $\Delta y_n/\Delta x_n$ converges. Using this result find the tangent line to the function $y = x^2$ through the point $P = (20, 400)$.

3. The slope of the tangent for the function $y = x^2$ is $2x$. Find the equation of the tangent line at the point $x = 3$. Illustrate on a graph.

4. The slope of the tangent for the function $y = \sqrt{x}$ is $1/(2\sqrt{x})$. Find the equation of the tangent line at the point $x = 1$. Illustrate on a graph.

5.2 Definition of the Derivative and the Differential

The derivative of a function $y = f(x)$ at some point in the domain of the function is simply the slope of the tangent line. The notation used to depict the derivative at a point, x, in the domain of the function $f(x)$ varies but is usually written either as dy/dx or $f'(x)$. Since, as we saw in the previous section, the slope of the tangent line generally depends on the value of the variable x, then so does the value of the derivative. Therefore, the derivative function, f', is the function which indicates the value of the derivative of the function at each point of the domain of f. Only in the case of a linear function, $y = mx + b$, is the value of the derivative independent of the value of x. In this case the derivative is $f'(x) = m$ at every point x in the domain and so the derivative function is $f' = m$, the constant function. (Note that, as in Chapter 4, the domain of a function is assumed to be $\mathbb{R}$ unless otherwise stated.)

Definition 5.3

The **derivative** of a function $y = f(x)$ at the point $P = (x_1, f(x_1))$ is the slope of the tangent line at that point.

$$f'(x_1) = \lim_{\Delta x \to 0} m_{PQ} = \lim_{\Delta x \to 0} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

where $\Delta x = x_2 - x_1$. We can also write this as

$$f'(x_1) = \lim_{\Delta x \to 0} m_{PQ} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

In the previous section, we showed that the slope of the tangent line for the function $y = x^2$ is $2x$ and so this is also its derivative; i.e., $f'(x) = 2x$ or $dy/dx = 2x$. From an intuitive perspective, notice that the dy and dx reflect the idea of changes in y and x, as do Δy and Δx, respectively. In fact, for a specific value of dx we can think of dy/dx as an estimate of the ratio $\Delta y/\Delta x$, and so $dy/dx = 2x$ can be written as $dy = (2x)\,dx$ with dy representing an estimate of Δy for the value of dx chosen to be equal to Δx. The expression $dy = (2x)\,dx$ is known as the differential of the function $y = x^2$. A formal definition for the differential is given below.

Definition 5.4

If $f'(x^0)$ is the derivative of the function $y = f(x)$ at point x^0, then the **total differential** at a point x^0 is

$$dy = df(x^0, dx) = f'(x^0)\,dx$$

Thus the differential is a function of both x and dx.

The differential provides us with a method of *estimating* the effect of a change in x of amount $dx = \Delta x$ on y, where Δy is the *exact* change in y while dy is the *approximate* change in y. Given the definition of the derivative, this is equivalent to using the tangent line of a function to estimate the impact of a change in x on y. For the function $f(x) = x^2$, the differential is $dy = f'(x)\,dx = 2x\,dx$. To see that this expression represents only an approximation to the *true* relationship between the change in x and the change in y, refer to figure 5.8. We see that from the point $P = (2, 4)$, an increase in x of amount $\Delta x = dx = 2$ leads to a change in y of $\Delta y = 12$. If we use the tangent line at the point $P = (2, 4)$ to estimate the change in y resulting from a change in x of 2 units, we find that $dy = 8$ (i.e., $dy = f'(x)\,dx = (2x)\,dx = 4\,dx$, with $dx = 2$). In this case, using the differential leads to an estimate of Δy that is too low.

In fact, in either the derivative or differential form, the relationship between dx and dy can be thought of as an estimate for the true relationship between changes in x and y (i.e., Δx and Δy). This is clearer if we think of the relationship in the following form:

$$\Delta y = dy + \epsilon \tag{5.3}$$

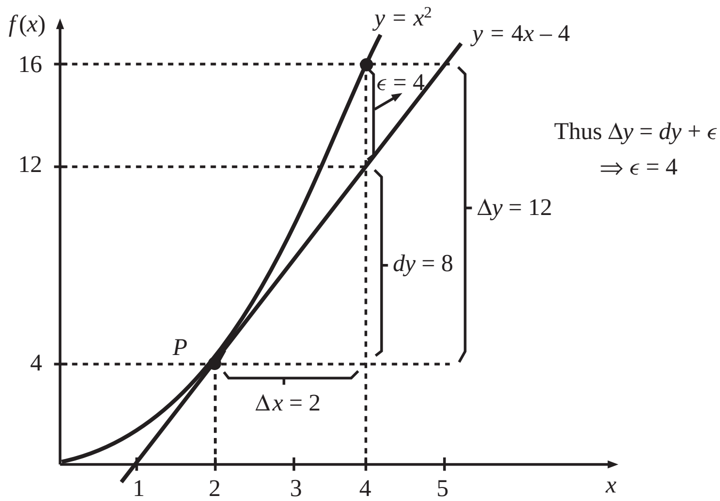


Figure 5.8 $dy = f'(x)\,dx$ as an approximation to the change in y

where ϵ is the approximation error. For the example illustrated in figure 5.8 the error is $\epsilon = 4$. Thus, as an approximation of the true change in y, the formula of equation (5.3) is not very impressive. However, one can see that for smaller changes in x, the expression $dy = f'(x)\,dx$ offers a *better* approximation. For example, beginning again with $x = 2$ (i.e., $P = (2, 4)$), we find that choosing $\Delta x = dx = 1$ leads to $\Delta y = 5$ and $dy = 4$ (see figure 5.9). Not only is dy closer to Δy in absolute terms, but the percentage error is reduced from 33% $(4/12)$ to 20% $(1/5)$ when going from $\Delta x = 2$ to $\Delta x = 1$. Furthermore we can show that

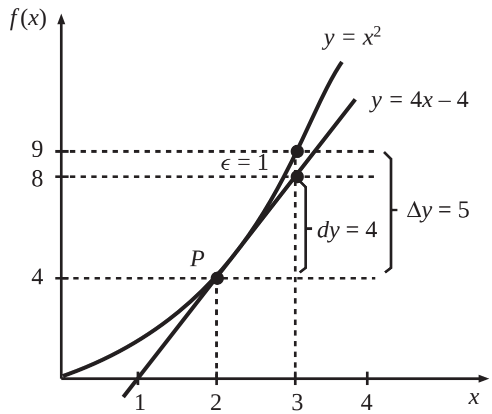


Figure 5.9 Accuracy of $dy = f'(x)\,dx$ as an approximation to Δy

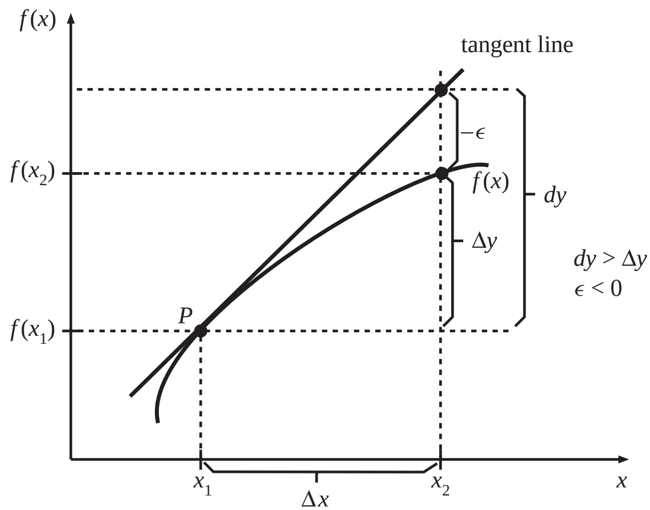


Figure 5.10 Case where the differential is an overestimate

equation (5.3) can be made arbitrarily accurate (i.e., the percentage error $\epsilon/\Delta y$ can be made arbitrarily small) by requiring that the change in x be small. Formally this means that $\lim_{\Delta x = dx \to 0} \epsilon/\Delta y = 0$.

Notice that for the example $y = x^2$, using the tangent line or the expression $dy = f'(x)\,dx$ to approximate the impact of a change in x on y led to an *underestimate*. It is of course also possible that the use of the tangent line will

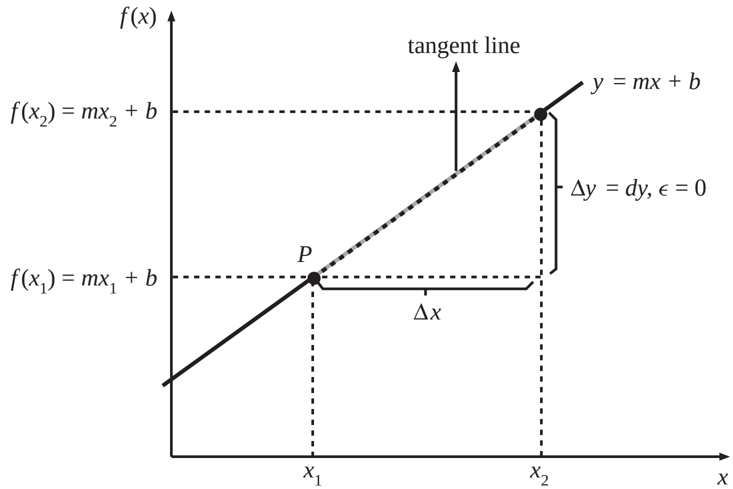


Figure 5.11 Case where the differential is an exact approximation

lead to an *overestimate* of the impact of Δx on Δy. Such is clearly the case for the function illustrated in figure 5.10. For the case of a linear function, $y = mx + b$, the expression $dy = f'(x)\,dx = m\,dx$ provides an *exact* approximation of the impact of a change in x of amount dx or Δx on y (i.e., $dy = m\,dx$ and $\Delta y = m\Delta x$). This is illustrated in figure 5.11.

The Total- and Marginal-Cost Functions

A firm's total-cost function, $C = C(y)$, indicates the cost of producing amount of output, y. Thus, given $C = C(y)$, the ratio $\Delta C/\Delta y = (C(y + \Delta y) - C(y))/\Delta y$ reflects the (average) rate of change in cost per added unit of output produced. If we take the limit of this ratio as $\Delta y \to 0$, we get the *instantaneous* rate of change, which is generally referred to as the **marginal-cost of production:**

$$\lim_{\Delta y \to 0} \frac{\Delta C}{\Delta y} = \lim_{\Delta y \to 0} \frac{C(y + \Delta y) - C(y)}{\Delta y} = C'(y)$$

and is the derivative of the total-cost function.

Let us begin with the simplest type of example, the case of a linear cost function. In particular, let $C = 80y$. This function implies that whatever is the current level of output produced, the cost of producing an extra unit of output is 80 (i.e., since $C'(y)$ or $dC/dy = 80$). The differential $dC = C'(y)\,dy$ becomes $dC = 80\,dy$, indicating that any change in y of amount dy leads to a change in cost of 80 times dy (e.g., producing $dy = 3$ more units of output leads to an increase in cost of $dC = 80(3) = 240$). As indicated earlier in this section (see figure 5.11), in the case of a linear function the differential represents an exact estimate of the relationship between the actual change in C (i.e., ΔC) and the actual change in y (i.e., Δy). Moreover the fact that the derivative is a constant implies that the marginal cost of production is independent of the existing amount of output being produced. This is illustrated by figure 5.12.

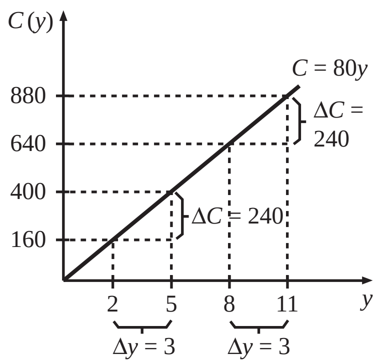


Figure 5.12 A linear cost function

A firm will have a linear cost function only if additional units of output require the same extra amount of inputs even as the scale of production becomes very large. This is the case of *constant returns to scale*. This may not be true for all production processes. It is especially unlikely to be true in the short run, when certain inputs, such as capital equipment, are fixed in amount. Suppose that we consider labor as the only variable input used. As the firm uses greater amounts of labor to increase production, the additional units of labour eventually become less productive due to the fact that there is less and less capital for each unit of labor to work with. It then becomes more expensive to produce extra units of output when higher levels of output are already being produced. The function $C(y) = y^2$ possesses this property.

The function $C(y) = y^2$ has derivative $C'(y) = 2y$ (see figure 5.13). According to this derivative function, the marginal cost of production is increasing in y. For example, if the amount of output being produced is $y = 200$ units, then the marginal cost of producing an additional unit of output is $C'(y) = 2(200) = 400$, while if the current level of output is 300, then the marginal cost of producing an additional unit of output is $C'(y) = 2(300) = 600$. Of course, using the derivative to represent the increased cost of producing an extra *discrete* unit of output is subject to error (refer to figures 5.8 and 5.9). For example, if $y = 300$ and one more unit of output is produced, then the exact increase in cost is $\Delta C = C(301) - C(300) = 90{,}601 - 90{,}000 = 601$.

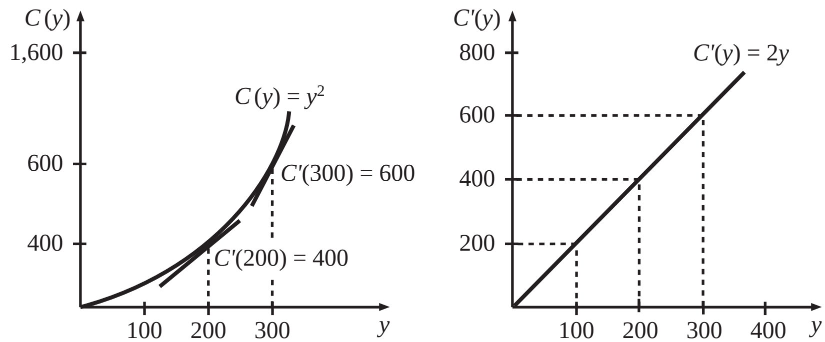


Figure 5.13 Relationship between the marginal and total cost functions for $C(y) = y^2$

Notice that the exact cost of producing a larger additional amount, say 100 more units of output, is $\Delta C = C(400) - C(300) = 160{,}000 - 90{,}000 = 70{,}000$ while using the differential to estimate this increased cost leads to the result $dC = C'(y)\,dy = 2y\,dy = 2(300)(100) = 60{,}000$. The error for the larger change is $(10{,}000/70{,}000) \times 100 = 14.3\%$ while the error for the smaller change is only $(1/600) \times 100 = 0.17\%$. This example demonstrates that as smaller changes in y are considered (i.e., $\Delta y \to 0$), the percentage error tends to zero.

The basic message of this section is that any function which has a derivative at a given point can be approximated by a linear function (the graph of which is represented by its tangent line) in some suitably *small* neighborhood of that point. Since linear functions are relatively easy to understand and work with, this result is very useful. In particular, by concentrating on some small neighborhood of a solution for a problem which involves nonlinear functions we are able to use the simple mathematics of linear functions to generate useful results. Thus the techniques of linear algebra (chapters 7 through 10) can be used to generate comparative statics results for systems of relationships which are described by nonlinear functions. This will become evident in later chapters.

EXERCISES

1. From the definition of the derivative (definition 5.3), find the derivative for each of the following functions:

(a) $f(x) = 3x - 5$

(b) $f(x) = 8x$

(c) $y = 3x^2$

2. From the definition of the derivative (definition 5.3), find the derivative for each of the following functions:

(a) $f(x) = 6x$

(b) $f(x) = 12x - 2$

(c) $f(x) = kx^2$ for k a constant

3. Return to the example in question 1 of exercise 5.1. Note that the derivative of this function, $f(x) = x^2$, is $f'(x) = 2x$. Use the differential to estimate the changes in y between $P = (20, 400)$ and each of the 5 points $Q_n, n = 1, 2, 3, 4, 5$. Find the percentage error defined as $\epsilon = (\Delta y - dy)/(\Delta y) \times 100$ for each case, where $dy = f'(x)\,dx$ and $dx \equiv \Delta$x. Use the following table:

Q_i	(25, 625)	(24, 576)	(23, 529)	(22, 484)	(21, 441)
$\Delta x \equiv dx$					
Δy					
$dy = f'(x)\,dx$					
ϵ					

What does this example suggest about the use of the differential as an estimate of the actual change in the function value as x changes? Discuss.

4. Return to the example in question 2 of exercise 5.1. Recall (or compute) the sequence of changes Δy_n associated with changes Δx_n with reference to the point $P = (20, 400)$. Compare the results of this formula for Δy_n, the actual change in y, with the estimated change in y using the differential $dy = f'(x)\,dx$ where $dx = \Delta x_n$. In particular, compute the percentage error term

$$\epsilon_n = \frac{\Delta y_n - dy}{\Delta y_n} \times 100$$

What does this formula suggest about the use of the differential as an estimate of the actual change in the function value as x changes? Discuss.

5.3 Conditions for Differentiability

In section 5.2 we defined the derivative of a function $f(x)$ as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Therefore the derivative of a function at a point x exists provided this limit exists. To begin, from our discussion of limits in chapter 4 we know that Δx may be either positive or negative and so existence of $f'(x)$ requires that f must be defined at every point in some neighborhood (i.e., open interval) of x. Having established this condition, we then need to check that $\lim_{\Delta x \to 0} f(x + \Delta x)$ exists, which means that the left-hand and right-hand limits

$$\lim_{\Delta x \to 0^-} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \qquad \lim_{\Delta x \to 0^+} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

are equal to each other. We refer to these two limits as the **left-hand derivative** and **right-hand derivative** of the function $f(x)$. These conditions are summarized in the following definition.

Definition 5.5 A function $f(x)$, which is defined on an open interval including the point $x = a$, is **differentiable** at that point if

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

exists and is finite. That is,

$$\lim_{\Delta x \to 0^-} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

The value of this expression is the value of the derivative function $f'(x)$ at the point $x = a$.

The best way to understand the condition for differentiability is through a series of examples of functions that are not differentiable at some point.

Example 5.1 The following simple example of a function which is *not* differentiable illustrates the meaning of the conditions in definition 5.5. Consider the function defined by

$$f(x) = \begin{cases} x, & \text{if } x < 1 \\ 2 - x, & \text{if } x \geq 1 \end{cases}$$

drawn in figure 5.14. This function is continuous at the point $x = 1$ but is not differentiable at this point. The reason is that for $x < 1$ we use $f(x) = x$ to generate the expression for the left-hand limit,

$$\lim_{\Delta x \to 0^-} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{(1 + \Delta x) - (1)}{\Delta x} = 1$$

while for $x > 1$ we use $f(x) = 2 - x$ to generate the expression for the right-hand limit,

$$\lim_{\Delta x \to 0^+} \frac{f(1 + \Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{(2 - (1 + \Delta x)) - (2 - 1)}{\Delta x}$$
$$= \frac{-\Delta x}{\Delta x} = -1$$

That is, the right- and left-hand derivatives do not equal each other and so the function is not differentiable at the point $x = 1$.

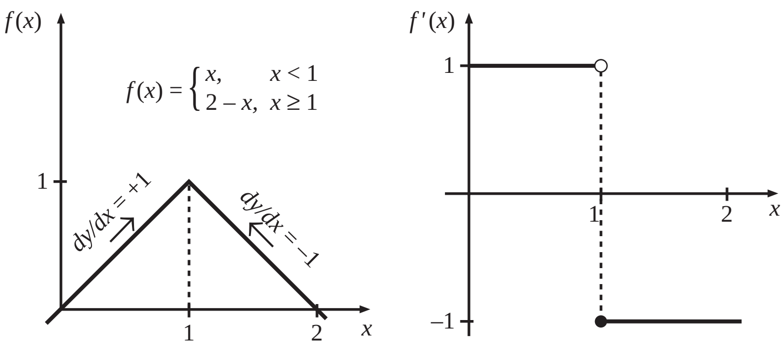


Figure 5.14 A function with different left-hand and right-hand derivatives at point $x = 1$ ■

Example 5.2 Typically the marginal rate of income tax varies across income ranges. For example, suppose that the first \$5,000 income earned is not taxed at all, the next \$10,000 earned is taxed at the rate of 15%, and any further income is taxed at the rate of 25%. Let $T(y)$ represent the income tax schedule for this example. The simplest way to graph this function is to think about how the marginal tax rate changes over the various income ranges:

$$\text{Marginal tax rate} = \begin{cases} 0, & \text{for } 0 < y \leq 5{,}000 \\ 0.15, & \text{for } 5{,}000 < y \leq 15{,}000 \\ 0.25, & \text{for } y > 15{,}000 \end{cases}$$

Since no tax is paid on zero income, we have $T(0) = 0$. Knowing the rate at which the tax increases allows us to draw the tax function (see figure 5.15). Note that the tax function is not discontinuous. However, it is not differentiable at the points $y = 5{,}000$ or $y = 15{,}000$ because the marginal tax rate changes at these points, with the result that the left- and right-hand derivatives are not the same. Notice that if we graph the derivative function, $T'(y)$, on the intervals $[0, 5{,}000]$, $(5{,}000, 15{,}000]$, and $(15{,}000, \infty)$, as is done in figure 5.16, we see that it is discontinuous at the points where $T(y)$ is not differentiable. This result illustrates that since the left- and right-hand limits of the derivative are not equal at a point of nondifferentiability, the derivative function is not defined at that point.

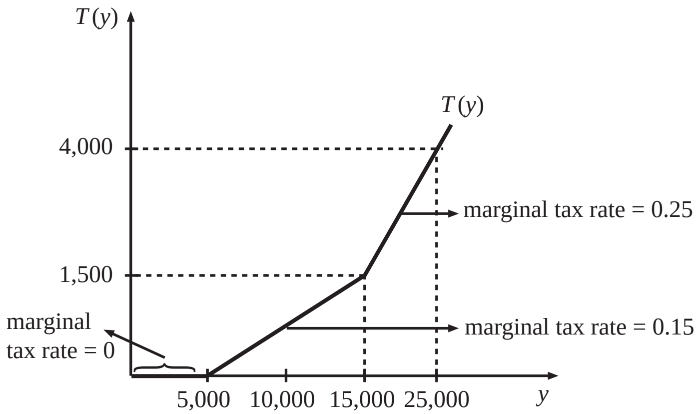


Figure 5.15 Income tax schedule for example 5.2

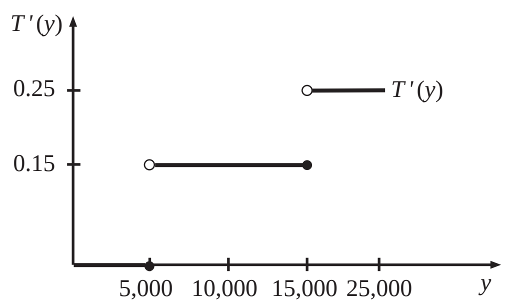


Figure 5.16 Marginal income tax function for example 5.2

The tax function, $T(y)$, is defined in equation (5.4). The first part of the function, $T(y) = 0$, for $0 \leq y \leq 5{,}000$ reflects the fact that the first \$5,000 earned is not taxed and so anyone earning less than this amount pays zero tax. The second part, $T(y) = 0.15(y - 5{,}000)$, for $y \in (5{,}000, 15{,}000]$ indicates that any income earned in excess of \$5,000, but less than \$15,000, is taxed at the rate of 15%. The third part, $T(y) = 0.15(y - 5{,}000) + 0.10(y - 15{,}000)$, for $y >$ \$15,000 indicates that income earned in excess of \$15,000 is taxed at a rate of 25% and so an additional $0.10(y - 15{,}000)$ must be added to the 15% which is charged on income earned over \$5,000.

$$T(y) = \begin{cases} 0, & 0 \le y \le 5{,}000 \\ 0.15(y - 5{,}000), & 5{,}000 < y \le 15{,}000 \\ 0.15(y - 5{,}000) + 0.10(y - 15{,}000), & y > 15{,}000 \end{cases} \tag{5.4}$$

This tax schedule can be simplified to get

$$T(y) = \begin{cases} 0, & 0 \le y \le 5{,}000 \\ 0.15y - 750, & 5{,}000 < y \le 15{,}000 \\ 0.25y - 2{,}250, & y > 15{,}000 \end{cases} \tag{5.5}$$

■

Returning to the definition for differentiability (definition 5.5), we can see a close relationship to the condition for continuity (refer to chapter 4.1). For the function f to be differentiable at the point x, the $\lim_{\Delta x \to 0} f(x + \Delta x)$ must exist; that is, the left-hand limit, $\lim_{\Delta x \to 0^-} f(x + \Delta x)$, must equal the right-hand limit, $\lim_{\Delta x \to 0^+} f(x + \Delta x)$. Moreover, as we noted earlier in this section, it must also be the case that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$. These conditions imply that the function $f(x)$ is continuous. Thus, if a function is not continuous, then it is not differentiable and so we have demonstrated the following theorem.

Theorem 5.1 If $f'(x)$ exists (i.e., the function $f(x)$ is differentiable) at the point $x = a$, then the function $f(x)$ must also be continuous at this point.

It is important to note that continuity is a necessary but not a sufficient condition for differentiability. In other words, a function may be continuous at some point yet not differentiable there. This is clearly established by example 5.2 above. At the point $x = 1$, we find that

$$\lim_{\Delta x \to 0^-} f(x + \Delta x) = 1, \quad \lim_{\Delta x \to 0^+} f(x + \Delta x) = 1, \quad f(1) = 1$$

which implies that the function is continuous. However, it is not differentiable at the point $x = 1$.

We can employ the concepts of left- and right-hand derivatives in defining whether a function is differentiable in the case in which the domain of the function is a closed interval, $[a, b]$, or has only one boundary point, $[a, +\infty)$ or $(-\infty, b]$. Note that if the domain of a function is the closed interval $[a, b]$, then the question arises of defining its differentiability at a and b. To handle such cases, we ask whether the relevant one-sided limits exist as one approaches $x = a$ from the right or $x = b$ from the left. The following definition for differentiability of a function defined on a closed interval $[a, b]$ can be extended in an obvious way for functions defined on the sets $[a, \infty)$ and $(-\infty, b]$.

Definition 5.6 A function $f(x)$ defined on the domain $x \in [a, b]$ is differentiable on $[a, b]$ if (i) the right-hand derivative for $f(x)$ exists at $x = a$, (ii) the left-hand derivative exists at $x = b$, and (iii) $f(x)$ is differentiable at every point in the open set (a, b).

An example of a situation in which it is natural to use a function defined on a closed interval is the case of trade quotas. If a firm can export up to a maximum amount, say b, of some product into a country, then one would define sales revenue for this activity by $R(x)$, $x \in [0, b]$. Sales, x, must not exceed the value b and cannot be negative.

Example 5.3 **Modeling Enforcement as a Probability**

Consider an enforcement agency choosing a level of enforcement of a law against some crime (e.g., tax evasion). We can represent the level of enforcement by the probability with which the crime will be detected. Let p be this probability of detection. By definition, p must lie in the closed interval $[0, 1]$. If the function $c(p)$ represents the resource cost of achieving detection level p, then $p \in [0, 1]$ is the domain of the function. For example, $c(p) = kp^2$ (see figure 5.17) could represent an enforcement cost function which has right-hand derivative of value zero at $p = 0$ and left-hand derivative of value $2k$ at $p = 1$. ■

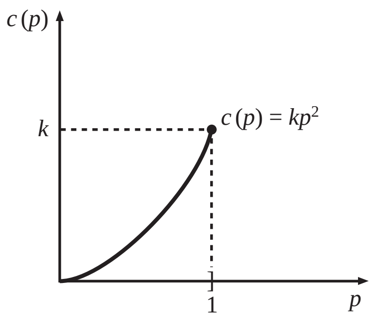


Figure 5.17 Function with a domain being a closed interval

EXERCISES

1. The following are examples of functions that are not differentiable at some point. Explain in each case why the function is not differentiable according to definition 5.5. That is, find the left- and right-hand derivatives at the point of nondifferentiability.

(a) $f(x) = \begin{cases} 3x + 2, & x \leq 5 \\ x + 12, & x > 5 \end{cases}$

(b) $f(x) = \begin{cases} -x, & x \leq 0 \\ x, & x > 0 \end{cases}$

(c) $f(x) = \begin{cases} 4x + 1, & x < 2 \\ 11 - x, & x \geq 2 \end{cases}$

2. The following are examples of functions which are not differentiable at some point. Explain in each case why the function is not differentiable according to definition 5.5. That is, find the left- and right-hand derivatives at the point of nondifferentiability.

(a) $f(x) = \begin{cases} -2x + 20, & x \leq 4 \\ -x + 16, & x > 4 \end{cases}$

(b) $f(x) = \begin{cases} -2x + 5, & x \leq 0 \\ x + 5, & x > 0 \end{cases}$

(c) $f(x) = \begin{cases} 3x, & x < 2 \\ 8 - x, & x \geq 2 \end{cases}$

3. Consider the following income tax scheme:

The first \$6,000 of income is not subject to any tax.
The next \$10,000 is subject to a tax rate of 20%.
The next \$30,000 is subject to a tax rate of 30%.
Any additional income is subject to a tax rate of 40%.

(a) Find and graph the tax function, $T(y)$, defined on $y \geq 0$.

(b) Determine the points of nondifferentiability for this function and indicate according to definition 5.5 why each is a point of nondifferentiability.

(c) Graph the marginal tax function and the average tax function $(T(y)/y)$ on the same graph.

4. Consider the following income tax scheme:

The first \$5,000 of income is not subject to any tax.
The next \$15,000 is subject to a tax rate of 20%.
The next \$30,000 is subject to a tax rate of 35%.
Any additional income is subject to a tax rate of 50%.

(a) Find and graph the tax function, $T(y)$, defined on $y \geq 0$.

(b) Determine the points of nondifferentiability for this function and indicate according to definition 5.5 why each is a point of nondifferentiability.

(c) Graph the marginal tax function and the average tax function $(T(y)/y)$ on the same graph.

5. Suppose that a salesperson has the following contract relating monthly sales, S, to her monthly pay, P. She is given a basic monthly amount of \$600, regardless of her sales level. On the first \$10,000 of monthly sales she earns a 10% commission. On any additional sales she earns a commission of 20%.

(a) Find and graph the function relating her pay to sales, $P(S)$, $S \geq 0$.

(b) Determine the point of nondifferentiability of $P(S)$ and indicate according to definition 5.5 why this is so.

5.4 Rules of Differentiation

In section 5.2 several examples were presented showing how to derive the derivatives of some simple functions by using the definition of the derivative. It would be tedious, however, to have to do this every time we wanted to find the derivative of a function. Since such derivations have been done for various general classes of functions, we can use these results and so avoid repeating the exercise each time. These rules or methods of finding derivatives are collected below in summary form and then presented more fully along with some economic applications. Only in some of the simpler cases do we show how to generate results from first principles.

Rules of Differentiation

Rule 1 Derivative of a constant function:

If $f(x) = c$, *a constant, then* $f'(x) = 0$.

Rule 2 Derivative of a linear function:

If $f(x) = mx + b$, *with* m *and* b *constants, then* $f'(x) = m$.

Rule 3 Derivative of a power function:

If $f(x) = x^n$, *then* $f'(x) = nx^{n-1}$.

Rule 4 Derivative of the constant multiple of a function:

If $g(x) = cf(x)$, *with* c *constant, then* $g'(x) = cf'(x)$.

Rule 5 Derivative of the sum or difference of a pair of functions

If $h(x) = g(x) + f(x)$, *then* $h'(x) = g'(x) + f'(x)$, *while if* $h(x) = g(x) - f(x)$, *then* $h'(x) = g'(x) - f'(x)$.

Rule 6 Derivative of the sum of an arbitrary but finite number of functions:

If $h(x) = \sum_{i=1}^{n} g_i(x)$, *then* $h'(x) = \sum_{i=1}^{n} g_i'(x)$.

Rule 7 Derivative of the product of two functions:

If $h(x) = f(x)g(x)$, *then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Rule 8 Derivative of the quotient of two functions:

If $h(x) = \dfrac{f(x)}{g(x)}$, $g(x) \neq 0$, *then* $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

Rule 9 Derivative of a function of a function—the chain rule:

If $y = f(u)$ *and* $u = g(x)$ *so that* $y = f(g(x)) = h(x)$, *then* $h'(x) = f'(u)g'(x)$ *or*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Rule 10 Derivative of the inverse of a function:

If $y = f(x)$ *has the inverse function* $x = g(y)$, *that is, if* $g(y) = f^{-1}(y)$ *and* $f'(x) \neq 0$, *then*

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad \text{or} \quad g'(y) = \frac{1}{f'(x)} \quad \text{where } y = f(x)$$

Rule 11 Derivative of the exponential function:

If $y = e^x$, *then* $dy/dx = e^x$.

Rule 12 Derivative of the logarithmic function:

If $y = \ln x$, *then* $dy/dx = 1/x$.

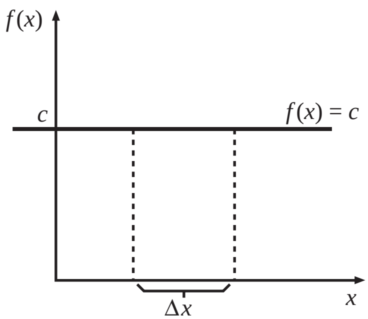


Figure 5.18 Constant function $f(x) = c$ has a zero slope

Rule 1 Derivative of a Constant Function, $f(x) = c$

If $f(x) = c$, a constant, then $f'(x) = 0$.

The reason that $f'(x) = 0$ when $f(x) = c$ is easy to see intuitively by looking at the graph of the function $f(x) = c$ (see figure 5.18). Regardless of which point x is chosen, $\Delta y = 0$ for any value of Δx. Here $\Delta y = 0$ for any size of Δx.

Example 5.4 Marginal Revenue Function for a Competitive Firm

A competitive firm believes that if it sells more output there will not be a reduction in the market price. The extra revenue generated by producing and selling one more unit of output is therefore simply the price of the product. This is, of course, a sensible attitude if the firm is a small producer in a large market. Thus the extra revenue generated by an extra unit of output is constant regardless of the level of output of the firm. If we let $\bar{p}$ be market price, and $\text{MR}(q)$ represent marginal revenue as a function of output, it follows that $d\text{MR}(q)/dq = 0$ (see figure 5.19).

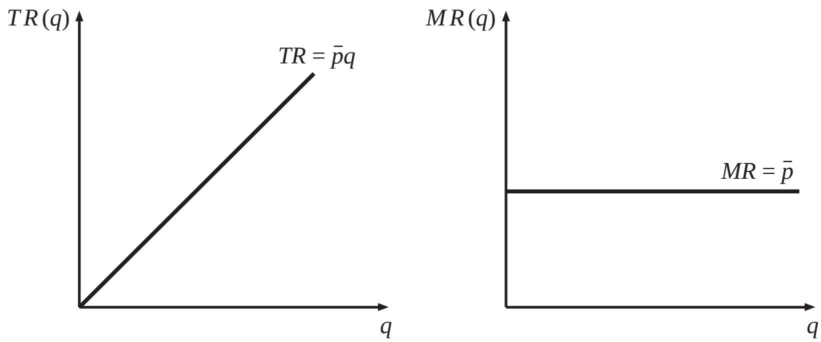


Figure 5.19 Total revenue and marginal revenue of a competitive firm (example 5.4)

Of course, if the firm produced a level of output equal to a substantial fraction of the output generated by all firms taken together, say one-half, then it would no longer make sense for the firm to believe that the extra revenue generated by additional sales was independent of the amount of extra output produced. In this case, if a firm's increments in output are *large* relative to the size of the market, then it would be appropriate to assume $d\text{MR}(q)/dq$ is negative. ■

Rule 2 Derivative of a Linear Function, $f(x) = mx + b$

If $f(x) = mx + b$, with m and b constants, then $f'(x) = m$.

This result follows because $\Delta y = f(x+\Delta x) - f(x) = m(x+\Delta x) + b - [m(x) + b] = m\Delta x$. Then $\Delta y/\Delta x = m$, and so $\lim_{\Delta x \to 0} \Delta y/\Delta x = m$.

For example, the derivative of the function $f(x) = 3x - 5$ is $f'(x) = 3$. The important implication of this result is that for a linear function the rate at which the variable y changes with respect to a change in x is the same at every value x. We explore this property of linear functions in detail with an economic example.

Example 5.5 **Slope of a Linear Demand Function**

If p represents price and q represents quantity demanded, then $q = a - bp$, $b > 0$ is a general form of a linear demand function. This could reflect either the demand for a product by a consumer or by the market as a whole. More specifically, if $q = 30 - 2p$, then $dq/dp = -2$ means that quantity demanded falls by 2 units for every one unit increase in the price regardless of the current price level or quantity demanded. This demand function is drawn in figure 5.20. Notice that in describing this function $q(p) = 20 - 2p$ one might well expect the variable p to be drawn on the horizontal axis with q on the vertical axis. However, it is standard practice to do the reverse of this, which corresponds more intuitively and conforms more to standard practice to think graphically in terms of the inverse of this function, namely $p = 15 - (q/2)$, which is often referred to as the inverse demand function.

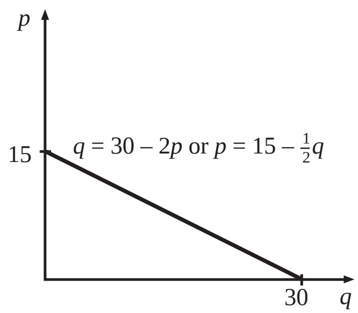


Figure 5.20 Linear demand has a constant slope (example 5.5)

This last condition may well seem rather unrealistic in certain circumstances. For example, suppose that we consider the case of electricity demanded by a household. Let p be the price of electricity, measured in cents per kilowatt-hour, and let q be quantity demanded by a single household, measured in hundreds of kilowatt-hours per month. It seems a reasonable possibility that if the price changed from $p = 8$ to $p = 9$, demand may fall by 200 kilowatt-hours per month (from $q = 1{,}400$ kilowatt-hours to $q = 1{,}200$ kilowatt-hours) as it would if the price changed from $p = 9$ to $p = 10$ (from $q = 1{,}200$ kilowatt-hours to $q = 1{,}000$ kilowatt-hours), as predicted by the demand function. If, however, the price were at the level $p = 14$ and were increased by one unit to $p = 15$, it seems less reasonable to expect that the household would reduce consumption by 200 kilowatt-hours, since that would take the consumer to a zero level of demand.

For many commodities the rate at which consumers will reduce consumption of a good (dq/dp) as price rises will depend on the current consumption level or, equivalently, the current price of that good. This is one reason economists often consider the presumption that demand is linear in price to be just a rough approximation that is reasonable only over a certain range of prices. ■

Rule 3 Derivative of a Power Function, $f(x) = x^n$

$$\textit{If } f(x) = x^n, \textit{ then } f'(x) = nx^{n-1}.$$

For example, the derivative of the function $f(x) = x^2$ is $f'(x) = 2x$, as we derived from the definition of the derivative in section 5.2. The following list of examples illustrates various types of results using this rule.

For $f(x) = x^{3/2}$, $x > 0$, we get $f'(x) = (3/2)x^{(3/2-1)} = (3/2)x^{1/2}$. The function $f(x) = x^{-2}$ has derivative $f'(x) = -2x^{-3}$. The derivative of $f(x) = x^0$, $x \neq 0$, is $f'(x) = 0x^{0-1} = 0/x = 0$ for $x \neq 0$. Since for $x \neq 0$, $x^0 = 1$, this is clearly the correct result. (Recall that 0^0 is not defined.)

Example 5.6 Total and Marginal Product of Labor

Consider the production function $y = L^a$, $a > 0$ which relates the level of input labor, L, to output, y. This function is often called the total product of labor, TP(L). The marginal product of labor is then $\text{MP}(L) = dy/dL = aL^{a-1}$. If the parameter value for a is greater than 1, then the marginal product of labor is increasing in L. For example, for $y = L^{3/2}$ we get $\text{MP}(L) = (3/2)L^{1/2}$. If $a = 1$, the function is simply the linear function $y = L$, and the derivative is the constant function $dy/dL = 1$. If the value for a is less than 1, then the marginal product of labor is decreasing in L. For example, for $y = L^{1/2}$ we get $\text{MP}(L) = (1/2)L^{-1/2}$ or $1/(2L^{1/2})$. These functions are graphed in figures 5.21, 5.22, and 5.23.

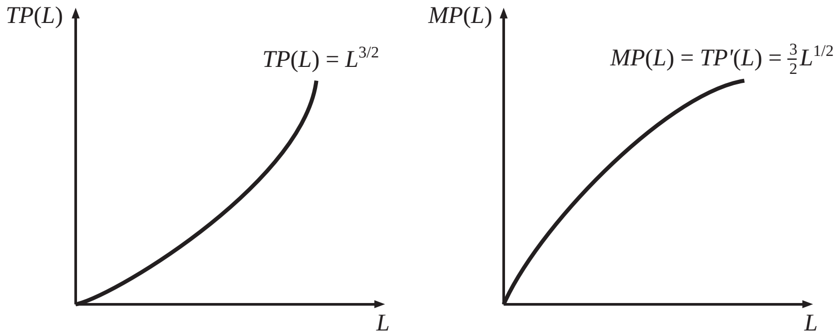


Figure 5.21 Total product of labor displaying increasing marginal product

It is reasonable to expect that if the amounts of inputs other than labor are held fixed, then, at least eventually, as more labor is used we obtain smaller increments in output per added unit of labor. This presumption is called **diminishing marginal productivity** of a variable input. The only time this is satisfied for the functional form $y = L^a$ is when $a < 1$, as the above examples illustrate. As a result we often see the restriction $a < 1$ imposed when the example of $y = L^a$ is used. We investigate this issue further in section 5.5. ■

As you will see throughout the chapters on optimization and multivariate calculus, power functions are frequently used to illustrate properties of various

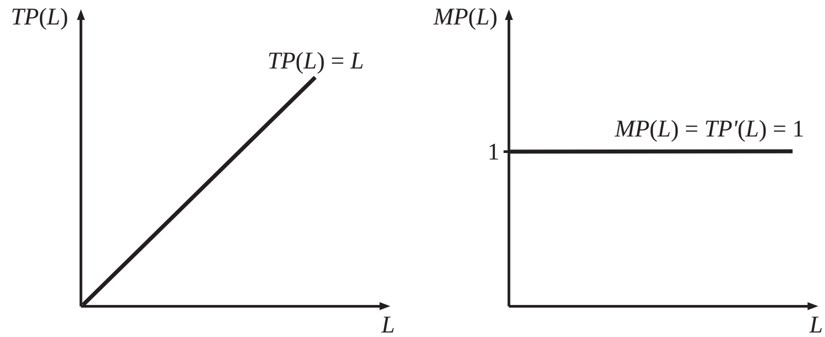


Figure 5.22 Total product of labor displaying constant marginal product

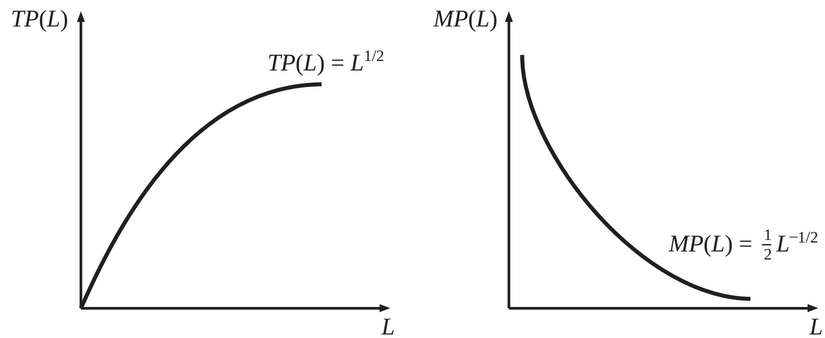


Figure 5.23 Total product of labor displaying decreasing marginal product

economic concepts such as production functions and utility functions. For example, the one variable case considered above, $y = L^a$, generalizes to the important class of examples referred to as Cobb-Douglas production or utility functions. It is very important, therefore, to understand the relationship between the size of a and the change in the value of the marginal product of labor as L increases.

Rule 4 Derivative of the Constant Multiple of a Function, $g(x) = cf(x)$

$$\textit{If } g(x) = cf(x), \textit{ then } g'(x) = cf'(x).$$

This rule is a very straightforward one to implement.

Example 5.7 Find the derivative of the function $g(L) = 5L^2$.

Solution

This function can be written as $5f(L)$ where $f(L) = L^2$ and so the derivative of $g(L)$ is $g'(L) = 5f'(L) = 5(2L) = 10L$. ■

This general rule is easy to prove. Note that

$$\begin{aligned} g'(x) &= \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{cf(x + \Delta x) - cf(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c[f(x + \Delta x) - f(x)]}{\Delta x} \\ &= c\left[\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}\right] = cf'(x) \end{aligned}$$

Rule 5 Derivative of the Sum or Difference of a Pair of Functions, $h(x) = g(x) \pm f(x)$

If $h(x) = g(x) + f(x)$, then $h'(x) = g'(x) + f'(x)$, while if $h(x) = g(x) - f(x)$, then $h'(x) = g'(x) - f'(x)$.

This rule is also straightforward to implement.

Example 5.8 Find the derivatives of (i) $h(x) = 5x + 3x^2$ and (ii) $h(x) = 7x^3 - 4x^5$.

Solution

For (i) we have $h'(x) = 5 + 6x$ and for (ii) we have $h'(x) = 21x^2 - 20x^4$. ■

Rule 6 Derivative of the Sum of an Arbitrary but Finite Number of Functions, $h(x) = \sum_{i=1}^{n} g_i(x)$

If $h(x) = \sum_{i=1}^{n} g_i(x)$, then $h'(x) = \sum_{i=1}^{n} g_i'(x)$.

This result also applies to the case where some or all of the operations involve subtraction rather than addition.

Rule 6 is a straightforward generalization of rule 5. That is, since the derivative of the sum of two functions is simply the sum of the derivatives of the functions taken separately, then doing this iteratively allows one to establish rule 6. Thus, for example, if $h(x) = x^4 + 8x^2 + 2x$, we can treat $h(x)$ as the sum of two functions, $f(x) = (x^4 + 8x^2)$ and $g(x) = 2x$, and write $h'(x) = f'(x) + g'(x) = f'(x) + 2$ using rule 5. Apply rule 5 again to the function $f(x)$ to obtain $f'(x) = 4x^3 + 16x$. Substitution gives $h'(x) = 4x^3 + 16x + 2$. Of course, if we treat each term x^4, $8x^2$, and $2x$ as separate functions, then we can simply apply rule 6 to establish directly that $h'(x) = 4x^3 + 16x + 2$.

Rule 6 offers a useful notation for writing out the derivative for a general polynomial function. That is, for the function

$$h(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n = \sum_{i=0}^{n} a_i x^i$$

we can use rule 6 to write

$$h'(x) = \sum_{i=0}^{n} i a_i x^{i-1}$$

Rule 7 Derivative of the Product of Two Functions

If $h(x) = f(x) \cdot g(x)$, *then* $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Example 5.9 Find the derivative of

$$h(x) = (6x^4 + 2x^3)(5x - 10x^2 + 18x^5 - 4)$$

Solution

Let

$$f(x) = (6x^4 + 2x^3)$$

and

$$g(x) = (5x - 10x^2 + 18x^5 - 4)$$

Since

$$f'(x) = (24x^3 + 6x^2)$$

and

$$g'(x) = (5 - 20x + 90x^4)$$

we get

$$\begin{aligned} h'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= (24x^3 + 6x^2)(5x - 10x^2 + 18x^5 - 4) + (6x^4 + 2x^3)(5 - 20x + 90x^4) \end{aligned}$$

■

This example shows the usefulness of this rule; that is, it is often much simpler to use rule 7 in this manner than it is to expand the original expression for $h(x)$ and find the derivatives term by term. Moreover, if we are faced with an expression of the form $h(x) = f(x)g(x)$ but only know certain properties of functions $f(x)$ and $g(x)$, we can sometimes gain some insights by using this technique. The following example illustrates this point.

Marginal Revenue Function for a Competitive Firm and a Monopoly Firm

The total revenue of either a competitive firm or a simple monopolist is the unit price of its output times the quantity it produces/sells. Thus we write $\text{TR}(q) = pq$. A competitive firm treats the price as a constant value, equal to the market price, $\bar{p}$. Thus we can write $\text{TR}(q) = \bar{p}q$ for total revenue, and so marginal revenue is $\text{MR}(q) = d\text{TR}(q)/dq = \bar{p}$ (recall example 5.4). A monopolist, however, is the only firm in the industry and so recognizes that the amount it can sell is determined by the price it sets according to the market demand function, $q = D(p)$. Writing the demand function in its inverse form, $p = D^{-1}(q) = p(q)$, we see that the monopolist's total revenue function is

$$\text{TR}(q) = pq = [p(q)]q$$

Thus the monopolist's total revenue function is the product of two functions, $p(q)$ and q, and so we need to use the product rule to find $\text{MR}(q)$. We then have

$$\text{TR}(q) = p(q)q$$

and so

$$\text{MR}(q) = \frac{d\text{TR}(q)}{dq} = \frac{dp}{dq}q + p(q)\frac{dq}{dq}$$

or

$$\text{MR}(q) = p'(q)q + p$$

where $p'(q)$ is the derivative of the inverse demand function. If we consider the usual case of price being negatively related to quantity sold, $p'(q) < 0$, then it follows that the term $p'(q)q$ is negative, and so $\text{MR}(q) < p$; that is, the marginal revenue of increased sales is less than the unit price being charged at any $q > 0$.

To understand the result above, work through the following steps, which are illustrated in figure 5.24. A firm sells output $\hat{q}$ at the price $\hat{p} = p(\hat{q})$. To sell an additional bit of output Δq, it must reduce price on *all the units it sells* by Δp. Thus the change in revenue is

$$\Delta R = \text{area } A - \text{area } B$$

and so we get

$$\Delta R = (\hat{p} - \Delta p)\Delta q - \Delta p\hat{q}$$

The first term is the *gain* in revenue from selling extra output Δq at the new price $(\hat{p} - \Delta p)$, and this is given by area A in figure 5.24. The second term is the loss in revenue resulting from having to take a price reduction of Δp on the output $\hat{q}$ previously sold at the higher price $\hat{p}$, area B in the figure. Whether ΔR is positive or negative then depends on the relative sizes of these two areas. Note that the lower is $\hat{p}$, the higher is $\hat{q}$, and so the larger is area B

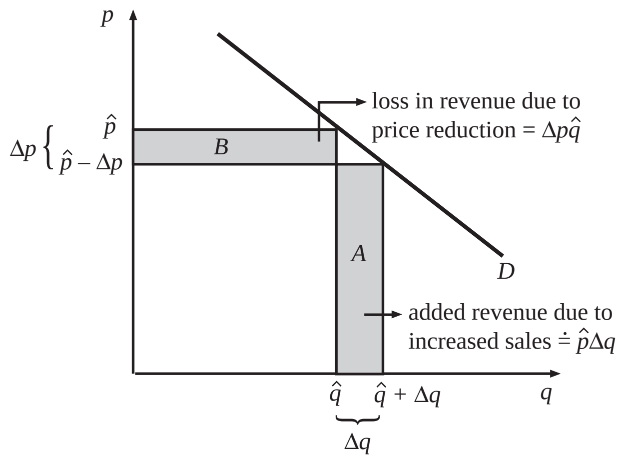


Figure 5.24 Impact on a monopolist's revenue of the sale of an additional Δq units

relative to area A. This explains why marginal revenue diverges increasingly from price as quantity sold increases. Finally note that this divergence only arises if the firm must charge the same price on all units it sells, that is, if it cannot price discriminate.

If we divide through the expression above for ΔR by Δq, we obtain

$$\frac{\Delta R}{\Delta q} = \hat{p} - \frac{\Delta p}{\Delta q}\hat{q} - \Delta p$$

Then marginal revenue is the limit of this expression as $\Delta q \to 0$. Since $\Delta p \to 0$ as $\Delta q \to 0$, the limit is

$$\text{MR} = \lim_{\Delta q \to 0} \frac{\Delta R}{\Delta q} = \hat{p} + \hat{q}\frac{dp}{dq}$$

Notice the change in sign to positive for the second term. This simply recognizes that in the expression for $\Delta R/\Delta q$, we treat the reduction in price as a positive value, while in the expression $\lim_{\Delta q \to 0} \Delta R/\Delta q$, the term dp/dq is the slope of the inverse demand function which is itself negative (i.e., $-\Delta p/\Delta q$ is the slope of the inverse demand function, dp/dq, as $\Delta q \to 0$). Thus the second term in the expression above for MR is indeed negative.

Example 5.10

An example with a specific functional form may illuminate further. Suppose that the demand function is linear and, in its inverse form,

$$p(q) = 40 - 2q$$

The monopolist's total revenue function becomes

$$\text{TR}(q) = p(q)q = [40 - 2q]q$$

Using the product rule, the marginal revenue function is

$$\text{MR}(q) = \frac{d[40 - 2q]}{dq}q + \frac{d[q]}{dq}[40 - 2q] = [-2]q + 1[40 - 2q] = 40 - 4q$$

As an example of the fact that MR $< p$, notice that for output level $\hat{q} = 5$, the price charged is $\hat{p} = 30$ but $\text{MR}(\hat{q}) = 20$, which is less than price. This is illustrated in figure 5.25. ■

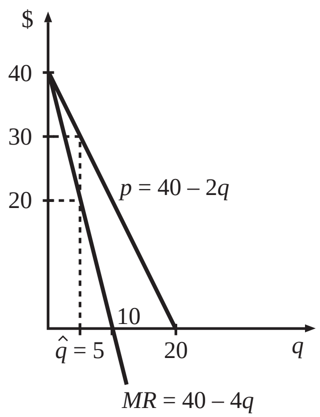


Figure 5.25 Marginal revenue for a monopolist facing the inverse demand function $p = 40 - 2q$

Rule 8 Derivative of the Quotient of Two Functions

If $h(x) = f(x)/g(x)$ *and* $g(x) \neq 0$, *then*

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Example 5.11 Find the derivative of

$$h(x) = \frac{10x^2 + 3x^4 + 5}{x^2 - 5x}$$

Solution

We can find the derivative by letting

$$f(x) = 10x^2 + 3x^4 + 5 \quad \text{and} \quad g(x) = x^2 - 5x$$

and then using the formula to get

$$f'(x) = 20x + 12x^3 \quad \text{and} \quad g'(x) = 2x - 5$$

and so

$$h'(x) = \frac{(20x + 12x^3)(x^2 - 5x) - (2x - 5)(10x^2 + 3x^4 + 5)}{[x^2 - 5x]^2}$$

■

Relation between Average and Marginal Values of a Function

Given a function $f(x)$ (which may be a cost function, total product function, etc.) its *average value function* is

$$A(x) = \frac{f(x)}{x}$$

Using the quotient rule, it follows that

$$A'(x) = \frac{(f'(x)x) - (1f(x))}{x^2} = \frac{1}{x}\left[f'(x) - \frac{f(x)}{x}\right] = \frac{1}{x}[f'(x) - A(x)]$$

where $f'(x)$ is the marginal value function. This tells us the following:

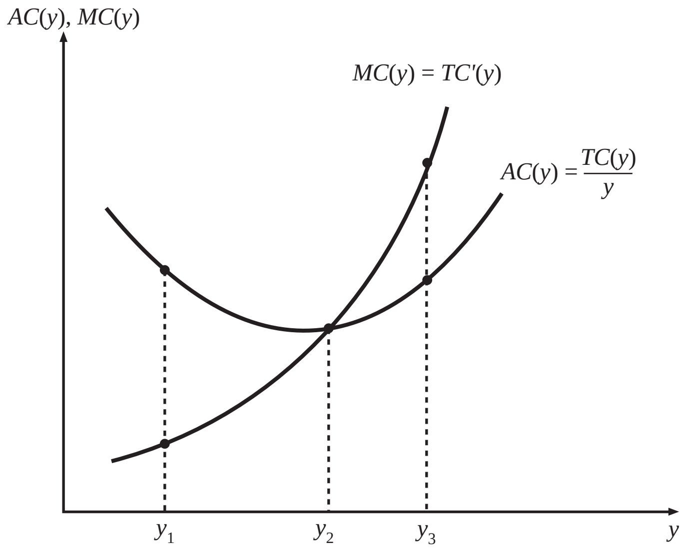


Figure 5.26 Marginal and average cost curves when the average cost is U-shaped

(i) When $f'(x) < A(x)$, then $A'(x) < 0$; that is, the average value function is falling.
(ii) When $f'(x) = A(x)$, then $A'(x) = 0$; that is, the average value function is horizontal or is at a point of horizontal tangency.
(iii) When $f'(x) > A(x)$, then $A'(x) > 0$; that is, the average value function is rising.

This relationship is illustrated for an average cost function, $AC(y)$, and marginal cost function, $MC(y)$, in figure 5.26. Note that when the cost of producing an extra unit of output, which is the marginal cost, is below the average cost, such as at point $y = y_1$, the average cost of production is falling in y. When the marginal cost is equal to the average cost, such as at point $y = y_2$, the average cost of production is not changing. When the marginal cost exceeds average cost, such as at point $y = y_3$, the average cost of production is rising.

Example 5.12 For the total cost function

$$TC(y) = y^2 + 10y + 25, \qquad y > 0$$

show that

(i) MC is less than AC where AC is falling
(ii) MC = AC at the point where the AC curve is horizontal
(iii) MC exceeds AC where AC is rising.

Solution

$$\mathrm{MC}(y) = 2y + 10$$

$$\mathrm{AC}(y) = y + 10 + \frac{25}{y}$$

$$\mathrm{AC}'(y) = 1 - \frac{25}{y^2}$$

and

$$\begin{aligned}\mathrm{AC}'(y) = 0 &\Rightarrow 1 - \frac{25}{y^2} = 0\\ &\Rightarrow 1 = \frac{25}{y^2}\\ &\Rightarrow y^2 = 25\\ &\Rightarrow y = 5 \qquad (\text{since } y > 0)\end{aligned}$$

Thus the AC curve is horizontal at the point $y = 5$, which corresponds to y_2 in figure 5.26. At $y = 5$, $\mathrm{MC} = 2(5) + 10 = 20$ and $\mathrm{AC} = 5 + 10 + 25/5 = 20$. This establishes result (ii).

Now

$$\begin{aligned}\mathrm{MC} < \mathrm{AC} &\Rightarrow 2y + 10 < y + 10 + \frac{25}{y}\\ &\Rightarrow y - \frac{25}{y} < 0\\ &\Rightarrow y < \frac{25}{y}\\ &\Rightarrow y^2 < 25\\ &\Rightarrow y < 5\end{aligned}$$

This establishes result (i). Similarly, $\mathrm{MC} > \mathrm{AC} \Rightarrow y > 5$ which establishes result (iii). ■

Rule 9 Derivative of a Function of a Function (the Chain Rule)

If $y = f(u)$ and $u = g(x)$ so that $y = f(g(x)) = h(x)$, then $h'(x) = f'(u)g'(x)$.

This result can also be written as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Thereason that this technique is called the chain rule is that a change in the value of the variable x affects the variable u, according to the function $u = g(x)$, and a change in the variable u, in turn, affects the variable y according to the function $y = f(u)$. Thus there is a chain of effects by which x affects y, which can be depicted by $x \to u \to y$.

The chain rule can be extremely convenient when one is faced with certain types of functions that would be very tedious to differentiate directly. For example, one way to differentiate the function

$$y = h(x) = (3x^4 + 5x^3 - 2x)^{30}$$

would be to expand the function by brute force and then find the derivative term by term. However, by recognizing that this function can be written as

$$y = f(u) = u^{30}, \quad \text{where } u = g(x) = 3x^4 + 5x^3 - 2x$$

it is easy to apply the chain rule to find its derivative. Noting that

$$\frac{dy}{du} = f'(u) = 30u^{29}$$

and

$$\frac{du}{dx} = g'(x) = 12x^3 + 15x^2 - 2$$

allows us to determine the result that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 30u^{29}[12x^3 + 15x^2 - 2]$$

Since the original problem is given in terms of the variable x, it is conventional also to give the final answer in terms of the variable x and so to make the substitution of $u = 3x^4 + 5x^3 - 2x$ to get

$$\frac{dy}{dx} = h'(x) = 30[3x^4 + 5x^3 - 2x]^{29}[12x^3 + 15x^2 - 2]$$

After a little practice one needn't make the explicit substitutions to solve such problems. For example, given the function $y = (3x - x^3)^6$, one can directly write down the derivative as

$$\frac{dy}{dx} = 6(3x - x^3)^5(3 - 3x^2)$$

This rule is also useful in describing certain economic relationships. The following example illustrates this point.

Rule 10 Finding the Derivative of the Inverse of a Function

If $y = f(x)$ has the inverse function $x = g(y)$, that is, $g(y) = f^{-1}(y)$, then

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Alternatively, we can write

$$g'(y) = \frac{1}{f'(x)}$$

This rule is easiest to understand in the context of a linear function. Suppose that $y = 2x$. Then $x = 0.5y$ is its inverse. Thus we can write $\Delta y/\Delta x = 2$ and $\Delta x/\Delta y = 1/(\Delta y/\Delta x) = 0.5$. Since the derivative of the function $y = f(x)$ is just the limit of $\Delta y/\Delta x$ as $\Delta x \to 0$ and the derivative of the function $x = f^{-1}(y)$ is just the limit of $\Delta x/\Delta y$ as $\Delta y \to 0$, the same logic applies to both the derivative of a function and its inverse.

The inverse function rule is especially useful for cases in which it is difficult to find explicitly the inverse of the function. For example, suppose that we have the function $y = x^5 + x^3$ and we want to know the derivative of the inverse of this function. Since $dy/dx = 5x^4 + 3x^2$, it follows that $dx/dy = 1/(dy/dx) = 1/(5x^4 + 3x^2)$. Thus, as long as one doesn't want the answer expressed in terms of the variable y, there is no need to try to write out the inverse function in the form $x = f^{-1}(y)$. The following example illustrates how to use the inverse function rule in a general context in order to demonstrate an important relationship between a firm's production function and its cost function.

Relationship between the Cost Function and the Production Function for the Case of One Input

Suppose that a firm produces quantity q of some product using a single input L according to the production function $q = q(L)$. The marginal product function is $\text{MP}(L) = dq/dL$. For example, if $q = L^{1/2}$, then $\text{MP}(L) = 1/2L^{-1/2}$. If the marginal product is falling as L increases, as it is in this example, then this means that for every extra unit of input used, the increment in output is smaller. Looking at this in reverse, we see that to produce an extra unit of output requires a larger increment of input the greater is the original output level. This is illustrated in figure 5.27. Note that at output level q^0 it takes ΔL_0 extra input to produce one more unit of output, whereas at output level q^* it takes ΔL^* extra input to produce an extra unit of output. At a given wage rate the cost of producing an extra unit of input must therefore be greater at output level $q = q^*$ than at $q = q^0$.

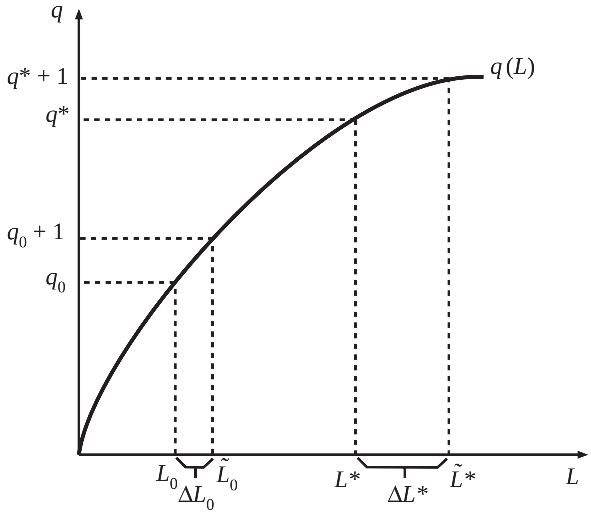


Figure 5.27 Case where more of an input is required to generate an extra unit of output, as the initial level of output becomes higher.

This example shows that the marginal cost of production rises whenever the marginal product of labor is falling. By a similar argument it follows that the marginal cost of production is falling whenever the marginal product of labor is rising. We develop this argument formally below.

If we let c_0 represent any fixed cost of production and w be the unit cost of L, then

$$C(L) = wL + c_0$$

is the cost when employing L units of labor. Using the inverse function notation, we can write the cost as a function of output as

$$C(q) = wL(q) + c_0$$

Thus marginal cost is

$$\frac{dC}{dq} = w\frac{dL}{dq}$$

Using the inverse function rule for differentiation, we see that

$$\frac{dC}{dq} = w\frac{1}{dq/dL}$$

Thus, since dq/dL appears in the denominator of the function dC/dq, we have the following results: if dq/dL is rising, then dC/dq is falling, while if dq/dL is falling, then dC/dq is rising.

Example 5.13 Use the inverse function rule of differentiation to show that the marginal cost curve associated with the production function $q = L^{1/2}$ is rising.

Solution

$$C(q) = wL(q) + c_0$$

and so

$$C'(q) = w\frac{dL}{dq} = \frac{w}{dq/dL}$$

Since $q = L^{1/2}$, we have

$$\frac{dq}{dL} = \frac{1}{2}L^{-1/2}$$

and so

$$C'(q) = \frac{w}{(L^{-1/2})/2} = 2wL^{1/2} = 2wq$$

which is increasing in output.

Notice that we can check this result by direct substitution

$$q = L^{1/2} \;\Rightarrow\; L = q^2$$

and so

$$C(q) = wL(q) + c_0 \;\Rightarrow\; C(q) = wq^2 + c_0$$

which implies that

$$C'(q) = 2wq.$$

The relationship between this production function and its associated cost function is illustrated in figure 5.28 for the special case of $c_0 = 0$.

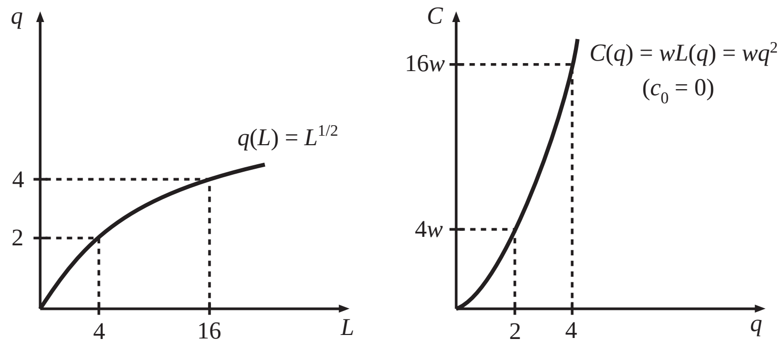


Figure 5.28 Production function and associated cost function for example 5.13 ■

Rule 11 Derivative of the Exponential Function

If $y = e^x$, *then* $dy/dx = e^x$.

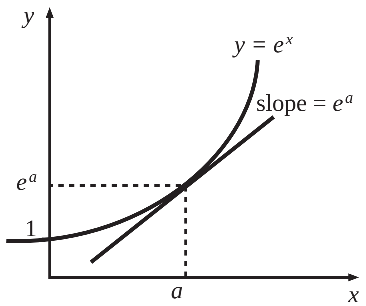


Figure 5.29 Function $y = e^x$

This rule states that the derivative of the function $f(x) = e^x$ at any point $x = a$ has the same value as the function itself, namely $f(a) = e^a$ and $f'(a) = e^a$. This is illustrated in figure 5.29.

The chain rule can also be applied to obtain the result that $f(x) = e^{g(x)}$ has derivative function $f'(x) = g'(x)e^{g(x)}$. To see how to get this result, let $u = g(x)$. Then $f(u) = e^u$, and it follows that $f'(x) = (dy/du)(du/dx) = e^u g'(x) = g'(x)e^{g(x)}$. For example, the derivative of $f(x) = e^{x^2}$ is $f'(x) = 2x(e^{x^2})$.

Example 5.14 **Exponential Growth in the Price of a Bottle of Wine**

Expert wine growers will argue that their best wines improve exponentially in the length of time they are stored and then so does the price. If we let g be the rate of growth in the price, t the number of years it is stored, and p_0 the price when the wine is sold immediately, then

$$p(t) = e^{gt} p_0$$

is the price as a function of t. Notice that

$$\frac{dp}{dt} = ge^{gt} p_0$$

is the rate at which price changes per period, and so the percentage growth rate in the price as $\Delta t \to 0$ is

$$\lim_{\Delta t \to 0} \frac{\Delta p/\Delta t}{p} \times 100 = \frac{dp/dt}{p} \times 100 = \frac{ge^{gt} p_0}{e^{gt} p_0} \times 100 = g \times 100$$

This illustrates that g is indeed the rate of growth in the price. ■

Rule 12 Derivative of the Logarithmic Function

If $y = \ln x$, then $dy/dx = 1/x$.

Since the function $y = \ln x$ is the inverse of the exponential function, this rule can be derived from rules 11 and 10 as follows: The statements $y = \ln x$ and $x = e^y$ are equivalent. From rule 11 we know that $dx/dy = e^y = x$, and so from the method of finding the derivative of the inverse function we know that

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{x}$$

That is,

$$\frac{d[\ln x]}{dx} = \frac{1}{x}$$

The chain rule can be applied to this rule as well to get the more general result that if $y = \ln[g(x)]$, then

$$\frac{dy}{dx} = g'(x)\frac{1}{g(x)}$$

Example 5.15 Find the derivative of the function

$$y = \ln[5x^2 - 3x]$$

Solution

The derivative is

$$\frac{dy}{dx} = (10x - 3)\frac{1}{[5x^2 - 3x]} = \frac{10x - 3}{5x^2 - 3x}$$ ∎

The result also extends easily to log functions with a base other than the natural number e. Since

$$\log_b x = \log_b e \ln x$$

it follows that

$$\frac{d[\log_b x]}{dx} = \log_b e\left(\frac{d[\ln x]}{dx}\right) = \log_b e\frac{1}{x}$$

As we illustrate now, this rule is very useful in expressing and finding elasticities.

Finding Elasticities

Consider a simple linear demand function $y = a - bp$, $b > 0$. The value a measures the quantity demanded at price $p = 0$, and $b = -dy/dp$ measures the amount by which demand falls as a result of a unit increase in price. One might be inclined to think that the slope, $-b$, is a good measure of the responsiveness of demand to price changes. However, the slope of the demand function is dependent on the units in which p and y are measured. Suppose, for example, that the demand function $y = 10 - 0.02p$ expresses demand for steel when y is measured in (metric) tonnes and p refers to the price in dollars per tonne. Thus ($p =$ \$100 per tonne, $y = 8$ tonnes) and ($p =$ \$200 per tonne, $y = 6$ tonnes) are two points on this demand function. If we measure y in kilograms (1,000 kg = 1 tonne) and p in dollars per kilogram, this same demand function becomes $y = 10{,}000 - 20{,}000p$. This can be checked by noting that the two points mentioned above are also on this demand function. That is, ($p =$ \$100 per tonne, $y = 8$ tonnes) = ($p =$ \$0.1 per kg, $y = 8{,}000$ kg) and ($p =$ \$200 per tonne, $y = 6$ tonnes) = ($p =$ \$0.2 per kg, $y = 6{,}000$ kg). The slopes of these two functions are very different, yet they explain the same demand behavior.

By defining a measure of responsiveness in terms of percentage changes in each of the variables, p and y, we escape the "units of measurement" problem above. First we develop the idea of **arc elasticity** which is the (average) elasticity of demand between two points on the demand function. Then we will indicate what is the more precise notion of elasticity, the so-called point elasticity of demand.

Letting (y_1, p_1) and (y_2, p_2) be two points on the demand function, the average percentage change in the price between two points is

$$\%\Delta p = \frac{p_2 - p_1}{(p_2 + p_1)/2} \times 100$$

and the average percentage change in the quantity is

$$\%\Delta y = \frac{y_2 - y_1}{(y_2 + y_1)/2} \times 100$$

Notice that these values are the same for the demand function above regardless of which equation we use to express it. That is,

$$\%\Delta p = \frac{200 - 100}{150} \times 100 = \frac{0.2 - 0.1}{0.15} \times 100 \doteq 66.7\%$$

and

$$\%\Delta y = \frac{6 - 8}{7} \times 100 = \frac{6000 - 8000}{7000} \times 100 \doteq -28.6\%$$

Using the standard notation that $\Delta y = y_2 - y_1$ and $\Delta p = p_2 - p_1$, we can define the **arc elasticity of demand** as

$$\left[-\frac{\%\Delta y}{\%\Delta p}\right] = -\frac{\dfrac{y_2 - y_1}{(y_2 + y_1)/2}}{\dfrac{p_2 - p_1}{(p_2 + p_1)/2}} = -\frac{\Delta y/(y_1 + y_2)}{\Delta p/(p_1 + p_2)}$$

There is, however, a difficulty or awkwardness with using this arc elasticity formula. Its value typically depends on the size of the price change (Δp) and the corresponding change in quantity demanded (Δy). There is no natural choice for the size of Δp. This issue is demonstrated in detail in the corresponding section of the Web page http://mitpress.mit.edu/math_econ3.

A method that avoids this difficulty is the **point elasticity of demand** formula. This formula simply uses the arc elasticity formula developed above but takes the limit as $\Delta p \to 0$ (and hence $\Delta y \to 0$ also). In doing so, one computes the elasticity at a given point, say (y_1, p_1), rather than between two points, (y_1, p_1) and (y_2, p_2). The point elasticity of demand formula is developed formally below.

$$\begin{aligned}\epsilon &= \lim_{\Delta p \to 0} (-) \frac{\Delta y/(y_1 + y_2)}{\Delta p/(p_1 + p_2)} \\ &= \lim_{\Delta p \to 0} (-) \frac{\Delta y}{\Delta p} \frac{p_1}{y_1} = (-) \frac{dy}{dp} \frac{p_1}{y_1}\end{aligned} \tag{5.6}$$

There are three things to note about this formula:

- As $\Delta p \to 0$, the two points (p_1, y_1) and (p_2, y_2) converge to a single point, hence the term *point* elasticity.
- If we are referring to the impact of the change in the (own) product's price on demand we generally refer simply to elasticity of demand. We could, of course, measure other elasticities such as the impact of a change in income on quantity demanded or the impact of a change in the price of some other good on quantity demanded. These other elasticities will be treated in chapter 11.
- The negative sign in formula (5.6) is used to convert the elasticity to a positive number in the standard case of a downward sloping demand function. Not all textbooks do this and it is only a matter of convenience that we do it here.

For a general linear demand function, $y = a - bp$, $b > 0$, we get

$$\epsilon = -\frac{dy}{dp}\frac{p}{y} = -(-b)\left(\frac{p}{y}\right) = b\left(\frac{p}{y}\right)$$

Thus the slope of a linear demand function is constant, but its elasticity is not. Another functional form commonly used to express the relationship between price and quantity demanded is the so-called constant elasticity demand function $y = \alpha p^{-\beta}$, $\alpha > 0$, $\beta > 0$. This demand function does not have a constant slope but rather has the same elasticity of demand at every point.

An alternative, easier method of computing the elasticity in this case is to first transform the demand function $y = \alpha p^{-\beta}$ using logarithms and then apply the rule for finding the derivative of the logarithmic function. Choice of base is irrelevant, and so we take the natural logarithm of both sides of $(y = \alpha p^{-\beta})$ and apply the rules for logarithms to get the following result:

$$\ln[y] = \ln[\alpha p^{-\beta}] = \ln[\alpha] - \beta \ln[p] \tag{5.7}$$

If we let $\hat{y}$ represent $\ln[y]$, $\hat{p}$ represent $\ln[p]$, and $\hat{\alpha}$ represent $\ln[\alpha]$, we can rewrite equation (5.7) in the simple linear form

$$\hat{y} = \hat{\alpha} - \beta \hat{p}$$

It follows that

$$\frac{d\hat{y}}{d\hat{p}} = -\beta \quad \text{or} \quad \frac{d\ln[y]}{d\ln[p]} = -\beta$$

which is the negative of the elasticity of demand. In fact, it can be shown that in general

$$-\frac{d\ln[y]}{d\ln[p]} = \epsilon = -\frac{dy}{dp}\frac{p}{y}$$

To see this, note that

$$\frac{d\ln[y]}{dy} = \frac{1}{y} \Rightarrow d\ln[y] = \frac{dy}{y}$$

and

$$\frac{d\ln[p]}{dp} = \frac{1}{p} \Rightarrow d\ln[p] = \frac{dp}{p}$$

By straightforward substitution we get

$$-\frac{d\ln[y]}{d\ln[p]} = -\frac{dy/y}{dp/p} = -\frac{dy}{dp}\frac{p}{y} = \epsilon \tag{5.8}$$

Therefore, if a demand relation is specified as a linear function in the logs of the variables y and p, rather than the variables y and p themselves, then it follows that the underlying demand function is presumed to have constant elasticity, and so the slope coefficient (on $\hat{p}$) is the elasticity of demand.

Example 5.16 Find the point elasticity of demand (with respect to own price) for the demand function $y = 50 - 2p$, at price $p = 5$. Over what range of prices is ϵ less than 1, and over what range of prices is it greater than 1?

Solution

We know from equation (5.6) that

$$\epsilon = -\frac{dy}{dp}\frac{p}{y}$$

Now $dy/dp = -2$, and so

$$\epsilon = \frac{2p}{y} = \frac{2p}{50 - 2p}$$

At $p = 5$ we have

$$\epsilon = \frac{2(5)}{50 - 2(5)} = \frac{10}{40} = \frac{1}{4}$$

For $\epsilon = 1$ we have

$$1 = \frac{2p}{50 - 2p}$$
$$2p = 50 - 2p$$
$$p = 12.5$$

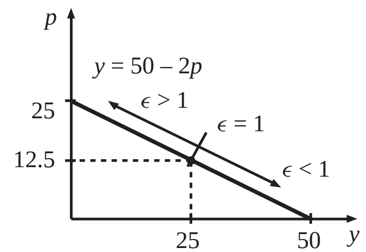


Figure 5.30 Elasticity changes along the demand function $y = 50 - 2p$

Thus $\epsilon = 1$ at price 12.5.

Similarly

$$\epsilon < 1 \quad \text{when} \quad \frac{2p}{50 - 2p} < 1 \qquad \text{(i.e., when } p < 12.5\text{)}$$

$$\epsilon > 1 \quad \text{when} \quad \frac{2p}{50 - 2p} > 1 \qquad \text{(i.e., when } p > 12.5\text{)}$$

These results are illustrated in figure 5.30. ■

Example 5.17 Find the point elasticity of demand ϵ (with respect to own price) for the demand function $y = 100p^{-2}$. Use both the direct approach, using equation (5.6), and the method of first taking logarithms, and then apply equation (5.8).

Solution

By equation (5.6),

$$\epsilon = -\frac{dy}{dp}\frac{p}{y}$$

where, in this case, $dy/dp = -200p^{-3}$, and so

$$\epsilon = -(-200p^{-3})\frac{p}{y} = \frac{200p^{-2}}{100p^{-2}} = 2$$

Alternatively, by first taking logs, we have

$$\ln y = \ln(100p^{-2}) = \ln 100 - 2\ln p$$

Noting that $\epsilon = -d\ln y/d\ln p$ gives

$$\epsilon = -\frac{d\ln y}{d\ln p} = 2$$

■

We can also define the (price) elasticity of supply as the ratio of the percentage change in quantity supplied divided by the percentage change in price. For example, given the linear supply function $y = -5+3p$, we get the elasticity of supply to be

$$\gamma = \frac{dy/y}{dp/p} = \frac{dy}{dp}\frac{p}{y} = 3\frac{p}{y}$$

In fact the concept of elasticity is very general. Given any variable x that affects some other variable z, according to the function $z = f(x)$, we can define the elasticity of variable x on z to be

$$\nu = \frac{dz/z}{dx/x} = \frac{dz}{dx}\frac{x}{z}$$

Rule 13 L'Hôpital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

This rule, which is more an application of differentiation than a rule for finding derivatives, is useful for finding the limiting value of the ratio of functions at a point $(x = a)$ where that value is undefined, such as $0/0$ or ∞/∞. For example, if $f(x) = x^2 - 1$ and $g(x) = x - 1$, then the ratio $f(x)/g(x)$ when evaluated at the point $x = 1$ is not defined (i.e., it is $0/0$). However, the ratio $f'(x)/g'(x)$ is defined at the point $x = 1$ and is easily computed as $f'(x)/g'(x) = 2x/1 = 2x = 2(1) = 2$.

Note that in the case above one could also determine this ratio by factoring, since $(x^2 - 1)/(x - 1) = [(x - 1)(x + 1)]/(x - 1) = x + 1$, which has value 2 when $x = 1$. However, it is often much easier to apply L'Hôpital's rule.

EXERCISES

1. Find the slope of each of the following production functions, $y = f(L)$. Graph the functions and their derivative functions. Give the economic significance of the sign of the slope of the derivative functions (i.e., whether the derivative is increasing or decreasing in L).

(a) $y = 10L$

(b) $y = 8L^{1/3}$

(c) $y = 3L^4$

2. Find the slope of each of the following production functions, $y = f(L)$. Graph the functions and their derivative functions. Give the economic significance of the sign of the slope of the derivative functions (i.e., whether the derivative is increasing or decreasing in L).

(a) $y = aL, \quad a > 0$

(b) $y = 10L^{2/3}$

(c) $y = 12L^2 - L^3$

3. Suppose that two firms, A and B, behave as competitive firms in deciding how much output to supply to the market. Firm A's cost function is $C^A = 10q + 2q^2$, $q \geq 0$, and firm B's cost function is $C^B = 15q + q^2$, $q \geq 0$.

(a) Find the supply functions, defined on $q \geq 0$, for each firm and draw them on the same graph. At which points of the domains are these functions differentiable?

(b) Find the total supply function for the two firms and graph it. Is this function differentiable? Discuss.

4. For the total cost function

$$TC(y) = 3y^2 + 7y + 24, \qquad y > 0$$

show that (and illustrate on a graph):

(a) MC is less than AC where AC is falling.

(b) MC = AC at the point where the AC curve is horizontal.

(c) MC exceeds AC where AC is rising.

5. A firm uses one input, L, to generate output, q, according to the production function $q = 16L^2$. The input price is w and fixed costs are $c_0 > 0$. Show that dq/dL is rising while dC/dq is falling. How does your result relate to the inverse function rule for differentiation?

6. Suppose that a monopolist faces inverse demand function $p = a - bq$. Find its marginal revenue function. Plot both the demand function and the marginal revenue function on a single graph.

7. A bakery advertises its bagels by noting either the price per dozen (i.e., 12 bagels—not a "baker's dozen" of 13) or per bagel and doesn't offer any quantity discounts. Thus, for example, if the price is \$4.80 per dozen (i.e., for 12 bagels), then it is \$0.40 per bagel. Since these prices are the same, the baker is not surprised to find that demand is the same no matter how she decides to quote the price. By using the per dozen price, the baker finds the demand function to be $y = 100 - 2p$, where y is the number of dozens of bagels sold per day.

(a) Find the (own) price elasticity of demand for bagels.

(b) Find the demand function for bagels, $\hat{y} = a - b\hat{p}$, where $\hat{y}$ is the number of bagels sold per day and $\hat{p}$ is the price per bagel for this example (i.e., conversion of units). Find the (own) price elasticity of demand for bagels using this demand function and show that the answer is the same as for part (a).

8. A bakery advertises its bagels by noting either the price per dozen or per bagel and doesn't offer any quantity discounts. Thus, for example, if the price is \$4.80 per dozen (i.e., for 12 bagels), then it is \$0.40 per bagel. Since these prices are the same, the baker is not surprised to find that demand is the same no matter how she decides to quote the price. By using the per dozen price, the baker finds the demand function to be $y = 100/p^2$, where y is the number of dozens of bagels sold per day.

(a) Find the (own) price elasticity of demand for bagels by first taking logs and using the result that $\epsilon = -d\ln[y]/d\ln[p]$.

(b) Rewrite the demand function for bagels in terms of the variables $\hat{y}$ and $\hat{p}$ where $\hat{y}$ is the number of bagels sold per day and $\hat{p}$ is the price per bagel. Find the (own) price elasticity of demand for bagels using this demand function and show that the answer is the same as for part (a).

5.5 Higher Order Derivatives: Concavity and Convexity of a Function

Since the derivative of a function is also a function, we can write $dy/dx = f'(x)$ and find its derivative, $d[f'(x)]/dx$. We call f' the **first derivative function** of the function f and the derivative of the first derivative,

$$\frac{d(dy/dx)}{dx} \quad \text{or} \quad \frac{d[f'(x)]}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} \quad \text{or} \quad f''(x)$$

the **second derivative function.** Since the second derivative is also a function, we can also find its derivative. We call $d[f''(x)]/dx$ the **third derivative function** and write it as f''' or $f^{(3)}(x)$ to indicate that this function is found by three successive operations of differentiation, starting with the function f. Of course, this process may continue indefinitely and so we use the general notation $f^{(n)}$ to indicate the nth derivative of the function f.

Example 5.18 Find the first four derivatives of the function $f(x) = x^4$.

Solution

Taking successive derivatives gives

$$f'(x) = 4x^3, \quad f''(x) = 12x^2, \quad f'''(x) = 24x, \quad \text{and} \quad f^{(4)}(x) = 24$$

Upon taking the derivative of the fourth derivative, we get the fifth derivative to be $f^{(5)}(x) = 0$. Every further order derivative for this example is also the constant function zero (i.e., $f^{(n)}(x) = 0$ for $n \geq 5$). ■

If the first two derivatives of a function exist, we say the function is *twice differentiable*. In economics we obtain many useful results by concentrating on the first and second derivatives of a function. In particular, certain results often depend on whether the second derivative is positive or negative. Thus we will discuss at some length below what it means for the second derivative of a function to be negative or positive. This leads us to a simple method of determining whether a function is *convex* or *concave* (see chapter 2).

Consider the function $f(x) = x^2$ for domain $x > 0$. On $x > 0$ this function is upward sloping and, from its graph (figure 5.31), we can see that its slope increases as x increases. This means that the first derivative function is increasing in x, and so the derivative of this, the function's second derivative, must be positive valued for all $x > 0$. Upon finding the derivatives, we get $f'(x) = 2x$ and $f''(x) = 2$. Thus the second derivative is indeed positive for any value of x.

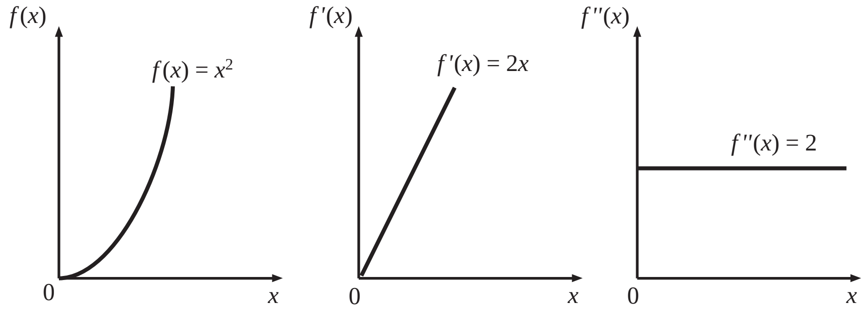


Figure 5.31 Function $f(x) = x^2$, $x \geq 0$ and its first two derivatives

Now, consider the graph of this same function defined on $x < 0$ (figure 5.32). On this set of values the function is negatively sloped $f'(x) = 2x < 0$ on $x < 0$. The greater the value of x, the less steep is the curve. Thus, as x increases, the slope falls in absolute value which means that since the slope is negative, the value of the slope is actually increasing in x. Thus the second derivative is positive.

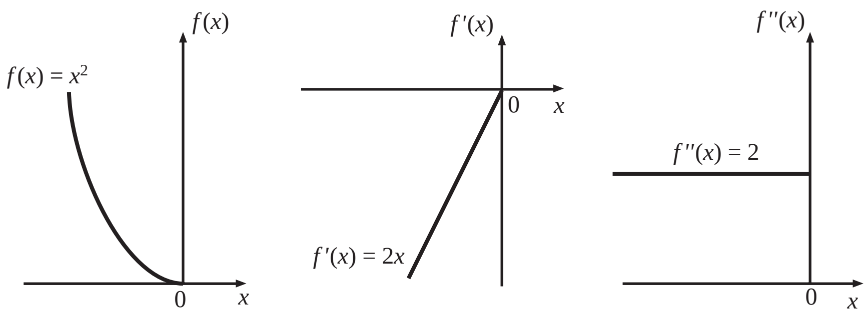


Figure 5.32 Function $f(x) = x^2$, $x \leq 0$ and its first two derivatives

Defining this function on the domain $\mathbb{R}$ (and putting the graphs over $x \leq 0$ and $x \geq 0$ together) we see that the second derivative is positive throughout (see figure 5.33). A function with this shape, as determined by the second derivative being positive, is convex.

Definition 5.7 A twice differentiable function $f(x)$ is **convex** if, at all points on its domain, $f''(x) \geq 0$.

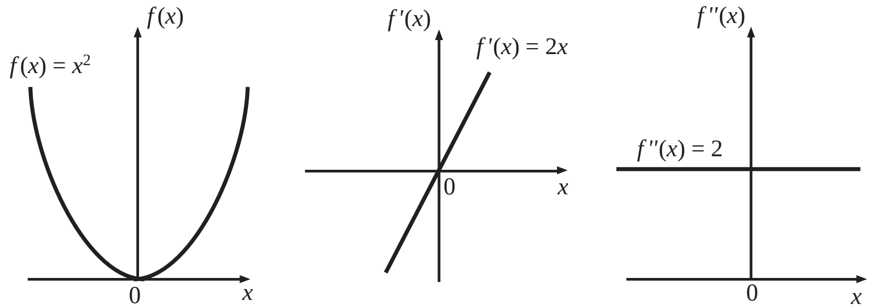


Figure 5.33 Function $f(x) = x^2$, $x \in \mathbb{R}$, and its first two derivatives

A linear function is convex according to definition 5.7. In many instances, however, we will want to consider linear functions separately from functions with positive or nonzero second derivatives. Thus we often use the concept of strict convexity to exclude linear functions. This is done by replacing the weak inequality ($\geq$) in definition 5.7 with the strict inequality ($>$).

Definition 5.8 A twice differentiable function $f(x)$ is **strictly convex** if $f''(x) > 0$ except possibly at a single point.

Notice that the function $f(x) = x^4$ has the second derivative $f''(x) = 12x^2$ which is positive for all x except $x = 0$ where the second derivative becomes zero. This function is, however, strictly convex, and hence the qualification in definition 5.8 regarding the requirement $f''(x) > 0$ except possibly at one point.

The set of diagrams in figure 5.34 illustrates the shape of strictly convex functions for the cases where the function is monotonic increasing, monotonic

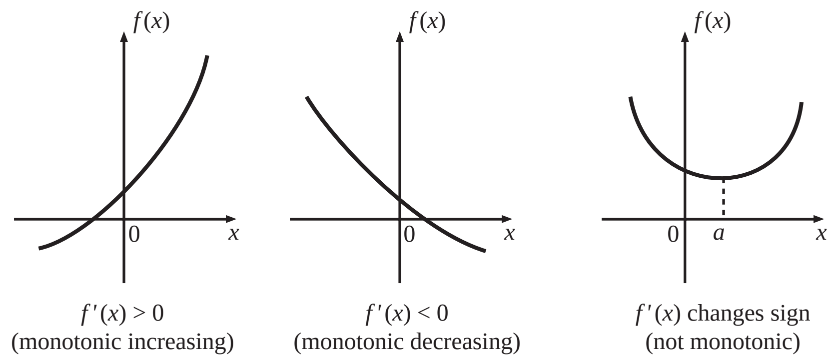


Figure 5.34 Possible shapes of strictly convex functions

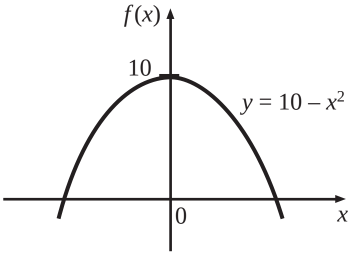


Figure 5.35 A strictly concave function: $y = 10 - x^2$

decreasing, or not monotonic. If we now consider the function $f(x) = 10 - x^2$, we find that its first derivative is $f'(x) = -2x$ and its second derivative is $f''(x) = -2$ (see figure 5.35). Thus this function is increasing in x for $x < 0$ and is decreasing in x for $x > 0$. The slope, however, is falling for all values of x. This means that when $f'(x) > 0$, it is becoming less steep while when $f'(x) < 0$, the function is becoming more steep in absolute value but the slope is becoming *more negative*.

Since the function $y = 10-x^2$ has a negative second derivative, it exemplifies properties which are the opposite of those of a convex function. It is an example of a concave function.

Definition 5.9 A twice differentiable function $f(x)$ is **concave** if $f''(x) \leq 0$ on all points of its domain.

As for the case of convex functions, a linear function also satisfies the definition of concavity ($f''(x) = 0$ for all x). To exclude it, we have the following definition for strictly concave functions.

Definition 5.10 A twice differentiable function $f(x)$ is **strictly concave** if $f''(x) < 0$ on all points of its domain except possibly at a single point.

Alternatively, since multiplying through by -1 reverses an inequality, we could say that $f(x)$ is (strictly) concave if $-f(x)$ is (strictly) convex.

A function whose second derivative is sometimes positive and sometimes negative is neither convex nor concave *everywhere*. However, we can sometimes find intervals over which the function is one or the other. This information is useful in determining the shape of the graph of a function, as is illustrated in the following example.

Example 5.19 Use the sign of the second derivative to help in graphing the function

$$f(x) = -\left(\frac{1}{3}\right)x^3 + 3x^2 - 5x + 10 \quad \text{on } x \geq 0$$

Solution

The first two derivatives of this function are

$$f'(x) = -x^2 + 6x - 5$$

and

$$f''(x) = -2x + 6$$

Thus, since $f''(x) > 0$ for $x < 3$ and $f''(x) < 0$ for $x > 3$, the function is convex on the interval $[0, 3]$ and concave on the interval $[3, +\infty)$. This helps us to draw the function. The following table gives the values of the function as well as the values for the first and second derivatives at the points $x = 0, 1, 2, \ldots, 8$. Notice that at any values of x where $f'(x) = 0$, the function is neither rising nor falling through that point, and hence is *flat* at that point.

x	$f(x)$	$f'(x)$	$f''(x)$
0	10.00	−5.00	6.00
1	7.67	0.00	4.00
2	9.33	3.00	2.00
3	13.00	4.00	0.00
4	16.67	3.00	−2.00
5	18.33	0.00	−4.00
6	16.00	−5.00	−6.00
7	7.67	−12.00	−8.00
8	−8.67	−21.00	−10.00

Upon factoring the first derivative, we get

$$f'(x) = -(x - 5)(x - 1)$$

and so it is easy to derive algebraically that the first derivative is zero at the points $x = 1$ and $x = 5$. Also it is easy to see that the function has a positive second derivative, and so is *convex-shaped* for $x < 3$ and has a negative second derivative, and so is *concave-shaped* for $x > 3$. This information allows us to draw the function in figure 5.36. ■

Concavity and Convexity of Production Functions

Consider the case of a short-run production function. Let x represent a single (variable) input which is used to produce output y according to the production function $y = f(x)$. All other inputs are assumed to be fixed in the short run. The first derivative of the production function is the marginal product of x, $\text{MP}(x) = f'(x)$. If $f'(x)$ is decreasing in x, this means that the marginal product is falling as more x is used, which is the law of the diminishing marginal product of a variable input.

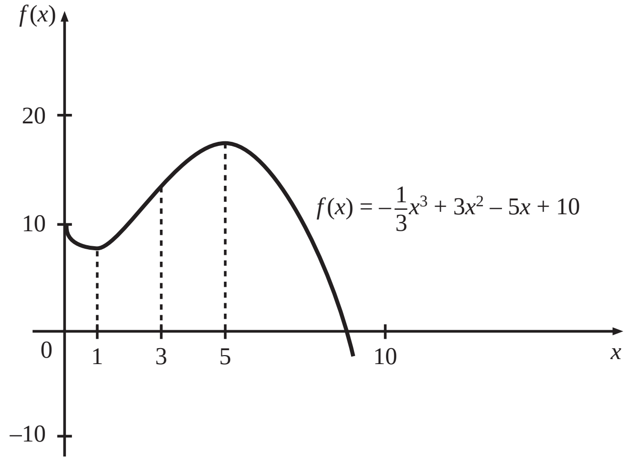


Figure 5.36 Function $f(x) = -\frac{1}{3}x^3 + 3x^2 - 5x + 10$ (example 5.19)

If $f'(x)$ is falling as x is increasing, then the second derivative is negative:

$$\frac{d[f'(x)]}{dx} = f''(x) < 0$$

means the production function is strictly concave (definition 5.10). For example, the second derivative of the function $y = x^{1/2}$, $x \geq 0$ is

$$f''(x) = -\frac{1}{4}x^{-3/2}$$

which is negative (for $x > 0$). Thus this is an example of a concave production function.

If a production function $y = f(x)$ has a positive second derivative (everywhere), $f''(x) > 0$, then the marginal product of labor, $\text{MP}(x) = f'(x)$, is increasing in x. Thus a strictly convex production function exhibits the property that the marginal product of the variable input is increasing. An example is $y = x^2$. From an economic perspective, this property seems implausible. If all inputs but one, say labor, are available in fixed amounts, it seems unlikely that adding more labor will continually lead to greater increments in output, as sooner or later the productivity of additional amounts of labor will fall because of inadequate levels of the fixed inputs. Thus, when constructing an economic model involving short-run production, it is common to assume that the production function is concave (i.e., $f''(x) < 0$).

It is plausible, however, that for certain production processes increasing the level of the variable input may lead to increasingly large increments in output at “low to moderate" levels of production. For example, suppose that there are only a few workers available to operate the machines in a large factory. Given how much each worker would have to run from machine to machine, these workers may not be very productive and adding an extra worker may also not lead to a very large increase in output. Consequently the marginal product may increase more rapidly only when there is a sufficient number of workers to operate the various machines effectively. One would, however, expect the extra output generated by employing additional workers to fall eventually as more labor is added. Thus it seems reasonable to expect that for some production processes the marginal product of labor may rise initially and then fall.

Example 5.20 Show that the production function

$$f(x) = -\left(\frac{2}{3}\right)x^3 + 10x^2 + 5x$$

has both a concave and a convex section. Draw its graph.

Solution

Since the second derivative of this function is

$$f''(x) = -4x + 20$$

it follows that the function is convex (marginal productivity of x increasing) on the interval $[0, 5)$ and concave (marginal productivity of x decreasing) on the interval $(5, \infty)$. That is,

$$f''(x) = -4x + 20 > 0 \Rightarrow x < 5 \qquad \text{(convex)}$$

$$f''(x) = -4x + 20 < 0 \Rightarrow x > 5 \qquad \text{(concave)}$$

The graph of $f(x)$ is provided in figure 5.37. ■

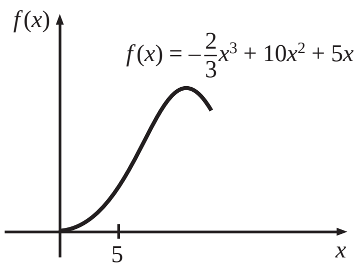


Figure 5.37 Function $f(x) = -(2/3)x^3 + 10x^2 + 5x$ (example 5.20)

Concavity and Convexity of Cost Functions

Recall that there is an inverse relationship between marginal productivity of an input and the short-run marginal cost function. The reason is that if the marginal productivity of an input is decreasing as more input is used, then to produce an extra unit of output requires a larger increment of the input the higher the production level is to begin with. Therefore, if the marginal product of x (the input) is decreasing

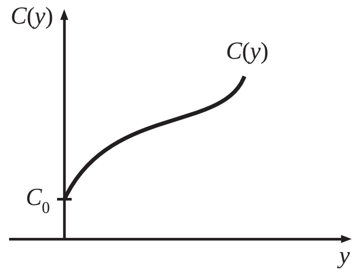


Figure 5.38 Shape of the cost function associated with the production function in figure 5.37

in x, then the marginal cost of production is increasing in y (the output). We can see this point by using $y = x^{1/2}$ as the production function. The inverse function is $x = y^2$. If we assume that the cost of fixed inputs is c_0, and we let r represent the unit cost of the variable input (x), then we get the total cost function to be $C(x) = c_0 + rx$ in terms of the input and $C(y) = c_0 + ry^2$ in terms of the output. The marginal cost function is $C'(y) = 2ry$, which is increasing in y. An equivalent way to see this is to note that the second derivative is $C''(y) = 2r$, which is positive. Thus for this example the production function is concave while the cost function is convex. By starting with the convex production function $y = x^2$, we can follow the same steps to get $x = y^{1/2}$, the inverse of the production function, and $C(y) = c_0 + ry^{1/2}$, which is a concave function. This relationship also applies to ranges of the production function over which it switches from being concave to convex. The shape of the cost function in figure 5.38 corresponds to the shape of the production function in figure 5.37.

Example 5.21 A single input, x, is used to produce output y. Show that if the production function is $y = x^{1/3}$, $x > 0$, then the cost function, $C(y)$ is convex while the production function is concave.

Solution

$$\text{Production function:} \quad f(x) = x^{1/3}$$

$$\text{First derivative:} \quad f'(x) = \frac{1}{3}x^{-2/3}$$

$$\text{Second derivative:} \quad f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}} < 0 \qquad \text{if } x > 0$$

and so the production function is concave. The cost of production is $C(x) = c_0 + rx$ in terms of the input where c_0 is fixed cost and r is the cost per unit of the input. Since $y = x^{1/3}$, we have $x = y^3$ and substituting into $C(x)$ gives the cost function, in terms of output y, to be $C(y) = c_0 + ry^3$ with

$$C'(y) = 3ry^2$$

$$C''(y) = 6ry > 0 \qquad \text{on } y > 0$$

and so the cost function is convex. ■

We complete this section with a couple of applications of derivatives from macroeconomics.

The "Keynesian" Consumption Function

Consider the general relationship between aggregate national income and aggregate consumption given by

$$C = C(Y) \qquad \text{with } Y \geq 0,\ 0 < C'(Y) < 1$$

The derivative function C' is usually called the *marginal propensity to consume*, and the *average propensity to consume* is defined as C/Y.

In the simplest textbook specification, we have the linear form for $C(Y)$, say

$$C(Y) = a + bY, \qquad a > 0,\ 0 < b < 1$$

where a and b are constants. The marginal propensity to consume is, in this case,

$$\text{MPC} \equiv \frac{dC}{dY} = C'(Y) = b \qquad \text{a constant fraction}$$

So, in the aggregate, consumers in this economy consume a constant *additional* amount out of *additional* income, regardless of the level of income. However, consumers do *not* consume a constant fraction of total income. The fraction of income consumed, or the average propensity to consume is, in this case,

$$\text{APC}(Y) \equiv \frac{C(Y)}{Y} = \frac{a}{Y} + b$$

with

$$\frac{d\text{APC}(Y)}{dY} = -\frac{a}{Y^2} < 0$$

so the consumption ratio decreases as income increases.

Example 5.22 **The Proportional Consumption Function**

Derive the functional form of the proportional consumption function, for which the average propensity to consume is constant.

Solution

For the average propensity to consume to be constant, we require that

$$\text{APC}(Y) = k \qquad \text{for some constant fraction } k$$

Now substitute the definition for APC(Y):

$$\frac{C(Y)}{Y} = k$$

or

$$C(Y) = kY$$

Thus the consumption function with a constant APC has a zero intercept, and in this case the average propensity to consume and marginal propensity to consume are the same. ■

Naturally, when considering actual aggregate consumption functions, we need not restrict attention to linear forms. This is particularly true when comparing consumption patterns between countries at different stages of development or those in one country over a long period of income growth. A candidate functional form in these cases is one which allows both the APC and the MPC to fall with increases in income. That is, a concave function

$$C = C(Y), \quad 0 < C'(Y) < 1, \quad \text{and} \quad C''(Y) < 0 \quad \forall\, Y$$

Note that the concavity assumption implies that for the lowest income level, the marginal propensity to consume is closest to one, while for the highest income level it is closest to zero. (How "close" depends on the parameters and the precise functional form of $C(Y)$.)

EXERCISES

1. Show that the function $f(x) = x^4$ satisfies the definition of a strictly convex function (definition 5.8).

2. Show that the function $f(x) = x^{1/4}$, $x > 0$ satisfies the definition of a strictly concave function (definition 5.10).

3. Let $y = x^{1/3}$, $x > 0$, be a production function, where y is output and x is a single input. Derive the cost function, $C(y) = c_0 + rg(y)$, where $x = g(y)$ is the inverse of the production function, c_0 is the fixed cost, and r is the unit cost of the input. Show that the production function is strictly concave while the cost function is strictly convex. Illustrate with a graph and discuss the economic intuition underlying the result.

4. Let $C(y) = y^3 - 9y^2 + 60y + 10$, $y \geq 0$ be a firm's cost function. Find the interval over which this function is concave and the interval over which it is convex. Use this information and a table such as that of example 5.20 to draw this function.

5. Suppose that a firm in a competitive market sells a product at price $\bar{p} = 45$ and has the same cost function as in question 4 above, namely $C(y) = y^3 - 9y^2 + 60y + 10$, $y \geq 0$. For this firm's profit function find the interval over which the function is concave and the interval over which it is convex. Use this information and a table such as that of example 5.20 to draw the function.

5.6 Taylor Series Formula and the Mean-Value Theorem

Recall that the differential, $dy = f'(x)\,dx$, can be used to provide an approximation to the change in the y variable, $dy \doteq \Delta y$, for a given change in the x variable, $dx \equiv \Delta x$ (see figure 5.39). As we saw in section 5.2, the percentage error from using dy as an approximation to the actual change in y, Δy, can be made arbitrarily small if we are willing to consider changes in x that are made arbitrarily small. However, we are not always satisfied with the restriction that Δx be *small* and for noninfinitesimal changes in the x variable this approximation may not be very accurate. The Taylor series expansion formula allows us to investigate this issue more fully.

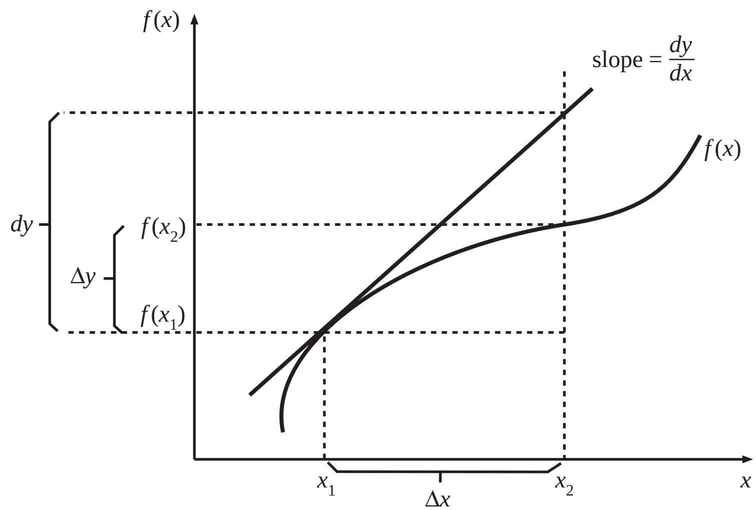


Figure 5.39 The use of the total differential to approximate a change in the value of a function

The idea of the Taylor series formula is to use information about the value of a function, $y = f(x)$, at a specific point, $x = x_0$, in conjunction with information about the value of the derivative functions of $f(x)$ at the point $x = x_0$, in order to obtain the value of the function at a different value of x, $x = x_1$, within some neighborhood of the point $x = x_0$. The following is sometimes called the *remainder* form of the Taylor series expansion formula:

$$
\begin{aligned}
f(x_1) = f(x_0) &+ \frac{f'(x_0)(x_1 - x_0)}{1!} \\
&+ \frac{f''(x_0)(x_1 - x_0)^2}{2!} \\
&+ \frac{f^{(3)}(x_0)(x_1 - x_0)^3}{3!} \\
&+ \frac{f^{(4)}(x_0)(x_1 - x_0)^4}{4!} + \cdots \\
&+ \frac{f^{(n-1)}(x_0)(x_1 - x_0)^{n-1}}{(n-1)!} + R_n
\end{aligned}
$$

where $R_n = f^{(n)}(\xi)(x_1 - x_0)^n/n!$ with ξ lying between x_0 and x_1.

The last term, R_n, is called the *remainder term* and is computed in the same manner as the previous terms except the (nth order) derivative is to be evaluated at some (unknown) number between x_0 and x_1. The same expression using summation notation is given below.

Definition 5.11

The **Taylor series expansion** of the function $f(x)$ in a neighborhood of the value $x = x_0$ in the remainder formula is

$$
f(x_1) = f(x_0) + \sum_{k=1}^{n-1} \left[\frac{f^k(x_0)(x_1 - x_0)^k}{k!} \right] + R_n
$$

where $R_n = f^{(n)}(\xi)(x_1 - x_0)^n/n!$ and ξ lies between x_0 and x_1. The function is assumed to possess derivatives to the nth order.

Now, knowing the value of a function at some point, $x = x_0$, and then using this formula to find the value of the function at some other point, $x = x_1$, is the same exercise as finding how the function f changes as a result of changing x by amount $\Delta x = x_1 - x_0$. This is formally seen by simply moving the term $f(x_0)$ of the expression above to the left side to get $f(x_1) - f(x_0) \equiv \Delta y$.

At first glance the formula may appear complicated to use. Moreover the value $x = \xi$ at which the term $f^{(n)}(\xi)$ is to be evaluated is not known. However, under certain conditions we can show that $R_n \to 0$ as $n \to \infty$, and so for specific functions we can ignore the remainder term and simply add up the series as $n \to \infty$, provided that this limit exists.

Example 5.23 Find the Taylor series expansion for $f(x) = e^x$ around the point $x_0 = 0$.

Solution

First, we have $f(x_0) = e^0 = 1$. Since the derivative of e^x is just e^x itself, each successively higher order derivative is simply e^x; that is, $f^{(k)}(x) = e^x$, and so $f^{(k)}(x_0) = e^0 = 1$. Thus the Taylor series expansion for e^x evaluated about the point $x = 0$ gives the result

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n$$

where $R_n = \xi^n/n!$, and ξ is between 0 and x. ■

The term R_n can be made as small as one wishes by taking n large, and so this formula can be used to approximate e^x to any desired level of accuracy. For example, suppose that one wants to use this formula to estimate e^x for values of x between 0 and 1. The remainder term, $R_n = \xi^n/n!, \xi \in [0, 1]$, will obtain its highest possible value at $\xi = 1$, and so we know that by ignoring it we will always be within $1/n!$ of the true result. Thus, to get an estimate that is guaranteed to be correct to 6 decimal points, we can choose n large enough so that $1/n! < 0.000001$; that is, we want $n! >$1,000,000. It turns out that $n = 10$ will do.

Many of the important uses of the Taylor series formula in economics can be illustrated by dealing with only two terms ($n = 2$). In this case we get

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(\xi)(x_1 - x_0)^2}{2} \tag{5.9}$$

for ξ between x_0 and x_1.

By taking $f(x_0)$ to the left side of this equation and using the notation $dx = (x_1 - x_0), dy = f'(x)\,dx$ and $\Delta y = f(x_1) - f(x_0)$, we get the result in equation (5.10):

$$\Delta y = dy + \frac{f''(\xi)(x_1 - x_0)^2}{2} \tag{5.10}$$

for ξ between x_0 and x_1. This explains more fully how the differential *dy* is an estimate of the actual change in y, Δy, (see figure 5.40). The error is in fact the remainder term of the Taylor series formula (i.e., $\epsilon = \Delta y - dy = f''(\xi)(x_1 - x_0)^2/2$).

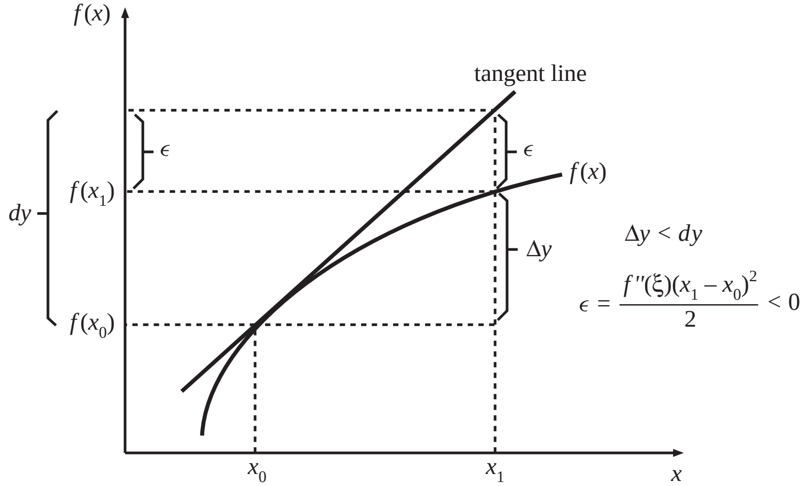


Figure 5.40 Example where the total differential overestimates the change in the function's value

Suppose that $f(x)$ is a strictly concave function (everywhere) so that $f''(x) < 0$. Since $(x_1 - x_0)^2$ is positive for any value $x_1 \neq x_0$, it turns out that using

$$dy = f'(x_0)(x_1 - x_0) = f'(x_0)\,dx$$

always provides an overestimate of Δy, the actual value of the change in y. This is seen to be the case for the function in figure 5.40. If the function $f(x)$ is strictly convex ($f''(x) > 0$), then the opposite holds, as illustrated in figure 5.41. Using equation (5.10) also turns out to be very useful in the understanding of optimization, as will be seen in the next chapter.

The mean value theorem for the derivative can be illustrated by taking only one term in the Taylor series formula:

$$f(x_1) = f(x_0) + f'(\xi)(x_1 - x_0) \tag{5.11}$$

for ξ between x_0 and x_1.

By rearranging terms we can illustrate the next theorem.

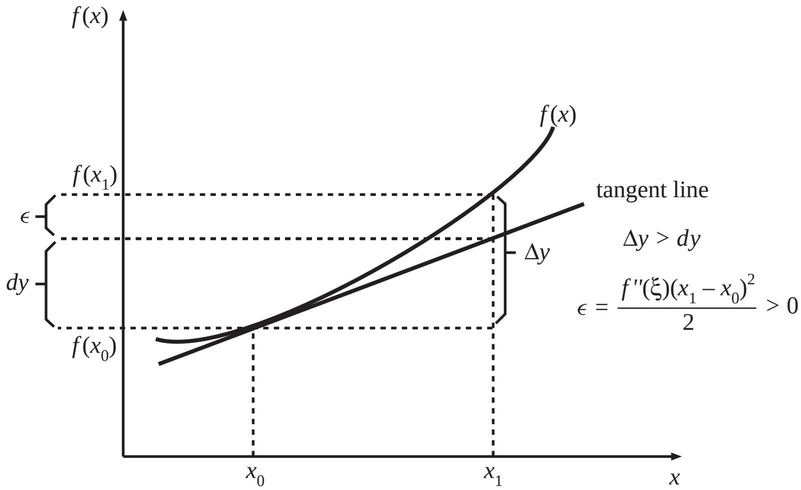


Figure 5.41 Example where the total differential underestimates the change in the function's value

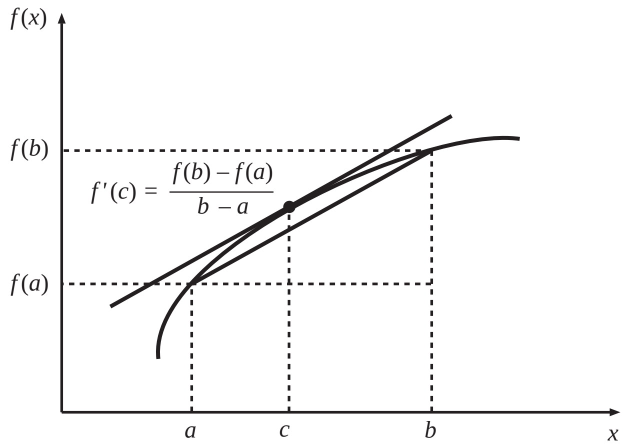


Figure 5.42 Illustration of the mean-value theorem

Theorem 5.2 **(Mean-value theorem)** If the function $f(x)$ is continuous and differentiable on some closed interval $[a, b]$, then there must be a number $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which is the slope of the secant line joining a and b.

This theorem is illustrated in figure 5.42.

EXERCISES

1. Use the Taylor series expansion formula to find an estimate for the function $f(x) = e^{-x}$ for any value x belonging to the interval [0, 1]. Choose $x_0 = 0$ and ensure that your computation is correct to within 0.001 (see example 5.24).

2. Use the Taylor series expansion formula to find an estimate for the function $f(x) = \ln(1 + x)$, $x > -1$, for any value x belonging to the interval [0, 1]. Choose $x_0 = 0$, and ensure that your computation is correct to within 0.001 (see example 5.24).

3. For the function $f(x) = x^{1/2}$, $x \geq 0$, find the Taylor series formula for $n = 2$ (i.e., the remainder term involves the second derivative as in equation 5.9). Show that using the differential $dy = f'(x)\,dx$ to estimate the impact on y of a change in x of amount dx leads to an overestimate of the actual change in y.

4. For the function $f(x) = x^2$, find the Taylor series formula for $n = 2$ (i.e., the remainder term involves the second derivative as in equation 5.9). Show that using the differential $dy = f'(x)\,dx$ to estimate the impact on y of a change in x of amount dx leads to an underestimate of the actual change in y.

CHAPTER REVIEW

Key Concepts

chain rule
concave function
convex function
derivative
differentiable function
first derivative function
instantaneous rate of change
left-hand derivative

marginal cost of production
mean-value theorem
product rule
quotient rule
right-hand derivative
rules of differentiation
secant line
strictly concave function
strictly convex function
tangent line
Taylor series expansion
total differential

Review Questions

1. What is the difference between a secant line and a tangent line?
2. How can a tangent line be defined in terms of a sequence of secant lines?
3. What is the relationship between a tangent line, the derivative of a function, and total differential?
4. Define and explain left-hand and right-hand derivatives of a function $f(x)$ at a point $x = x_0$.
5. Use the concepts of left-hand and right-hand derivatives to indicate when a function $f(x)$ is differentiable at a point $x = x_0$.
6. Why is it the case that if a function $f(x)$ is differentiable at a point $x = a$, then it must also be continuous at that point?
7. Write out the 13 rules of differentiation given in this chapter.
8. Explain with the use of a graph why the second derivative of a differentiable convex function is greater than or equal to zero.
9. Explain with the use of a graph why the second derivative of a differentiable concave function is less than or equal to zero.
10. Use the Taylor series expansion to show that using the tangent line (or differential) at a point $x = x_0$ to estimate the value of the function at some other point $x \neq x_0$ leads to an overestimate if the function is strictly concave and an underestimate if the function is strictly convex. Illustrate your answer with graphs.

Review Exercises

1. From the definition of the derivative (definition 5.3), find the derivative of the function

$$f(x) = x^2 + 3x - 4$$

2. Suppose that a salesperson has the following contract relating monthly sales, S, to her monthly pay, P. She is given a basic monthly amount of \$500, regardless of her sales level. On the first \$10,000 of monthly sales she earns a

10% commission. On the next \$10,000 of monthly sales she earns a commission of 20% while on any additional sales she earns a commission of 25%.

(a) Find and graph the function relating her pay to sales, $P(S)$, $S \geq 0$.

(b) Determine the points of nondifferentiability of $P(S)$ and indicate according to definition 5.5 why this is so.

3. Find the slope of each of the following production functions, $y = f(L)$. Graph the functions and their derivative functions. Give the economic significance of the slope of the derivative functions (i.e. whether the derivative is increasing or decreasing in L). In each case $L > 0$.

(a) $y = 64L^{1/4}$

(b) $y = 10L + 2L^{1/2}$

(c) $y = 5L^3$

(d) $y = -L^3 + 12L^2 + 3L$

4. Suppose that two firms, A and B, behave as competitive firms in deciding how much output to supply to the market. Firm A's cost function is $C^A = \alpha q + \beta q^2$ and firm B's cost function is $C^B = \gamma q + \omega q^2$. Assume $\alpha, \beta, \gamma, \omega > 0$ and $\gamma \geq \alpha$.

(a) Find the supply functions, defined on $q \geq 0$, for each firm and draw them on the same graph. At which points of their domains are these functions differentiable?

(b) Find the total supply function for the two firms and graph it. Under what restrictions on the parameters α, β, γ and ω is this function differentiable? Discuss.

5. Find the expression for the point elasticity of demand ϵ (with respect to own price) for the demand function $y = 200 - 5p$. Determine the ranges of prices for which ϵ is less than 1 and greater than 1. Illustrate on a graph of this demand function.

6. Suppose that a firm's total product function is $y = 40L^2 - L^3$. Show that the average product of labor, AP(L), rises when marginal product of labor, MP(L), exceeds AP(L), falls when MP(L) is less than AP(L), and is horizontal at the point where MP(L) = AP(L).

7. A firm uses one input (L) to generate output (q) according to the production function $q = aL^b$, $a > 0$, and $b > 0$ (also $L \geq 0$). The input price is w and fixed costs are c_0. Show that dq/dL is rising if dC/dq is falling, dq/dL is falling if dC/dq is rising, and dq/dL neither rises nor falls if dC/dq neither rises nor falls. How does your answer relate to the value of b? How does your result relate to the inverse function rule for differentiation?

8. For the same production function as in question 7, $q = aL^b$, show that the cost function is convex (concave) if the production function is concave (convex). Relate your answer to the answer in question 7.

9. Let $C(y) = y^3 - 12y^2 + 50y + 20$, $y \geq 0$ be a firm's cost function. Find the interval over which it is concave and the interval over which it is convex. Use this information and a table such as that of example 5.19 to draw this function.

10. For the following functions, find the Taylor series formula for $n = 2$ (i.e., the remainder term involves the second derivative as in equation (5.9)). Determine whether using the differential $dy = f'(x)\,dx$ to estimate the impact on y of a change in x of an amount dx leads to an underestimate or overestimate of the actual change in y.

(a) $f(x) = x^4$

(b) $f(x) = 1/x^2$, $x > 0$

Chapter 6 Optimization of Functions of One Variable

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Monopoly Equilibria I and II
- Monopoly Equilibrium I: Example
- Monopoly Equilibrium II: Example
- Monopoly with Constant-Elasticity Demand and Constant Costs
- Constant Elasticity of Demand Less Than One: Example
- Average and Marginal Functions Revisited
- The Labor-Managed Firm
- Competitive Firm with a Cubic Cost Function
- Short-Run Supply Function of a Competitive Firm
- The Competitive Firm with Cubic Costs Revisited
- The Excise Tax That Maximizes Total Tax Revenue: Example

Many economic models are based on the idea that an individual decision maker makes an *optimal choice* from some given set of alternatives. To formalize this idea, we interpret optimal choice as maximizing or minimizing the value of some function. For example, a firm is assumed to minimize costs of producing each level of output and to maximize profit; a consumer to maximize utility; a policy maker to maximize welfare or the value of national output; and so on. It follows that the mathematics of optimization is of central importance in economics, and in this and chapters 12 and 13 we will be studying optimization methods in some depth.

In this chapter we study the simplest case, the optimization of functions of one variable. We will emphasize the intuitive interpretation of the methods and their application to economic problems. More technical issues, such as the question of the existence of optimal solutions, are postponed until chapter 13.

6.1 Necessary Conditions for Unconstrained Maxima and Minima

Given some function f (i.e., $y = f(x)$), we optimize it by finding a value of x at which it takes on a maximum or minimum value. Such values are called **extreme values** of the function. If the set of x-values from which we can choose is the entire real line, the problem of finding an extreme value is **unconstrained,** while if the set of x-values is restricted to be a proper subset of the real line, the problem is **constrained.** To begin with we consider only unconstrained problems. We also assume that the function f is differentiable at least twice everywhere on its domain.

Of course, it is perfectly possible that a particular function may not have a maximum or minimum value. For example,

$$y = a + bx, \qquad a, b > 0$$

has neither a maximum nor a minimum, while

$$y = a + x^2$$

has a minimum at $x = 0$ but no maximum, and

$$y = a - x^2$$

has a maximum at $x = 0$ but no minimum (see figure 6.1).

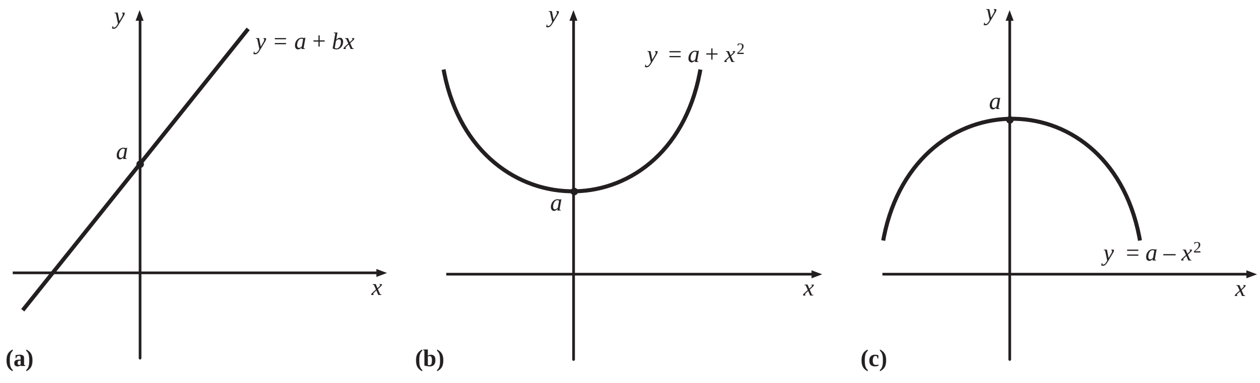


Figure 6.1 (a) Neither a maximum nor minimum exists; (b) no maximum exists; (c) no minimum exists

We begin by considering the problem of finding a maximum of a function, and assume that we are dealing with a function, $y = f(x)$, which certainly possesses one. It is important to distinguish between a *local* and a *global* maximum of the function.

Definition 6.1

At a **global maximum** x^*,

$$f(x^*) \geq f(x) \qquad \text{for all } x \tag{6.1}$$

whereas at a **local maximum** $\hat{x}$,

$$f(\hat{x}) \geq f(x), \qquad \hat{x} - \epsilon \leq x \leq \hat{x} + \epsilon \tag{6.2}$$

for x in an interval, perhaps very small, around $\hat{x}$.

The reason for this distinction is that in an optimization problem we are usually trying to find a *global* maximum, but the methods we have for finding solutions deliver only a *local* maximum. Usually some further work is then required to make sure we really have found a global solution. This need not be difficult. Note that a global solution *must* be a local one, since if $f(x^*) \geq f(x)$ for all x, this condition must also be true for those x in a small interval around x^*. So we find the global maximum by generating all the local maxima and comparing the values of the function at each of them to find the global solution, *assuming, of course, that it exists*. In many economic problems, assumptions are built in to ensure that there is only one local maximum, in which case this unique local solution must also be a global solution. In general, however, optimum solutions may not be unique, and this is recognized by having *weak* inequalities in equations (6.1) and (6.2).

If it is true that the function has a local maximum at x^*, then it must also be true that the first derivative of the function is zero at $x = x^*$; that is,

$$f'(x^*) = 0 \tag{6.3}$$

We refer to this requirement as the **first-order condition.** It is easy to see why it must hold. Take the differential of $y = f(x)$ at x^*:

$$dy = f'(x^*)\, dx \tag{6.4}$$

If the function is at a local maximum at x^*, it must be impossible to increase its value by small changes, dx, in either direction from x^*. This could not be true if $f'(x*) \neq 0$. For if $f'(x^*) > 0$, then choosing $dx > 0$ gives $dy > 0$ and the function has increased; if $f'(x^*) < 0$, then choosing $dx < 0$ again gives $dy > 0$ and increases the value of the function. Only if $f'(x^*) = 0$ does any $dx \neq 0$ give $dy = 0$, so that the function cannot be increased.

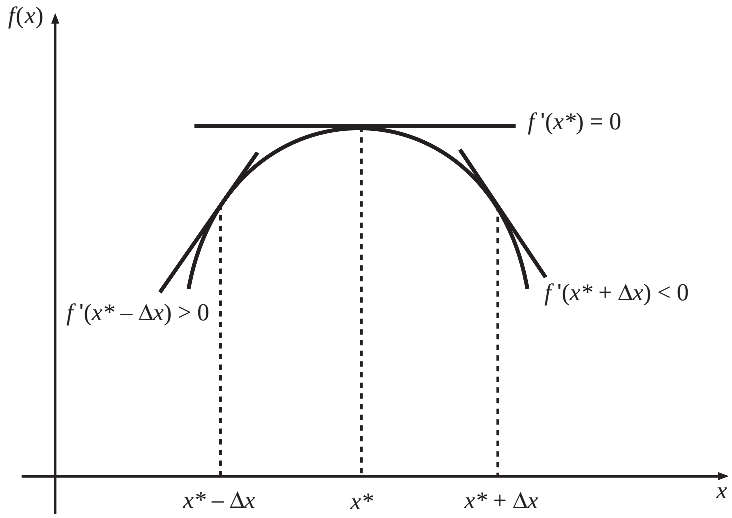


Figure 6.2 If x^* is a local maximum, then $f'(x^*) = 0$

This principle is even easier to see diagrammatically. In figure 6.2, x^* clearly gives a local maximum of the function. At the point $(x^*, f(x^*))$, the tangent line to the function is horizontal. But $f'(x^*)$ is the slope of this tangent line, and the slope of a horizontal line is zero. At the point $x^* + \Delta x$, we see that $f'(x^* + \Delta x) < 0$ and y can be increased by reducing x; at $x^* - \Delta x$, we have $f'(x^* - \Delta x) > 0$ and y can be increased by increasing x.

We emphasize that the equation (6.3), $f'(x) = 0$, does only identify—and allow us to solve for—a *local* maximum and not a global maximum. In figure 6.2, as we move further away from x^*, the function might start increasing and reach a value greater than $f(x^*)$—we cannot say without knowing the shape of the function across its entire domain.

Suppose now that we wish to minimize the function, f, and that a minimum of the function certainly exists. Then, by the same reasoning we used in the case of a maximum, we can establish that if the function is minimized at a point x^*, then at that point we must have the first-order condition

$$f'(x^*) = 0 \tag{6.5}$$

meaning that its first derivative is zero at that point. Again, taking the differential of the function at x^* as in equation (6.4), if $f'(x^*) \neq 0$, then we can always find a small change in x such that the value of the function is reduced. The only instance where this cannot be done is when $f'(x^*) = 0$. Figure 6.3 illustrates this. Again, the tangent line to the minimum point is horizontal, which is the geometrical equivalent to $f'(x^*) = 0$, while at any point with a nonzero derivative we can always find a change in x that reduces the value of the function.

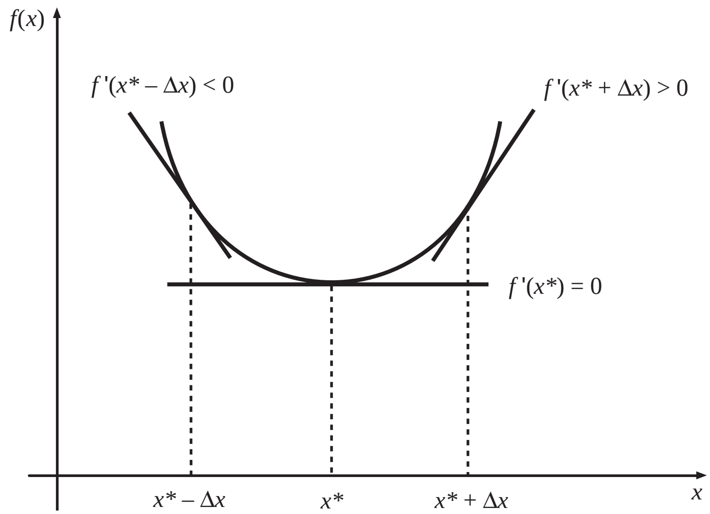


Figure 6.3 If x^* is a local minimum, then $f'(x^*) = 0$

We have therefore established

Theorem 6.1 If the differentiable function f takes an extreme value (maximum or minimum) at a point x^*, then $f'(x^*) = 0$.

The first-order condition $f'(x^*) = 0$ is a **necessary condition** for x^* to yield an extreme value of the function because any x that does *not* satisfy it *cannot* yield an extreme value. If we are trying to find, say, a maximum of the function, it is clearly not sufficient that a point, x^*, satisfy $f'(x^*) = 0$, since that point could actually be a minimum of the function. Indeed, the fact that x^* satisfies this condition is not even sufficient to guarantee that it yield an extreme value of the function, since there is another class of points, called **points of inflection,** at which it is possible that the derivative $f'(x) = 0$. Thus consider the function

$$y = 16x - 4x^3 + x^4$$

We have

$$\frac{dy}{dx} = 16 - 12x^2 + 4x^3 = f'(x)$$

and at $x = 2$ we have $f'(2) = 0$. Yet, as the graph of the function in figure 6.4 shows, $x = 2$ does not yield an extreme value of the function: it happens that the tangent to the function is horizontal at that point.

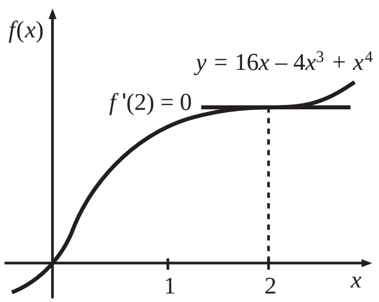


Figure 6.4 A point of inflection

Definition 6.2

For a differentiable function f, point x^*, at which $f'(x^*) = 0$, yields a **stationary value** of the function. Such stationary values may be extreme values or points of inflection. Every extreme value of a function is a stationary value, but *not* every stationary value need be an extreme value.

It is then immediately clear that if we wish to use equation (6.3) on its own to locate or solve for a value x^* which maximizes or minimizes a function, we have a problem. That condition alone does not tell us whether we have found a maximum, minimum, or point of inflection. So some further work is required. We examine what that is in the next section. First, we consider some examples and economic applications of the mathematics covered so far.

Example 6.1

Find the extreme values of the functions

(i) $y = 2x^3 - 0.5x^2 + 2$
(ii) $y = 4x^2 - 5x + 10$
(iii) $y = 6x/(x^4 + 2)$
(iv) $y = 0.5x^4 - 5x^3 + 2x^2$

and state in each case whether we have a local maximum or minimum.

Solution

(i) $dy/dx = 6x^2 - x = 0 \Rightarrow x^* = 0.167$ or $x^* = 0$

At $x = 0.167$, the value of y is 1.995. We check whether it is a minimum or maximum by taking small deviations around this point. At $x = 0.15$, the value of y is 1.996. At 0.19, the value of y is 1.996. Thus $x^* = 0.167$ yields a local minimum of the function. At $x = 0$, the value of y is 2. At $x = 0.1$, the value of y is 1.997, while at $x = -0.1$, the value of y is 1.993. Thus $x = 0$ yields a local maximum of the function. Note that had we chosen $x = 0.3$ for comparison, we would have found $y = 2.009$, which would have led us to reject $x = 0$ as a local maximum. This emphasizes the fact that we are dealing only with a local extremum and the neighborhood over which a point yields an extreme value may be *small* (see figure 6.5).

(ii) $dy/dx = 8x - 5 = 0 \Rightarrow x^* = 0.625$

At $x = 0.625$, the value of y is 8.4375. At $x = 0.5$, the value of y is 8.5, while at $x = 0.8$, the value of y is 8.56. Thus $x = 0.625$ yields a local minimum of the function (see figure 6.6).

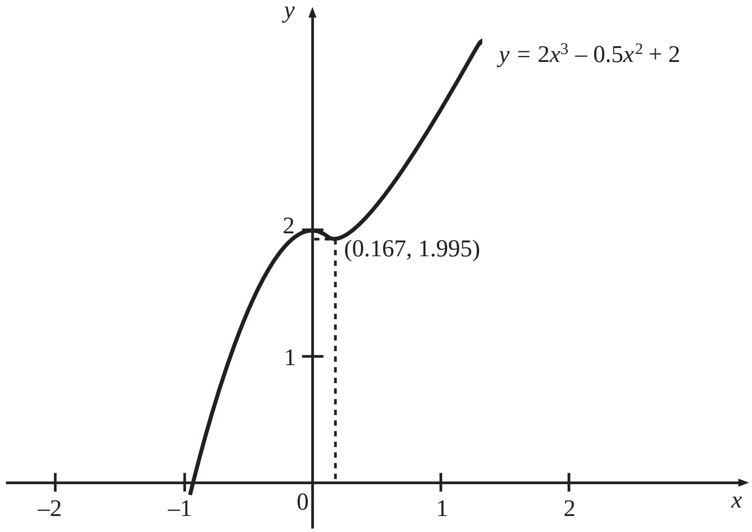


Figure 6.5 Graph of $y = 2x^3 - 0.5x^2 + 2$ for example 6.1(i)

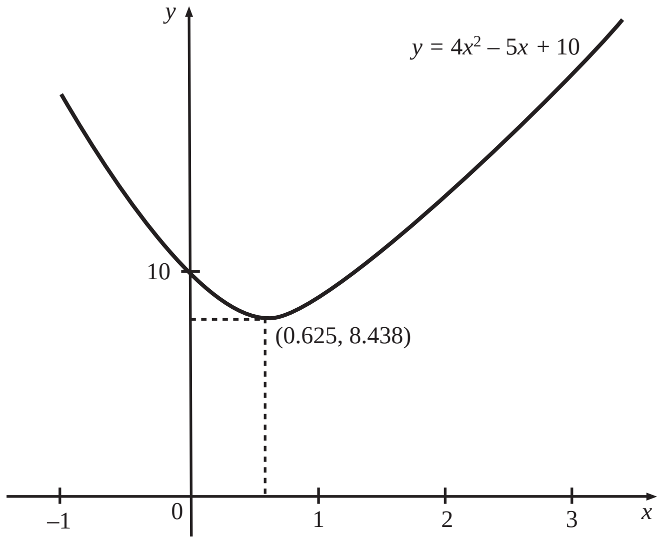


Figure 6.6 Graph of $y = 4x^2 - 5x + 10$ for example 6.1(ii)

(iii)

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(x^4+2)^2}[6(x^4+2) - 6x(4x^3)] \\ &= \frac{1}{(x^4+2)^2}(12-18x^4) \\ &= 0 \Rightarrow x^* = \left(\frac{12}{18}\right)^{1/4} = \pm 0.9\end{aligned}$$

At $x = 0.9$, $y = 2.03$. At $x = 1.0$, $y = 2.0$. At $x = 0.8$, $y = 1.99$. Thus $x = 0.9$ yields a local maximum of the function (see figure 6.7).

At $x = -0.9$, $y = -2.03$. At $x = -1$, $y = -2.0$. At $x = -0.8$, $y = -1.99$. Thus $x = -0.9$ yields a local minimum of the function (see figure 6.7).

(iv) $dy/dx = 2x^3 - 15x^2 + 4x = 0$ and dividing through by x gives the quadratic

$$2x^2 - 15x + 4 = 0$$

which has roots $x = 0.277$ and $x = 7.222$.

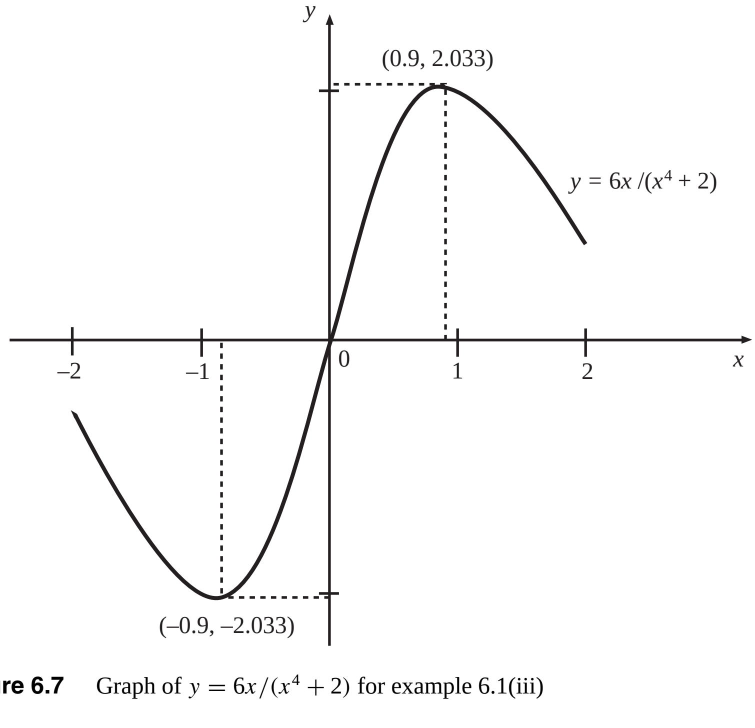


Figure 6.7 Graph of $y = 6x/(x^4+2)$ for example 6.1(iii)

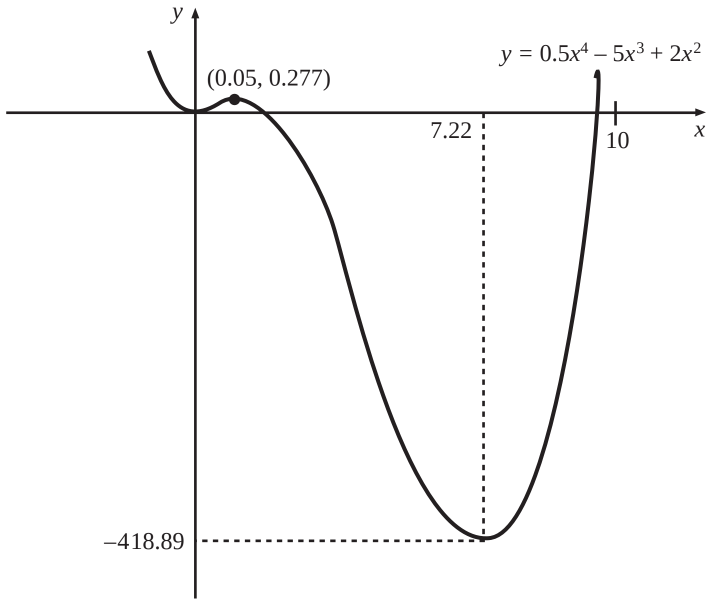


Figure 6.8 Graph of $y = 0.5x^4 - 5x^3 + 2x^2$ for example 6.1(iv)

At $x = 0.277$, $y = 0.05$. At $x = 0.4$, $y = 0.01$. At $x = 0.2$, $y = 0.04$. Thus $x = 0.277$ yields a local maximum of the function.

At $x = 7.222$, $y = -418.89$. At $x = 7$, $y = -416.5$. At $x = 8$, $y = -384.0$. Thus $x = 7.222$ yields a local minimum of the function (see figure 6.8). Note that $dy/dx = 0$ also at the point $x = 0$. At $x = 0$, $y = 0$. At $x = \pm 0.01$, $y = 0.0002$. Thus $x = 0$ yields a local minimum of the function. ■

Monopoly with Linear Demand and Costs

A monopolist faces a linear demand function $x = 100 - p$, where x is demand (output) and p is price. This function means that if the monopolist sets a price of \$100 or more, it can sell no output because no one is willing to pay that much, while for $p <$ \$100, a reduction in price of \$1 leads to an increase in demand and sales of one unit. The firm's cost function is $C = 25x$. That is, each additional unit of output costs \$25 to produce, no matter what total level of output is already being produced. We wish to work in terms of output, x, as the variable in the problem, and so we transform the demand function into an *inverse* demand function

$$p = 100 - x$$

and write the firm's profit as

$$\pi(x) = px - C = 100x - x^2 - 25x \tag{6.6}$$

If x^* is used to denote the output at which the firm's profit is maximized, applying equation (6.3) gives

$$\pi'(x^*) = 100 - 2x^* - 25 = 0 \tag{6.7}$$

or

$$x^* = \frac{100 - 25}{2} = 37.5$$

Then the firm's profit-maximizing price, and the maximum amount of profit it can make, are respectively

$$p^* = 100 - x^* = \$62.50$$
$$\pi^* = \$(62.50 - 25)37.5 = \$1{,}406.25$$

The diagrams in figure 6.9 illustrate this example, which should be familiar to the economics student. In figure 6.9(a) we show the graphs of

- the total-revenue function $R(x) = px = 100x - x^2$
- the total-cost function $C(x) = 25x$
- the total-profit function $\pi(x) = (100 - 25)x - x^2 = 75x - x^2$

In figure 6.9(b) we show the graphs of

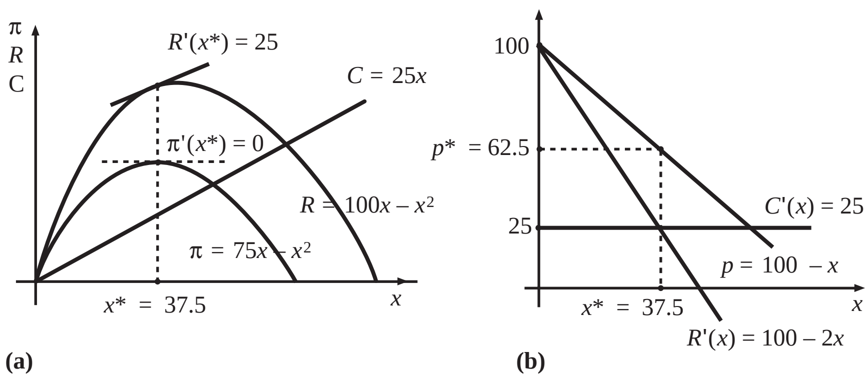


Figure 6.9 Monopoly equilibrium

- the demand function $x = 100 - p$ or $p = 100 - x$
- the marginal-revenue function $R'(x) = 100 - 2x$
- the marginal-cost function $C'(x) = 25$

Figure 6.9(b) shows that profit-maximizing price and output are at the point of equality of marginal cost and marginal revenue. This can easily be seen from equation (6.6)

$$100 - 2x^* = 25$$

$$R'(x^*) = C'(x^*)$$

Figure 6.9(a) shows that the slope of the tangent to $R(x)$ at x^* is equal to the slope of $C(x)$ at that point.

Note that we could just as well have worked with price as the firm's choice variable (though the corresponding diagrams would then look less familiar). We can express profit as a function of price by writing

$$R(p) = px = p(100 - p) = 100p - p^2$$

$$C(p) = 25(100 - p) = 2{,}500 - 25p$$

$$\begin{aligned}\pi(p) &= R(p) - C(p) = 100p - p^2 - (2{,}500 - 25p)\\ &= 125p - p^2 - 2{,}500\end{aligned}$$

Then maximizing with respect to p gives

$$\pi'(p^*) = 125 - 2p^* = 0$$

giving $p^* = \$62.50$, just as before.

Competitive Firm with Linear Costs

Suppose that a firm has the cost function $C = 5x$, and sells into a perfectly competitive market. This means that it can take the market price p as a given constant, rather than a function of its output. This assumption shows that a *competitive* firm is so small relative to the size of the market that its output decision has essentially no effect on the market price. Suppose that the market price is $p = \$8$. Then we set up the firm's profit as follows:

$$R(x) = px = 8x$$

$$C(x) = 5x$$

$$\pi(x) = (8 - 5)x = 3x$$

If we now apply equation (6.3) to find profit-maximizing output, we have

$$\pi'(x^*) = 3 = 0$$

which is nonsense! What went wrong?

Figure 6.10 shows the mathematical answer to this question. In figure 6.10(a), the firm's profit function always increases with output—it has no maximum. Thus, this firm would want to increase output indefinitely, since doing so always increases profit. In Figure 6.10(b), this is seen as resulting from price (= marginal revenue) being always greater than marginal cost, implying that an extra unit of output always adds more to revenue than it does to cost.

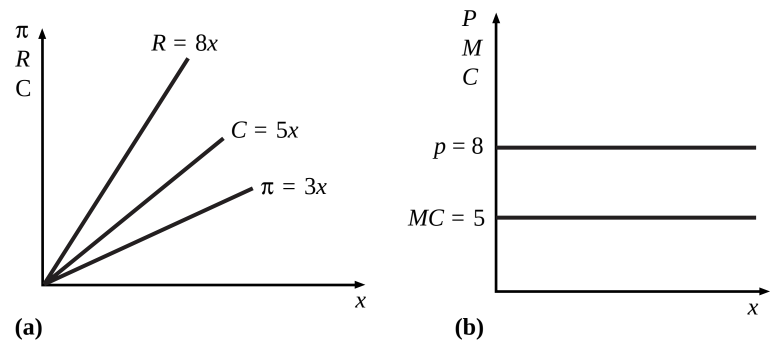


Figure 6.10 Competitive firm with constant cost

Mathematically this example serves as a warning. If we want to solve a problem by applying the $f'(x^*) = 0$ condition, we must be sure that the function really does possess a maximum. In other words, this is an example of a function that does not possess a maximum. (Compare figure 6.10(a) with figure 6.1(a).) But does the example have any *economic* meaning? In fact, it does. Essentially it says that *if* at least one firm in a competitive market has constant returns to scale (implying a linear cost function of the kind used here), then we might expect perfect competition to break down and be replaced by monopoly or oligopoly, since it pays the firm to expand output indefinitely, and as this happens the market structure must become less competitive as firms are driven out of the market. Then the assumption that firms face a "horizontal demand curve" becomes untenable: we can no longer assume that firms are "too small" to affect price. Thus we need to turn to other market models such as oligopoly and monopoly. In a sense, figure 6.10 shows a logically inconsistent situation: if a firm has constant costs below the market price, then it cannot regard price as a given parameter for *all possible* output levels.

A Publisher Will Always Set a Higher Price and Sell Fewer Copies of a Book Than the Author Would Like

Take the monopoly firm illustrated in figure 6.9 and assume the good concerned is a book. The author of the book receives a royalty of 10% of the purchase price for each book sold, so her income is

$$Y(x) = 0.1px = 0.1R(x) = 0.1(100x - x^2)$$

The publisher's profit must now take into account the royalty paid to the author:

$$\begin{aligned}\pi(x) &= R(x) - C(x) - Y(x) = 75x - x^2 - (10x - 0.1x^2)\\ &= 65x - 0.9x^2\end{aligned}$$

We assume that the author would like to set price and quantity to maximize her income. We then have the condition

$$Y'(x_A) = 10 - 0.2x_A = 0$$

giving $x_A = 50$ as the author's desired sales, and $p_A = 100 - 50 = \$50$ as her desired price. The publisher, however, chooses the number of books to satisfy

$$\pi'(x_p) = 65 - 1.8x_p = 0$$

giving desired sales of 36.1 at a price of \$63.90.

This "conflict of interest" always arises and is not due to the special example chosen. If we let r, $0 < r < 1$, denote the royalty rate, we can write the author's income and the publisher's profit respectively as

$$Y(x) = rR(x)$$

$$\begin{aligned}\pi(x) &= R(x) - C(x) - rR(x)\\ &= (1 - r)R(x) - C(x)\end{aligned}$$

Maximizing $Y(x)$ gives

$$Y'(x_A) = rR'(x_A) = 0 \quad \text{or} \quad R'(x_A) = 0 \tag{6.8}$$

Thus the author essentially wishes to maximize sales revenue. Maximizing the publisher's profit gives

$$\pi'(x_p) = (1 - r)R'(x_p) - C'(x_p) = 0$$

or

$$R'(x_p) = \frac{C'(x_p)}{1-r} \tag{6.9}$$

Then, as long as marginal cost is greater than zero, $C' > 0$, the publisher's desired output x_p must differ from the author's desired output x_A. Given that we usually assume marginal revenue decreasing with output, that is, $R''(x) < 0$, equations (6.7) and (6.8) must imply that $x_A > x_p$.

Example 6.2 A publisher pays the author of a book a royalty of 15%. Demand for the book is $x = 200 - 5p$ and the production cost is $C = 10 + 2x + x^2$. Find the optimal sales from both the author's and the publisher's perspective.

Solution

The inverse demand function is

$$p = 40 - 0.2x$$

The author's income is

$$Y(x) = 0.15px = 0.15(40 - 0.2x)x = 6x - 0.03x^2$$

The optimal level of sales from the author's viewpoint, x_A, leads to a maximum value for $Y(x)$, so

$$Y'(x_A) = 0 \Rightarrow 6 - 0.06x_A = 0 \Rightarrow x_A = 100$$

The profit for the publisher is

$$\begin{aligned}\pi(x) &= R(x) - C(x) - Y(x)\\ &= px - (10 + 2x + x^2) - (6x - 0.03x^2)\\ &= (40 - 0.2x)x - (10 + 2x + x^2) - (6x - 0.03x^2)\\ &= 32x - 1.32x^2 - 10\end{aligned}$$

The optimal level of sales from the publisher's viewpoint, x_P, leads to a maximum value for $\pi(x)$, so

$$\pi'(x_P) = 32 - 2.46x = 0 \;\Rightarrow\; x_P = \frac{32}{2.46} = 13$$

Clearly, the author seeks a level of sales that maximizes sales revenue, while the publisher take costs, including royalty costs, into account and prefers a lower level of sales. ■

Revenue and Elasticity

A monopolist with inverse demand function $p = p(x)$ has a revenue function $R = p(x)x = R(x)$, and marginal revenue

$$R'(x) = p + x\frac{dp}{dx}$$

The price elasticity of demand is

$$\epsilon = -\frac{p}{x}\frac{dx}{dp}$$

and marginal revenue may be written as

$$R'(x) = p + \frac{x}{p}\frac{dp}{dx} = p\left(1 - \frac{1}{\epsilon}\right)$$

This little formula turns out to be very useful in many contexts. For example, it establishes in a simple way the relationship between the elasticity of demand at a point on the demand curve and the effect on sales revenue of a change in output or price. Thus, taking $p > 0$ and using the formula tells us that

$\epsilon < 1 \Rightarrow R'(x) < 0$, so when demand is inelastic an increase in output (decrease in price) reduces revenue;
$\epsilon > 1 \Rightarrow R'(x) > 0$, so when demand is elastic an increase in output (decrease in price) increases revenue.

Moreover the output x^* at which revenue is *maximized* must satisfy

$$R'(x^*) = p\left(1 - \frac{1}{\epsilon}\right) = 0$$

implying that revenue is maximized at the point on the demand curve where $\epsilon = 1$.

Finally, the formula gives us an easy way of establishing the proposition: a profit-maximizing monopolist with positive marginal costs will always be in equilibrium at a point on the demand curve where $\epsilon > 1$. To see this, note that at the equilibrium

$$R'(x) = C'(x) > 0$$

and for $R'(x) > 0$ we must have $\epsilon > 1$. This is what underlies the conflict of interest between author and publisher discussed in the previous example.

EXERCISES

1. Find the stationary values of the following functions and state whether they yield a local maximum, local minimum, or point of inflection (sketch the function in the neighborhood of the stationary value):

(a) $y = x^3 - 3x^2 + 1$

(b) $y = x^4 - 4x^3 + 16x - 2$

(c) $y = 3x^3 - 3x - 2$

(d) $y = 3x^4 - 10x^3 + 6x^2 + 1$

(e) $y = 2x/(x^2 + 1)$

2. Show that a profit-maximizing monopolist's output is unaffected by a proportional profit tax, but is reduced by a tax of $\$t$ per unit of output. Explain these results.

3. Find the supply curve of a competitive firm with the total-cost function

$$C = 0.04x^2 + 3x + 80$$

4. The demand function facing a monopolist is

$$x = ap^{-b}$$

What range of values must b lie in for a solution to the profit-maximization problem to exist?

5. A monopolist faces a linear demand function. Show that if it maximizes sales revenue, it sets an output exactly half that it would produce if it "sold" its output at a zero price.

6.2 Second-Order Conditions

We saw in section 6.1 that the condition $f'(x^*) = 0$ does not in itself tell us whether x^* yields a maximum, a minimum, or a point of inflection of the function f. Since this condition is stated in terms of the first derivative of the function it is usually referred to as the first-order condition. We now go on to examine how conditions on the second derivative of a function, namely **second-order conditions,** can be developed to help us distinguish among the three kinds of stationary value.

In figure 6.11 we show the three possible cases of stationary values. Associated with each graph of a function f is, directly below it, the graph of its first derivative f', in the neighborhood of a stationary value. The *curvature* of the function f determines the *slope* of its derivative f'.

In figure 6.11(a), the function is strictly concave in the neighborhood of x^*. This implies that in figure 6.11(d), at x-values below x^*, we have $f'(x) > 0$, while at x-values above x^*, $f'(x) < 0$. The slope of f' is the second derivative $f''(x)$, and so we have $f''(x^*) < 0$. Since, in this case, x^* yields a maximum of the function, we have

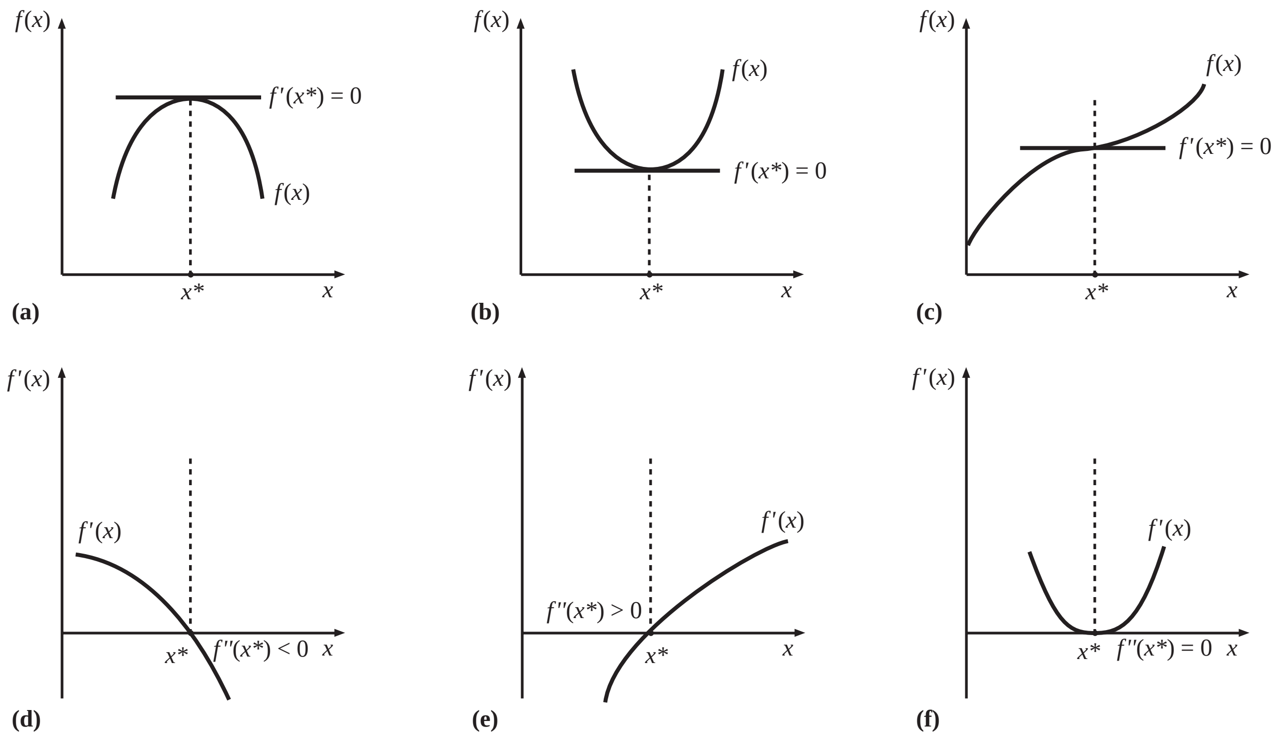


Figure 6.11 Second-order conditions

Theorem 6.2 If $f'(x^*) = 0$, and $f''(x^*) < 0$, then f has a local maximum at x^*.

In figure 6.11(b), the function is strictly convex in the neighborhood of the minimum point, implying that as x increases through x^*, $f'(x)$ is increasing, and its slope at x^* is positive. Thus $f''(x^*) > 0$, and we have

Theorem 6.3 If $f'(x^*) = 0$, and $f''(x^*) > 0$, then f has a local minimum at x^*.

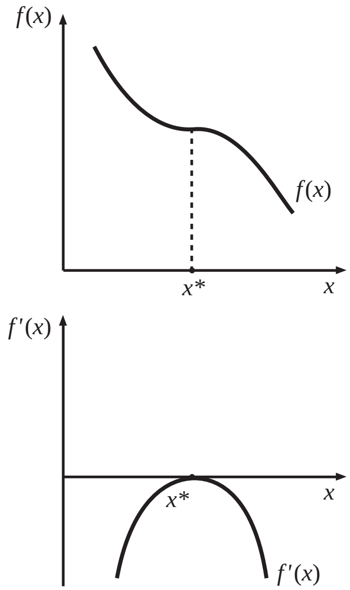


Figure 6.12 An alternative point of inflection

In the case of a point of inflection, we see that at that point the function changes its curvature from concave to convex. The derivative $f'(x)$ is positive everywhere except at x^*, which is where it takes its minimum value. This change in curvature is characteristic of points of inflection and implies that the derivative f' always takes on an extreme value (in this case a minimum) at points of inflection. Of course, the opposite case is also possible and shown in figure 6.12 as here $f'(x)$ is negative in a neighborhood of x^* except at x^* itself, where $f'(x^*) = 0$.

From now on we focus on extreme values as illustrated in figure 6.11(a) and (b). The introduction of the condition on the second derivative $f''(x^*)$ gives us a *sufficient condition* for a maximum. Satisfaction of $f'(x^*) = 0$ *and* $f''(x^*) < 0$ guarantees that x^* yields a maximum of the function. Can we then say that these conditions are also *necessary*, in that they must be satisfied by all points which do yield a maximum of the function? The answer is no. We may have points which maximize a function, but at which $f''(x^*) = 0$. Consider the function

$$y = -x^4$$

The graph of this function shows that it has a maximum at $x^* = 0$ (see figure 6.13), but

$$f''(0) = -12(0)^2 = 0$$

If we want to have conditions that are both necessary and sufficient, a little more work is necessary. We carry this out in the last part of this section. For the purposes of economic applications, it is usually safe to ignore such cases and simply apply the sufficient conditions in theorems 6.2 and 6.3.

Example 6.3 Use the second-order condition to determine whether the extreme values of the functions in example 6.1 are local maxima or local minima.

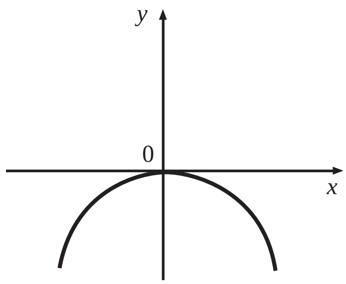


Figure 6.13 Graph of $y = -x^4$

Solution

(i)

$$y = 2x^3 - 0.5x^2 + 2$$

$$\frac{dy}{dx} = 6x^2 - x$$

$$\frac{d^2y}{dx^2} = 12x - 1 = 12(0.167) - 1$$
$$= 1.004 > 0 \quad \text{at } x = 0.167$$

and so $x = 0.167$ yields a local minimum of the function. We also have

$$y''(0) = -1 < 0$$

and so $x = 0$ yields a local maximum of the function.

(ii)

$$y = 4x^2 - 5x + 10$$

$$\frac{dy}{dx} = 8x - 5$$

$$\frac{d^2y}{dx^2} = 8 > 0$$

and so $x = 0.625$ yields a local minimum of the function.

(iii)

$$y = \frac{6x}{x^4 + 2}$$

$$\frac{dy}{dx} = \frac{(12 - 18x^4)}{(x^4 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(-(x^4 + 2)^2 72x^3 - (12 - 18x^4)8x^3(x^4 + 2))}{(x^4 + 2)^4}$$
$$= \frac{72x^7 - 240x^3}{(x^4 + 2)^3}$$

Since, $x^4 + 2$ is always positive, we can ignore it in determining the sign of d^2y/dx^2.

At $x = 0.9$, $72x^7 - 240x^3 = -140.5 < 0$, *where there is a local maximum.*
At $x = -0.9$, $72x^7 - 240x^3 = 140.5 > 0$, *where there is a local minimum.*

(iv)

$$y = 0.5x^4 - 5x^3 + 2x^2$$

$$\frac{dy}{dx} = 2x^3 - 15x^2 + 4x$$

$$\frac{d^2y}{dx^2} = 6x^2 - 30x + 4$$

At $x = 0.277$, $6x^2 - 30x + 4 = -3.85 < 0$, *where there is a local maximum.*
At $x = 7.222$, $6x^2 - 30x + 4 = 100.28 > 0$, *where there is a local minimum.*
At $x = 0$, $y' = 0$, *and* $y'' = 4 > 0$, *and so there is a local minimum.* ■

Maximizing Tax Revenue

In a market for a good, we have a linear demand function

$$D = a_0 - a_1 p_B, \qquad a_0, a_1 > 0$$

and a linear supply function

$$S = b_0 + b_1 p_S, \qquad b_0, b_1, > 0$$

where D is quantity demanded, p_B is the *price paid by buyers*, S is quantity supplied, and p_S is the price *received by sellers*. In the usual supply-demand model with no tax, $p_B = p_S = p$, and so we solve for market equilibrium by setting $D = S$:

$$a_0 - a_1 p = b_0 + b_1 p$$

giving equilibrium price as

$$p^* = \frac{a_0 - b_0}{a_1 + b_1}$$

Note that for this to give a *positive* equilibrium price, we require that $a_0 > b_0$, an assumption we now make explicitly. The corresponding equilibrium quantity traded is

$$D^* = S^* = b_0 + b_1\left(\frac{a_0 - b_0}{a_1 + b_1}\right) = a_0 - a_1\left(\frac{a_0 - b_0}{a_1 + b_1}\right)$$

found by inserting equilibrium price into the demand or supply functions. (Since by definition $D^* = S^*$ in equilibrium, we only need to compute one of $D^* = D(p^*)$ or $S^* = S(p^*)$, although computing both provides us with a check on our calculations.)

When the government imposes a tax on a good, say of \$$t$ per unit of the good sold, it drives a wedge between the price buyers pay and the price sellers receive. In fact, sellers receive what buyers pay *minus* the tax, and so

$$p_S = p_B - t$$

Note that we could also think of buyers paying the sellers' price plus tax:

$$p_B = p_S + t$$

Since the two formulations are mathematically equivalent, we work with the first.

For market equilibrium we still require that demand equal supply, and so we have

$$a_0 - a_1 p_B = b_0 + b_1(p_B - t)$$

where we have substituted for p_S in the supply function. Solving now for the equilibrium buyers' price $\hat{p}_B$ and sellers' price $\hat{p}_s$ gives

$$\hat{p}_B = \frac{a_0 - b_0}{a_1 + b_1} + \frac{b_1 t}{a_1 + b_1} = p^* + \frac{b_1 t}{a_1 + b_1}$$

$$\hat{p}_s = \hat{p}_B - t = p^* + \frac{b_1 t}{a_1 + b_1} - t = p^* - \frac{a_1 t}{a_1 + b_1}$$

Thus we see that imposing a tax raises price to buyers and reduces price to sellers, by an amount which depends on the relative values of the slope coefficients of the supply and demand functions, a_1 and b_1. Moreover

$$\frac{d\hat{p}_B}{dt} = \frac{b_1}{a_1 + b_1}, \quad \frac{d\hat{p}_s}{dt} = -\frac{a_1}{a_1 + b_1}$$

Since $a_1/(a_1 + b_1) + b_1/(a_1 + b_1) = 1$, this tells us that an extra \$1 in tax is divided between an increase in price to buyers, and a reduction in price to sellers, in a way determined by the slope coefficients, and this is called the *incidence* of the tax.

Consider now the amount of revenue raised by the tax. At the price $\hat{p}_B$, the quantity $\hat{D}$ is bought, where

$$\hat{D} = a_0 - a_1 \hat{p}_B = a_0 - a_1 p^* - \frac{a_1 b_1 t}{a_1 + b_1}$$

Since a tax of t is paid on each unit bought, tax revenue is therefore

$$\begin{aligned} T(t) &= t\hat{D} = (a_0 - a_1 p^*)t - \frac{a_1 b_1 t^2}{a_1 + b_1} \\ &= tD^* - \frac{a_1 b_1 t^2}{a_1 + b_1} \end{aligned}$$

Suppose that the government is interested in finding the value of the tax t that maximizes its tax revenue. Applying theorem 6.2, we have

$$T'(t^*) = D^* - \frac{2a_1 b_1 t^*}{a_1 + b_1} = 0$$

and so the revenue-maximizing tax is given by

$$t^* = D^* \frac{(a_1 + b_1)}{2a_1 b_1}$$

We can simplify this by substituting for D^*

$$\begin{aligned} t^* &= \left[a_0 - a_1 \frac{(a_0 - b_0)}{(a_1 + b_1)} \right] \frac{(a_1 + b_1)}{2a_1 b_1} \\ &= \frac{[a_0 a_1 + a_0 b_1 - a_0 a_1 + a_1 b_0]}{2a_1 b_1} \\ &= \left(\frac{a_0 b_1 + a_1 b_0}{2a_1 b_1} \right) \end{aligned}$$

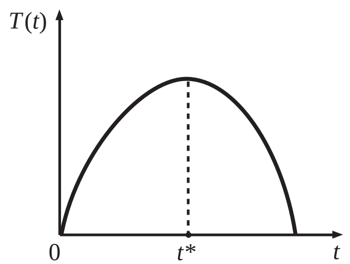


Figure 6.14 Tax revenue is a strictly concave function of the tax rate

We now have the question: Is t^* a true (local) maximum? We answer this by taking

$$T''(t^*) = \frac{-2a_1 b_1}{a_1 + b_1} < 0$$

and so t^* is a maximum. Note that since $T''(t)$ is always negative, $T(t)$ must be a strictly concave function (see figure 6.14).

This concave shape has interesting economic implications. As the tax rate goes up, the equilibrium quantity bought and sold goes down. Initially the increase in tax rate outweighs the effect of reduced quantity and generates increased tax revenue, but after a point this ceases to be true: reduced quantity traded outweighs the increased tax rate and produces lower tax revenue. If the *actual* tax rate exceeds t^*, tax revenue can be increased by reducing the tax rate. Given that the supply and demand curves are (approximately) linear, estimation of the four demand and supply parameters—a_0, a_1, b_0, b_1—would allow an assessment to be made of whether this was in fact the case.

Second-Order Conditions and the Taylor Series Expansion

A useful way of developing the second-order conditions for maxima and minima, and which also allows us to consider what happens when $f''(x^*) = 0$, is provided by the Taylor series expansion (see section 5.6). Let x^* be such that $f'(x^*) = 0$, and we wish to confirm that it is a maximum or minimum. Let $\hat{x}$ be any x in a small interval around x^*. Then a Taylor series expansion in remainder form gives

$$f(\hat{x}) = f(x^*) + \frac{f'(x^*)(\hat{x} - x^*)}{1!} + \frac{f''(\zeta)(\hat{x} - x^*)^2}{2!}$$

for some point ζ lying between x^* and $\hat{x}$. Since $f'(x^*) = 0$, if $f''(\zeta) < 0$, then $f(\hat{x}) - f(x^*) < 0$, or $f(x^*) > f(\hat{x})$, and so x^* yields a local maximum; while if $f''(\zeta) > 0$, then $f(\hat{x}) - f(x^*) > 0$, or $f(x^*) < f(\hat{x})$, and x^* yields a local minimum. Suppose, however, that $f''(\zeta) = 0$. To determine if x^* is a maximum or minimum, using the Taylor series expansion then requires taking the expansion to more than two terms. The next term in the sequence is $f'''(x^*)(\hat{x} - x^*)^3/3!$, but this tells us nothing conclusive since the sign of $(\hat{x} - x^*)^3$ may be positive or negative.

Suppose, however, that $f'''(x^*) = 0$, and indeed so is every derivative up until the nth, which we denote by $f^{(n)}(x)$. Then we have

$$f(\hat{x}) = f(x^*) + \frac{f^{(n)}(\zeta)(\hat{x} - x^*)^n}{n!}$$

Now, *if n is even*, we can use the sign of $f^{(n)}(\zeta)$ to tell us whether x^* yields a maximum or a minimum, since $(\hat{x} - x^*)^n > 0$, and we can use the same argument as in the case of $f''(\zeta)$. This is called the ***n*th derivative test** for a maximum or minimum.

Finally, note that the Taylor series expansion gives us formal confirmation of the "diagrammatically obvious" fact that if a function is strictly concave in a neighborhood of an extreme value x^*, then x^* must yield a local maximum, while if it is strictly convex in that neighborhood, then x^* yields a local minimum (excluding cases where $f'' = 0$ at some point). Recall from section 5.5 that if $f'' < 0$, then f is strictly concave, while if $f'' > 0$, the function f is strictly convex.

Example 6.4 Consider the function $f(x) = x^2 + 5$ where

$$f'(x) = 2x, \quad f''(x) = 2$$

The first derivative vanishes at $x^* = 0$ and $f'' > 0$ everywhere. Thus this function is strictly convex with an extreme value at $x^* = 0$, which implies there is a minimum here. From the Taylor series expansion we have, with $x^* = 0$,

$$f(\hat{x}) = f(0) + f'(0)(\hat{x} - x^*) + \frac{f''(\zeta)\hat{x}^2}{2!}$$

$$f(\hat{x}) = 5 + \frac{2\hat{x}^2}{2} = 5 + \hat{x}^2$$

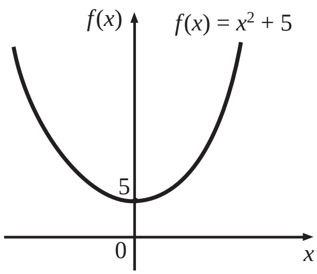


Figure 6.15 Graph of $f(x) = x^2 + 5$ for example 6.9

Obviously $f(\hat{x}) > f(0) = 5$ for any $\hat{x} \neq 0$. Since $f'' > 0$ for any value of x (and so for any ζ between x^* and $\hat{x}$), we can see from the Taylor series expansion that $x^* = 0$ delivers a global minimum in this case (see figure 6.15). ■

Example 6.5 Consider the function $f(x) = x^3$. We have

$$f'(x) = 3x^2$$

$$f''(x) = 6x$$

$$f'''(x) = 6$$

The first derivative vanishes at $x^* = 0$ and so does the second derivative. Therefore, with $f'''(x^*)$ being the first of the higher order derivatives not to vanish at x^*, it follows that we need to use the third derivative ($n = 3$) as the final term in the Taylor series expansion to investigate the nature of the function around $x^* = 0$; that is,

$$f(\hat{x}) = f(0) + \frac{f'''(\zeta)(\hat{x})^3}{6} \qquad \text{for } \zeta \text{ between } x^* \text{ and } \hat{x}$$

Since $f'''(x) = 6$ for all x, we can write this equation as

$$f(\hat{x}) = f(0) + \hat{x}^3$$

For $\hat{x} > 0$ we get $f(\hat{x}) > f(0)$, while for $\hat{x} < 0$ we get $f(\hat{x}) < f(0)$. Therefore $x^* = 0$ delivers neither a maximum nor a minimum. It is a point of inflection (see figure 6.16). ■

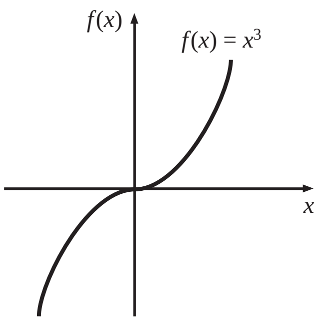


Figure 6.16 Graph of $f(x) = x^3$ for example 6.5

Example 6.6 Consider the function $f(x) = 10 - x^4$. We have

$$f'(x) = -4x^3$$

$$f''(x) = -12x^2$$

$$f'''(x) = -24x$$

$$f^{(4)}(x) = -24$$

The first derivative vanishes at x^* as do the higher order derivatives up to the fourth. It follows that we need to use the fourth derivative ($n = 4$) as the final term in the Taylor series expansion to investigate the nature of the function around $x^* = 0$:

$$f(\hat{x}) = f(0) + \frac{f^{(4)}(\zeta)(\hat{x})^4}{24}$$

Since $f^{(4)}(x) = -24$ for all x, we can write this as

$$f(\hat{x}) = f(0) - \hat{x}^4 \quad \text{or} \quad f(\hat{x}) - f(0) = -\hat{x}^4$$

For any value of $\hat{x} \neq 0$, we have $-\hat{x}^4 < 0$ and so $f(\hat{x}) - f(0) < 0$ or $f(0) > f(\hat{x})$. That is, the point $x^* = 0$ delivers a maximum value for this function (see figure 6.17. ■

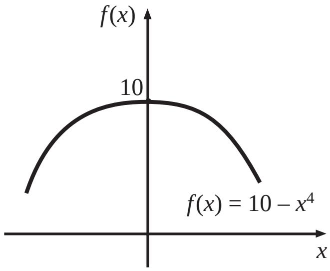


Figure 6.17 Graph of $f(x) = 10 - x^4$ for example 6.6

EXERCISES

1. For each function in question 1 of section 6.1 exercises, now use second-order conditions to determine whether each stationary value you found is a maximum, minimum, or point of inflection.

2. A firm in a competitive market discovers a new production process which gives it the total-cost function

$$C = 10 \log x$$

where x is output. Explain as fully as you can, in both mathematical and economic terms, why there may be a breakdown of perfect competition in this market.

3. A monopolist has the inverse demand function

$$p = a - bx$$

and the total-cost function

$$C = 10 \log x$$

Give conditions under which there will be a well-defined, profit-maximizing output and explain your answer in a diagram.

4. A monopsonist's revenue as a function of its only input is

$$R = az - bz^2, \qquad z \geq 0$$

It is faced with a supply function for the input

$$z = \alpha + \beta p, \qquad p \geq 0$$

where p is the input price, and $a, b, \alpha, \beta > 0$. Find the profit-maximizing price and quantity of the input the monopsonist will choose, and compare the analysis to that of the profit-maximizing monopoly.

5. A firm has the production function $x = f(L)$, where x is output and L is labor input. The firm buys the input in a competitive market.

(a) Assuming the firm sells its output in a competitive market, show that setting output where price equals marginal cost is equivalent to setting labor input where input price equals marginal value product.

(b) Assuming the firm is a monopoly, show that setting output where marginal revenue equals marginal cost is equivalent to setting labor input where input price equals marginal-revenue product.

(c) What restriction do we have to impose on the production function to ensure the second-order conditions in problems (a) and (b) are satisfied?

6. Use the nth derivative test to show that $-x^4$ has a maximum at $x = 0$.

6.3 Optimization over an Interval

The discussion of first- and second-order conditions in sections 6.1 and 6.2 has dealt exclusively with the unconstrained case, in which a solution to the problem can be anywhere on the real line. Often in economics, however, this is unacceptably general. For example, in problems in which firms choose outputs (as in several examples in the previous section) or consumers choose goods, we cannot assume that *negative* quantities are possible. In other problems, it may be reasonable to place an upper bound on a variable: for example, a firm may have a fixed production capacity that puts an upper limit on how much output it can produce. Or alternatively, a decision-taker may be choosing a proportion, for example, the share of a company to buy, and that naturally places the bounds zero and one on the variable.

Example 6.7 Solve the following problems involving optimization over an interval:

(i) max $y = 3 - 2x$ subject to $0 \leq x \leq 1$
(ii) max $y = 6$ subject to $-1 \leq x \leq 1$
(iii) min $y = 6x^2$ subject to $-2 \leq x \leq 2$
(iv) min $y = 2x^3 - 0.5x^2 + 2$ subject to $0 \leq x \leq 1$
(v) min $y = 4x^2 - 5x + 10$ subject to $1 \leq x \leq 10$

Solution

(i) This is a linear function with a negative slope. Therefore, we know an interior maximum cannot occur, and the solution is clearly at $x^* = 0$, with $y^* = 3$. Note that at $x = 0$ (and indeed at all points in the domain)

$$\frac{dy}{dx} = -2 \neq 0$$

(See figure 6.18.)

(ii) This is a constant function. All the points in the interval $[-1, 1]$ are solutions because they all yield the maximum *and* minimum of the function (see figure 6.19).

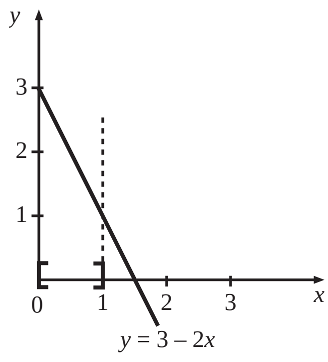


Figure 6.18 Graph of the problem in example 6.7(i)

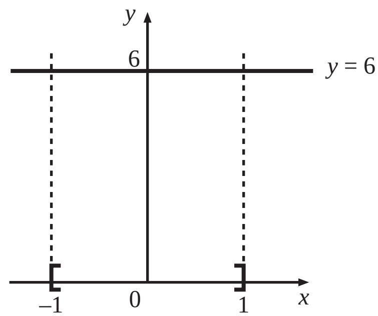


Figure 6.19 Graph of the problem in example 6.7(ii)

(iii) This function takes on a minimum at $x^* = 0$. Thus we have an interior solution (see figure 6.20).

(iv) We solved this problem first for *unrestricted* x as (i) of examples 6.1 and 6.3. The unconstrained minimum of the function occurs at $x^* = 0.167$. This value of x is *not* ruled out by the constraint. Hence we have an interior solution with $x^* = 0.167$. Note that $f(x^*) = 1.995$, which is less than $f(0) = 2$, so x^* does indeed deliver the minimum value on this interval (see figure 6.21).

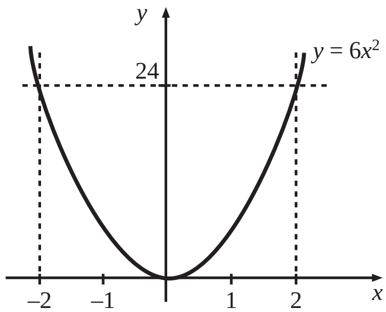


Figure 6.20 Graph of the problem in example 6.7(iii)

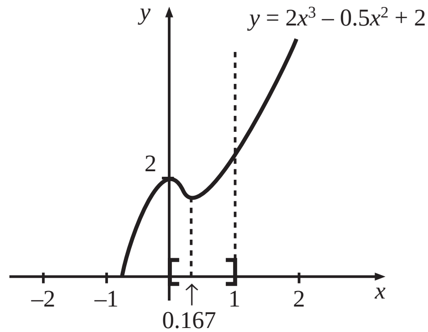


Figure 6.21 Graph of the problem in example 6.7(iv)

(v) We solved this problem for the unrestricted x as (ii) in examples 6.1 and 6.3. The unconstrained solution is $x = 0.625$, but this value is now ruled out by the constraint. The slope of this function is given by

$$\frac{dy}{dx} = 8x - 5$$

For $1 \leq x \leq 10$, this slope is always positive. Thus we know that the minimum value of the function over this interval occurs at $x = 1$ and the maximum at $x = 10$ (see figure 6.22). ■

Competitive Firm with Linear Costs Revisited

Suppose that a firm has the total-cost function $C = 20x$ and sells into a competitive market where the given price is \$10 per unit. Its profit function is

$$\pi(x) = 10x - 20x = -10x$$

and the first-order condition gives

$$\pi'(x^*) = -10 = 0$$

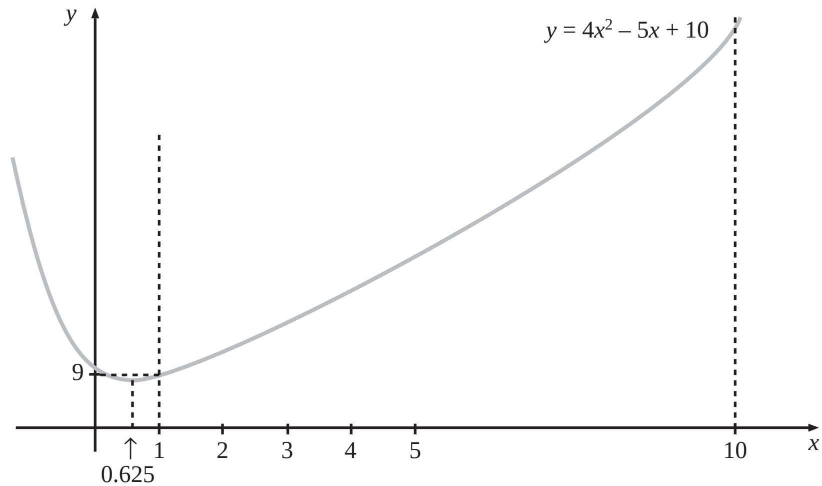


Figure 6.22 Graph of the problem in example 6.7(v)

which, of course, cannot be satisfied. The intuitive answer to the problem is obvious: market price is below unit cost (horizontal demand curve below horizontal marginal-cost curve), and so the firm does best by producing zero output. However, this possibility is not captured in the mathematics because, implicitly, we have allowed the whole real line. The mathematical solution is to set output at $-\infty$, because multiplying this by -10 gives the largest possible profit! If it is the case that negative outputs are impossible, we should incorporate this into the problem explicitly by specifying the constraint

$$x \geq 0$$

Then, as we will show below, the mathematics will produce the correct answer.

Returning to the general case, we assume that the possible values of x that are feasible for the problem are determined by a constraint

$$a \leq x \leq b$$

which defines an interval on the real line. We then write the problem as

$$\max f(x) \quad \text{subject to} \quad a \leq x \leq b \tag{6.10}$$

We are now trying to find the highest value of the function over this given interval. Figure 6.23 illustrates this situation. We see immediately that it is no longer a necessary condition for an extreme value that the derivative is zero.

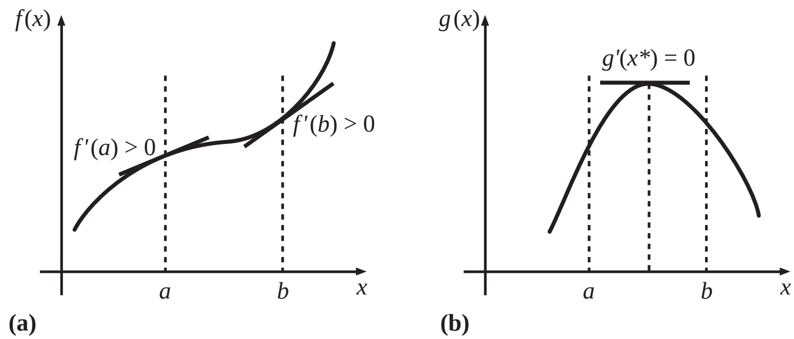


Figure 6.23 (a) Local constrained optima with interval constraint binding (b) Local constrained optima with interval constraint *not* binding

The function f shown in figure 6.23 (a) has a local (and global) maximum at b and a local (and global) minimum at a, while $f'(b) > 0$, $f'(a) > 0$. The reason b gives a local maximum, although $f'(b) > 0$, is that we cannot increase x above b: the only *feasible* direction of change in x is to go below b, and this reduces f. Likewise a is a local minimum because the only feasible change in x increases the value of the function. Then again, figure 6.23 (b) shows an example of a function g that has a maximum at x^*, and the standard condition $g'(x^*) = 0$ applies.

We start with the first-order necessary conditions for the maximization problem in equation (6.10). Let x^* denote a local maximum for this problem (so that necessarily $a \leq x^* \leq b$). There are then three possibilities:

1. $x^* = a$. In this case we must have $f'(x^*) \leq 0$. To see this, consider the differential

$$dy = f'(a)\,dx$$

   When $x = a$, the only permissible change in x is $dx > 0$. So to make $dy \leq 0$, we must have $f'(a) \leq 0$.
2. $a < x^* < b$. In this case we must have $f'(x^*) = 0$. This is because, when x^* is *inside* the interval, we can choose dx to be both positive or negative. So for dy not to be positive we must have $f'(x^*) = 0$.
3. $x^* = b$. In this case we must have $f'(x^*) \geq 0$. Again, consider the differential

$$dy = f'(b)\,dx$$

   When $x = b$, the only permissible change in x is $dx < 0$. For this not to produce $dy > 0$, we must have $f'(b) \geq 0$.

These arguments simply formalize somewhat the discussion of figures 6.22 and 6.23 and the results can be stated succinctly as follows:

Theorem 6.4 If x^* is a solution to the problem

$$\max f(x) \quad \text{subject to} \quad a \leq x \leq b$$

then it satisfies one or both of

$$f'(x^*) \leq 0 \quad \text{and} \quad (x^* - a)f'(x^*) = 0$$

$$f'(x^*) \geq 0 \quad \text{and} \quad (b - x^*)f'(x^*) = 0$$

where, if it satisfies *both* conditions and $f'(a) \neq 0$ and $f'(b) \neq 0$, we must have $a < x^* < b$.

These conditions essentially say that if x^* is inside the interval, meaning that, it is an **interior solution,** then its derivative must be zero, while if it is at one of the endpoints of the interval, the derivative must be nonpositive (at a) or nonnegative (at b). Note that we have not ruled out the cases in which the derivative just happens to be zero at an endpoint of the interval.

You should now sketch a figure corresponding to figure 6.23 for the case of a minimization problem, and set out the argument for

Theorem 6.5 If x^* is a solution to the problem

$$\min f(x) \quad \text{subject to} \quad a \leq x \leq b$$

then it satisfies one or both of

$$f'(x^*) \geq 0 \quad \text{and} \quad (x^* - a)f'(x^*) = 0$$

$$f'(x^*) \leq 0 \quad \text{and} \quad (b - x^*)f'(x^*) = 0$$

where, if it satisfies *both* conditions and $f'(a) \neq 0$ and $f'(b) \neq 0$, we must have $a < x^* < b$.

Competitive Firm with Linear Costs—Yet Again

To continue with this earlier example, we now explicitly impose the constraint $x \geq 0$. This gives the problem

$$\max \pi(x) \quad \text{subject to} \quad x \geq 0$$

which is a special case of the one just considered, with $a = 0$ and b set at $+\infty$ (i.e., dropped). Applying theorem 6.4 to the solution gives

$$\pi'(x^*) \leq 0 \quad \text{and} \quad x^*\pi'(x^*) = 0$$

and since in this case $\pi' = -10$, we have

$$-10 \leq 0 \quad \text{and} \quad x^*(-10) = 0$$

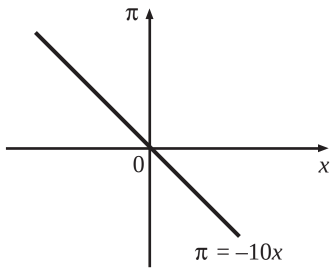


Figure 6.24 Profit maximum at $-\infty$

implying the solution $x^* = 0$. The profit function $-10x$ is graphed in figure 6.24, and with the positive half of the real line as the set of feasible x-values, we have the solution at the origin.

Monopoly with Output Quota

Consider a monopoly firm that sells to a foreign country, and suppose that the government of that country has imposed an upper limit L on its sales there. Let x denote sales in that country, $R(x)$ the revenue function from those sales, and $C(x)$ the cost function. Then the firm's problem is

$$\max \pi(x) = R(x) - C(x) \quad \text{subject to} \quad x \leq L$$

Strictly, of course, sales cannot be negative. We should also impose the constraint $x \geq 0$, but we assume that sales will turn out to be always positive. Applying theorem 6.4 gives

$$\pi'(x^*) = R'(x^*) - C'(x^*) \geq 0 \quad \text{and} \quad (L - x^*)\pi'(x^*) = 0$$

Since we do not have specific numerical functions, all we can do is consider logical possibilities:

1. $x^* < L$. In this case, $\pi'(x^*) = 0$. The constraint in this case is **nonbinding:** the firm is able to do what it wanted to do anyway, which is to set output at the profit-maximizing level.
2. $x^* = L$. In this case, $\pi'(x^*) \geq 0$. We now have two subcases:
   (a) $\pi'(x^*) > 0$. In this case, the firm *is* constrained by the output quota. It would like to expand output because its marginal profitability is positive

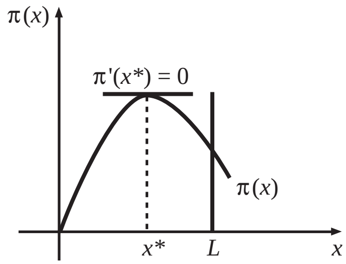


Figure 6.25 Monopoly subject to a quota: $x \leq L$ is not binding

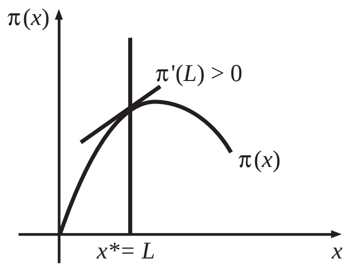


Figure 6.26 Monopoly subject to a quota: $x \leq L$ is binding

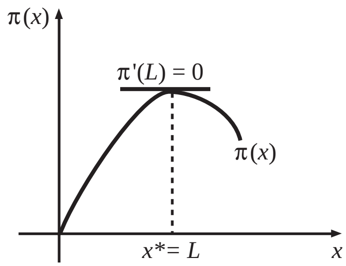


Figure 6.27 Monopoly subject to a quota: $x \leq L$ is a trivial constant

but it cannot exceed the quota. Note an important point of economic interpretation: if the quota were increased slightly, then the firm's profit would increase at the rate $\pi'(x^*)$. That is, taking the differential of profit and setting $x^* = L$,

$$d\pi = \pi'(L)\,dL, \quad \text{and so} \quad \frac{d\pi}{dL} = \pi'(L)$$

Thus we could say that the marginal value of a relaxation in the quota, or the **shadow price** of the quota, is $\pi'(L)$.

(b) $\pi'(x^*) = 0$. In this case, we would say that the firm is "trivially constrained" by the quota, in the sense that although $x^* = L$, the firm is able to set its profit-maximizing output. If the quota were slightly increased, the firm would not change its output, since x^* maximizes profit. Its profit would therefore also not increase. We would say that the shadow price of the output quota is zero in this case (at least for *increases* in the quota).

Figure 6.25 corresponds to case 1, figure 6.26 corresponds to case 2(a), and figure 6.27 corresponds to case 2(b).

Price-Regulated Monopoly

A monopoly has the linear demand function $x = a - bp$, where x is demand and p price, and a linear total-cost function $C = cx$. It is therefore just like the monopoly studied in section 6.1. However, this monopoly is not free to charge whatever price it likes. A regulatory agency sets a maximum price $\bar{p}$ that it may charge. We must assume that $\bar{p} \geq c$; otherwise, the firm would close down. We can write the monopoly's revenue function as

$$R(p) = px = ap - bp^2$$

and its cost function as

$$C = cx = c(a - bp) = ca - cbp$$

Then we could formulate its problem as

$$\max \pi(p) = ap - bp^2 - [ca - cbp] \quad \text{subject to} \quad p \leq \bar{p}$$

Applying theorem 6.4, we obtain

$$\pi'(p^*) = a - 2bp^* + cb \geq 0 \quad \text{and} \quad (\bar{p} - p^*)\pi'(p^*) = 0$$

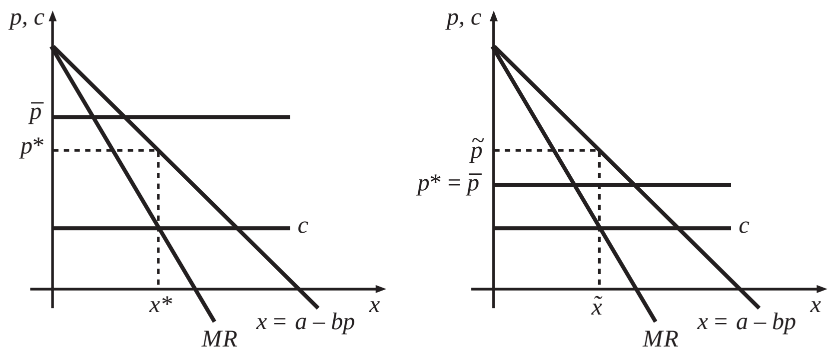
Then again, we have two cases:

1. $p^* < \bar{p}$. In this case, $\pi'(p^*) = 0$, implying that

$$p^* = \frac{a + cb}{2b}$$

   This is, of course, the monopoly's profit-maximizing price (confirm from the earlier example). The price regulation is nonbinding in that it allows the firm to maximize profit. So what appears to be good news for consumers—that the firm prices at a lower level than is permitted by the regulator—is simply a sign that the regulation is ineffective.
2. $p^* = \bar{p}$. In this case, either the regulation is a binding constraint, with $\pi'(\bar{p}) > 0$, or it is trivially binding, with $\pi' = 0$. In the first of these cases, we can take $\pi'(\bar{p})$ as the shadow price to the firm of the regulatory price constraint.

These solutions are illustrated in figure 6.28, where $\bar{p}$ and $\bar{x}$ are the monopoly's unconstrained profit-maximizing price and quantity respectively.



(a) **(b)**

Figure 6.28 (a) Price-regulation nonbinding; (b) price-regulation binding

EXERCISES

1. Solve the following problems:

(a) $\max 3 + 2x$ subject to $0 \leq x \leq 10$

(b) $\max 1 + 10x^2$ subject to $5 \leq x \leq 20$

(c) $\min 5 - x^2$ subject to $0 \leq x \leq 10$

Sketch your solutions. How do these solutions differ from the case in which x is unconstrained?

2. A firm has the cost function

$$C = \begin{cases} cx, & 0 \le x \le \bar{x} \\ \infty, & x > \bar{x} \end{cases}$$

That is, it has a fixed production capacity $\bar{x}$, below which marginal cost is constant, at c.

(a) Sketch the firm's marginal- and average-cost function.

(b) Solve for its profit-maximizing output if it sells in a perfectly competitive market.

(c) Describe the solution possibilities for output if the firm is a profit-maximizing monopoly with linear demand.

(d) Identify the "shadow price of capacity" in each of cases (b) and (c).

3. In a competitive but regulated agricultural market, the price of output is $1 per unit, a farm has the constant marginal cost of $0.20 per unit, and there is an output quota of 100 units.

(a) What is the shadow price of the output quota?

(b) Suppose that the farmer wishes to sell his output quota. How much can he ask for it? (Assume that all farmers have identical marginal costs.)

CHAPTER REVIEW

Key Concepts

constrained extrema
extreme values
first-order conditions
global maximum (minimum)
interior solution
local maximum (minimum)
necessary condition
nonbinding constraint
nth derivative test
points of inflection
second-order conditions
shadow price
stationary value
sufficient condition
unconstrained extrema

Review Questions

1. Distinguish between *local* and *global* optima.

2. What is the first-order condition for a maximum?

3. What is the first-order condition for a minimum?

4. What is a sufficient condition for a maximum or minimum, making use of the second derivative?

5. What is the nth derivative test and why is it necessary?

6. When x is restricted to an interval, why is it no longer a necessary condition for a maximum or a minimum that the first derivative is zero at the optimal point?

7. Explain what is meant by *binding* and *nonbinding* constraints.

8. State and explain the first-order conditions for maximization and minimization over an interval.

9. Explain what is meant by a *shadow price* of a constraint.

Review Exercises

1. Illustrate in a Venn diagram the relation between stationary values, extreme values, maxima, and minima.

2. Explain why the condition $f'(x^*) = 0$ is *necessary* but not *sufficient* for x^* to yield a maximum of f, while $f'(x^*) = 0$ and $f''(x^*) < 0$ is *sufficient* but not *necessary* for x^* to yield a maximum.

3. Find the stationary values of the following functions, and determine whether they give maxima, minima, or points of inflection.

(a) $y = 0.5x^3 - 3x^2 + 6x + 10$

(b) $y = x^3 - 3x^2 + 5$

(c) $y = x^4 - 4x^3 + 16x$

(d) $y = x + 1/x$

(e) $y = x^3 - 3x - 1$

(f) $y = 3x^4 - 10x^3 + 6x^2 + 5$

(g) $y = (1 - x^2)/(1 + x^2)$

(h) $y = (3 - x^2)^{1/2}$

(i) $y = (2 - x)/(x^2 + x - 2)$

(j) $y = x^{0.5}e^{-0.1x}$

4. The market value of a stock of wine grows over time, $t \in \mathbb{R}_+$, according to the function $v(t)$, with $v' > 0$, $v'' < 0$, all t. The present value of the stock of wine is given by

$$V = v(t)e^{-rt}$$

where r is the interest rate. Find and interpret an expression for the point in time at which the present value of the wine is at a maximum.

5. A profit-maximizing firm has the total-cost function

$$C = 0.5x^3 - 3x^2 + 6x + 50$$

and sells into a competitive market on which the price is \$1.00. What output should it produce? What would your answer be if the price were \$2?

6. A firm wants to bid for the monopoly franchise to sell hot dogs at a baseball game. It estimates the inverse demand function for hot dogs as

$$p = 5 - 0.5x$$

where p is the price in dollars and x is sales of hot dogs in thousands. It also estimates that it can supply the hot dogs at a constant unit cost of \$0.50 per hot dog. What is the largest bid it would make for the franchise?

7. In the case described in question 6, suppose that the stadium owners decide to levy a royalty of \$0.25 per hot dog sold. Show the effect this has on your answer.

8. Now suppose that in addition to the royalty, the stadium owners, to prevent “price-gouging,” set an upper limit of \$2 on the price that can be charged for a hot dog. Show the effect this has on the maximum bid for the franchise.

9. A student is preparing for exams in two subjects. She estimates that the grades she will obtain in each subject, as a function of the amount of time spent working on them are

$$g_1 = 20 + 20\sqrt{t_1}$$

$$g_2 = -80 + 3t_2$$

where g_i is the grade in subject i and t_i is the number of hours per week spent in studying for subject i, $i = 1, 2$. She wishes to maximize her grade average $(g_1 + g_2)/2$. She cannot spend in total more than 60 hours studying in the week. Find the optimal values of t_1 and t_2 and discuss the characteristics of the solution. Why is this essentially an *economic* problem? [Hint: Assume that 60 hours a week is a binding constraint and express the problem as one involving t_1 only.]

10. A firm sells in a competitive market at a price of $10 per unit and has the production function

$$x = 2L^{1/2}$$

where x is output and L is labor. It has a maximum available labor supply of 16 units. What is its shadow price of labor? Now suppose the firm could hire additional labor at a wage rate of $2 per unit. How much labor would it want to hire?

Part III Linear Algebra

Chapter 7 Systems of Linear Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Open-Economy IS-LM-BP Model
- Gauss-Jordan Elimination
- Open Economy Equilibrium with a Flexible Exchange Rate: Example

In chapter 2 we defined a *linear function* as one that takes the form

$$y = a + bx \tag{7.1}$$

for known constants a and b, and where x is the *independent variable* that takes on values over some specified domain, and y is the resulting value of the function at each x-value. We also know that by taking specific values of x, we can draw the graph of x and y in a two-dimensional picture. The graph is a straight line: hence the phrase *linear function*. There are many examples of functions in economics that can be represented in a linear form. The market demand for a product, for example, may simply be represented by a straight line. In this case, y would represent the quantity demanded and x would be the unit price. We would also expect a to be positive and b negative.

In this chapter we take the analysis of linear functions further by looking at solutions of systems of linear equations. In many economic problems a single linear equation identifies or characterizes a relationship between two variables, x and y. Given a value of x, we can deduce the associated value of y implied by the equation. Often, however, there are two or more equations that must be satisfied *simultaneously*. In the study of a simple market, for example, we may specify another linear relationship between quantity and price representing the supply. The solution is then a price that equates demand and supply.

This chapter is concerned with methods for finding solutions to two or more linear equations.

7.1 Solving Systems of Linear Equations

There are many ways of solving systems of linear equations. Some of these require the use of matrix algebra and are discussed in chapters 8 and 9. Others, particularly suitable for systems with small numbers of equations, simply require direct manipulation of the equations and are discussed here. These methods may be familiar already.

Graphing Solutions

We start by considering systems of equations with just two variables, x and y. This allows us to illustrate the solutions graphically in x, y-space.

Notice first that *any* linear equation with two variables x and y can be written in the form

$$\alpha x + \beta y = \gamma \tag{7.2}$$

where α, β, and γ are known constants.

Example 7.1 Equation (7.1) may be written

$$-bx + y = a$$

so that in terms of equation (7.2), $\alpha = -b$, $\beta = 1$, and $\gamma = a$. ■

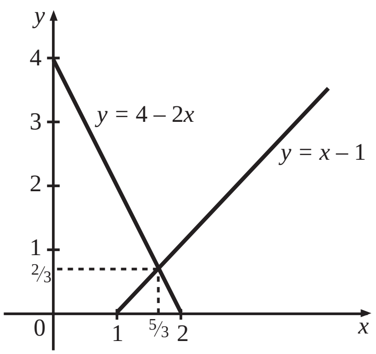


Figure 7.1 Graphs of $y = 4 - 2x$ and $y = x - 1$

Now consider the two linear equations

$$\begin{aligned} 2x + y &= 4 \\ x - y &= 1 \end{aligned}$$

If these equations are to be satisfied simultaneously, then there must be at least one pair of values for x and y which make both equations true. We can easily verify that $x = 5/3$ and $y = 2/3$ will satisfy both equations. However, $x = 1$ and $y = 2$ do not form a solution, since they hold for the first equation but not the second. The graphs of these two equations are shown in figure 7.1. The lines represent the equations $y = 4 - 2x$ and $y = x - 1$. Where the two lines intersect is exactly the solution we identified: $x = 5/3$ and $y = 2/3$. Notice that this is the *only* solution.

Example 7.2 Find a solution to the two linear equations

$$\begin{aligned} x + y &= 10 \\ x - y &= 0 \end{aligned}$$

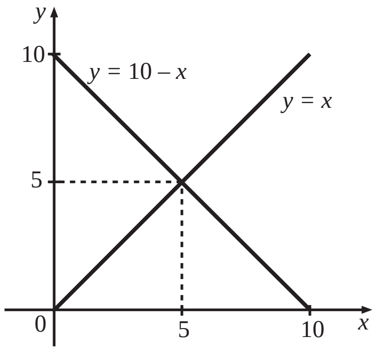


Figure 7.2 Graphs of $y = 10 - x$ and $y = x$

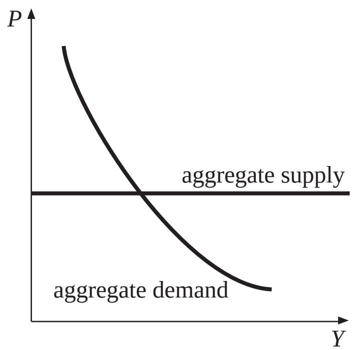


Figure 7.3 The aggregate demand and supply conditions implied by the simple IS-LM approach

Solution

Start by graphing the two equations, $y = 10 - x$ and $y = x$ (see figure 7.2). Clearly, the solution is $x = 5$ and $y = 5$, which can be verified by substituting these values into the two equations. Again, this is the *only* solution. ■

The Linear IS-LM Model

A common application of linear models in undergraduate economics courses is to the study of macroeconomic equilibrium. In particular, IS-LM models study the determination of national income, or national output, on the assumption that the amount supplied in the economy is determined solely by the level of aggregate demand. Figure 7.3 summarizes the aggregate demand and supply relationship presumed by this approach. The idea is that we can approximate some aspects of macroeconomic behavior by assuming that prices are fixed. Alternatively, we are assuming that the supply curve is perfectly elastic so that, whatever the price, output is determined by the position of the demand curve. The exercise then becomes one of finding the determinants of aggregate demand at a given price.

Typically the demand conditions of the economy are modeled as depending on aggregate expenditure components identified by their source. Thus we have expenditures by private consumers and private enterprises, expenditures by government, and net expenditures on home-produced goods by overseas residents—*net exports*. Ignoring the government and net exports for the moment, it is assumed that expenditures depend on both total income, Y, and the interest rate, R. The higher is income the higher are expenditures, while the higher is the interest rate, the lower are expenditures. The consumption function explains the former effect, while the investment decision may be used to explain the latter. Expenditures increase with income as long as output as a whole is regarded as a "normal good," while an increase in the cost of borrowing (or increase in the interest on savings) inhibits spending on large expenditure items such as machinery and consumer-durables. The so-called IS curve is a negative relation between the interest rate and output: the higher is the interest rate, the lower is demand, and the lower is output. A simple linear form is

$$R = \alpha - \beta Y$$

Of course, this only tells us what the interest rate will be if we know the level of income. Another relationship between R and Y is needed to determine both R and Y simultaneously.

The interest rate is assumed to be determined largely by money-market forces. Individuals hold money in order to make purchases (which depends on income), but their holding of money will otherwise be reduced if the cash could be earning a high interest rate. Thus the demand for money depends positively on income (expenditures) and negatively on the interest rate. If the amount of money available is fixed (determined by the government or the central bank), then total demand for money as income increases can only stay equal to the fixed supply of money if the interest rate increases. This gives a positive relationship between R and Y to maintain equilibrium in the money market. This is the so-called LM curve and a simple linear form is

$$R = -\gamma + \lambda Y$$

We will be using the expenditure components and the story of money market activity to derive the IS and LM curves in greater detail below. For now, we consider an example which starts with the IS and LM curves for a hypothetical economy.

Example 7.3 The Linear IS-LM Model

An economy has a linear IS curve given by the equation

$$R = 10 - 2Y \tag{7.3}$$

and an LM curve given by

$$R = -8 + 4Y \tag{7.4}$$

Here R is the interest rate (in percent), and Y is GDP or aggregate income (in billions of dollars). Find the equilibrium interest rate and level of GDP.

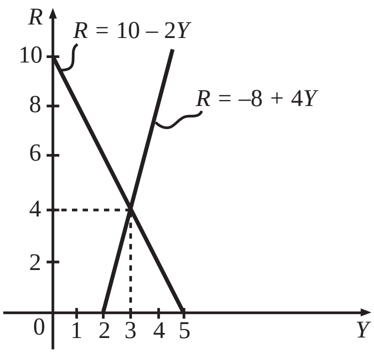


Figure 7.4 The linear IS-LM model of example 7.3

Solution

The equilibrium interest rate and GDP level solve both equations (7.3) and (7.4) simultaneously. The graphs of these two equations are shown in figure 7.4, where the solution is found to be $R = 4$ percent and $Y = \$3$ billion. Again, the solution is unique. ■

Now consider the two linear equations

$$y = 2 + x$$
$$y = 1 + x$$

If we draw these two equations, we see that they are parallel and so *never* cross. That is, there are *no* values of x and y that satisfy these equations simultaneously. There is *no solution*. In general, this will be true for any two equations whose graphs have the same slope, but different intercepts. (In this case, both equations have slopes of $+1$ but different intercepts [2 and 1 respectively].)

Example 7.4 **Linear Indifference Curves**

Consider a consumer of two goods, x and y, with preferences which give linear indifference curves

$$x + y = a \quad \text{or} \quad y = a - x \tag{7.5}$$

In other words, the consumer regards the two goods as *perfect substitutes*. Here the utility function is $u(x, y) = x + y$ and a is a given utility level. Now suppose that the individual has a budget constraint

$$M = p_x x + p_y y \tag{7.6}$$

where M is money income and p_x, p_y are the prices of the two goods. This says that total expenditures must equal total income. Rearranging this gives

$$y = m - px$$

where m $(= M/p_y)$ is *real income* measured in units of y and p $(= p_x/p_y)$ is the relative price ratio. Find the consumer's utility-maximizing choice of x and y.

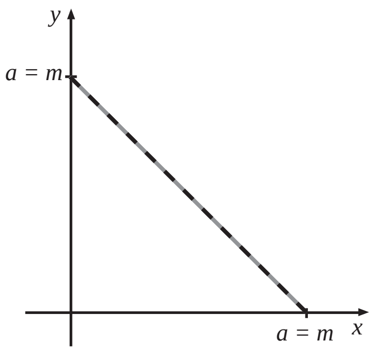


Figure 7.5 Linear indifference curves

Solution

If we suppose, first, that $p = 1$, then we see that the highest possible indifference curve occurs where $a = m$, and we see that the budget line and the highest possible indifference curve coincide exactly as shown in figure 7.5.

It is clear from this result that there are infinitely many values of x and y that satisfy these equations. The economic intuition here is as follows: The consumer's preferences indicate that the two goods are regarded as perfect substitutes with a marginal rate of substitution of -1. The budget constraint indicates that the goods have the same price: the slope is -1. Therefore there is nothing to choose between any combinations of the goods yielding the same level of utility and involving the same level of expenditures.

To continue with this example, we see that if $p_x > p_y$ so that $p > 1$, then there is exactly one solution in which only good y is bought, while if $p_x < p_y$ so that $p < 1$, then there is exactly one solution in which only good x is bought. The intuition here is very clear. Given that the goods are perfect substitutes, the consumer will only be concerned with their relative prices, and will simply consume only the relatively cheaper good. ■

Example 7.4 illustrates an important and quite general property of all linear systems.

Theorem 7.1 A system of two linear equations has either no solution, exactly one solution, or infinitely many solutions.

A moment's reflection shows this must be true. Two straight lines can be drawn so that they never intersect, or so they intersect just once, or so that they coincide. This exhausts the possibilities for graphing any two straight lines. (Try it!)

Solutions by Substitution and Elimination

Perhaps the simplest analytical (rather than diagrammatic) ways of solving these simple linear systems is by the *substitution* and *elimination* methods. Consider again the two equations represented in figure 7.1. These are

$$y = 4 - 2x \tag{7.7}$$

$$y = -1 + x \tag{7.8}$$

The analytical solution by substitution is obtained simply by solving one of these equations for either x or y and substituting the result in the other equation. For example, we can solve equation (7.8) for x to find $x = y + 1$ and then substitute this value for x in equation (7.7) to get $y = 4 - 2(y + 1)$, or $y = 2/3$. Substituting this value of y into either equation (7.7) or equation (7.8) yields the value for x of $x = 5/3$.

Example 7.5 Solve the following equations by substitution:

$$5x + 2y = 3$$

$$-x - 4y = 3$$

Solution

Rearrange the second equation to give $x = -3 - 4y$, and substitute this value into the first equation to give

$$5(-3 - 4y) + 2y = 3 \;\Rightarrow\; -15 - 20y + 2y = 3 \;\Rightarrow\; -18y = 18 \;\Rightarrow\; y = -1$$

Substituting this value of y into the second equation gives $x = 1$. ∎

The analytical solution by elimination can also be seen using equations (7.7) and (7.8) and in that case offers a more direct way of solving the equations, because the right-hand side of equation (7.8) can be set equal to the right-hand side of equation

(7.7) to *eliminate* y. We can extend the approach of solution by elimination by considering the equations in example 7.5 again, but now multiply the first of these through by 2 to give

$$10x + 4y = 6$$
$$-x - 4y = 3$$

and now *add* these two equations to eliminate y to give

$$9x = 9 \quad \text{or} \quad x = 1$$

Substitution of this value of x into either of the equations yields the solution for $y = -1$.

Example 7.6 Solve the following equations by elimination:

$$2x + 5y = -10$$
$$-2x + 4y = 0$$

Solution

To solve by elimination, simply add the two equations to obtain

$$9y = -10 \quad \Rightarrow \quad y = -\frac{10}{9}$$

Substituting back gives $x = -20/9$. ■

The elimination method is often more straightforward than the substitution method. With a large number of equations and variables removed, and thus solving for each variable in terms of the others and then substituting, long and cumbersome equations are avoided. The elimination method, however, reduces the number of variables by operations such as multiplication and addition performed on the equations. Later in this chapter we develop a more systematic approach to the manipulation of rows in equation systems.

Two very common applications of linear systems in undergraduate economics courses are in multimarket equilibrium and in the simple IS-LM model.

Equilibrium in Two Markets

In the model of multimarket equilibrium, the system of equations representing demand and supply in each market explicitly recognizes that the demand side, and possibly the supply side in each market may depend on prices in other markets.

For example, the demand for coffee may depend not only on the price of coffee but also on the price of *tea*—a substitute good. The demand for automobiles may depend on the price of automobiles and the price of *gasoline*—a complementary good. A firm's supply may also be affected by the prices of other goods for a variety of reasons. For example, a firm may use other produced goods as inputs, in which case that firm's supply is determined by the price it can obtain for its own output and the price charged by other firms producing goods required as inputs.

A general way of representing these interrelationships in a two-good model is

$$\left.\begin{aligned} q_1^s &= \alpha_1 + \beta_{11} p_1 + \beta_{12} p_2 \\ q_2^s &= \alpha_2 + \beta_{21} p_1 + \beta_{22} p_2 \end{aligned}\right\} \text{supply} \tag{7.9}$$

$$\left.\begin{aligned} q_1^d &= a_1 + b_{11} p_1 + b_{12} p_2 \\ q_2^d &= a_2 + b_{21} p_1 + b_{22} p_2 \end{aligned}\right\} \text{demand} \tag{7.10}$$

So, for example, if $\beta_{12} < 0$, then an increase in the price of good (input) 2 reduces the use of that good by firm 1 and so reduces the production of good 1. Setting supply equal to demand in each market gives the system of two equations to determine the two equilibrium prices p_1 and p_2. Note that some of the b_{ij}s and β_{ij}s may be zero. These equations are the basic building blocks of the model and are called the **structural equations:**

$$(b_{11} - \beta_{11}) p_1 + (b_{12} - \beta_{12}) p_2 = \alpha_1 - a_1$$

$$(b_{21} - \beta_{21}) p_1 + (b_{22} - \beta_{22}) p_2 = \alpha_2 - a_2$$

To solve by substitution, we first use the second equation to write p_1 in terms of p_2:

$$p_1 = \frac{(\alpha_2 - a_2) - (b_{22} - \beta_{22}) p_2}{(b_{21} - \beta_{21})}$$

Then we substitute into the first equation which, after rearranging, gives

$$p_2 = \frac{(b_{11} - \beta_{11})(\alpha_2 - a_2) - (b_{21} - \beta_{21})(\alpha_1 - a_1)}{(b_{11} - \beta_{11})(b_{22} - \beta_{22}) - (b_{21} - \beta_{21})(b_{12} - \beta_{12})}$$

This can then be substituted into the equation for p_1 to give

$$p_1 = \frac{(\alpha_1 - a_1)(b_{22} - \beta_{22}) - (\alpha_2 - a_2)(b_{12} - \beta_{12})}{(b_{11} - \beta_{11})(b_{22} - \beta_{22}) - (b_{21} - \beta_{21})(b_{12} - \beta_{12})}$$

These expressions for p_1 and p_2 are called the **reduced forms,** since they depend only on the parameters of the model. For particular values of the constant parameters a_i, α_i, b_{ij} and β_{ij}, $i, j = 1, 2$, we can then find values for the p_i, and subsequently, by substituting these values back into equations (7.9) and (7.10), we can find values for the equilibrium quantities. The following example illustrates how to find these values.

Example 7.7 Two-Market Equilibrium for Complementary Goods

Suppose that markets for two goods which are regarded by consumers as complements have the following demand and supply equations:

$$q_1^s = -1 + p_1, \quad q_1^d = 20 - 2p_1 - p_2 \qquad \text{(good 1)}$$

$$q_2^s = p_2, \qquad q_2^d = 40 - 2p_2 - p_1 \qquad \text{(good 2)}$$

where q_i^s and q_i^d are the quantities supplied and demanded of good i ($i = 1, 2$) and p_i is the price per unit of good i. The fact that the two goods are complements for each other is represented by the fact that the quantity demanded of each good falls the higher is the price of the other good. Derive the equilibrium prices of the two goods.

Solution

In equilibrium, $q_i^s = q_i^d$ for each good i. These conditions lead to the two equations

$$3p_1 + p_2 = 21$$
$$p_1 + 3p_2 = 40$$

Solving the second of these for $p_1 = 40 - 3p_2$ and substituting into the first gives

$$3(40 - 3p_2) + p_2 = 21 \Rightarrow 8p_2 = 99 \Rightarrow p_2 = 12.375$$

which, on substitution to find p_1, gives $p_1 = 2.875$. These two prices, and only these prices, will simultaneously give equilibrium in the markets for these two goods. ■

The Linear IS-LM Model—Again

We have already worked through one simple IS-LM example, but it is instructive to review where the equations representing the IS and LM sides of the economy originate. If C denotes aggregate consumption and Y aggregate income, then one form for the *consumption function* is

$$C = a + bY$$

where $a > 0$ and $0 < b < 1$. In the linear consumption function, b is called the *marginal propensity to consume*. The consumption function in this example is clearly linear in income. Ignoring the government for now, we identify investment spending by I and assume that it is a linear, decreasing function of the interest rate, R. The assumption that investment expenditure is negatively related to the interest rate can be related to the internal rate of return idea discussed in chapter 3. The higher is the cost of borrowing, R, the less likely it is that any given investment project will be profitable, and so investment spending falls as R increases. Specifically

$$I = e - lR, \qquad e, l > 0$$

We then use the idea that in equilibrium these expenditure components just absorb total output so that

$$\begin{aligned} Y &= C + I \\ &= a + bY + e - lR \\ &= \frac{a + e - lR}{1 - b} \end{aligned}$$

or

$$R = \frac{a + e - (1 - b)Y}{l}$$

This gives us the IS equation, and it clearly takes the linear form with $(a + e)/l$ being the intercept on the R axis and the slope being $-(1 - b)/l$. The consumption function and investment function are the building blocks or the *structural equations* of the IS curve. The graph of the IS curve traces out pairs of values for R and Y that are consistent with $Y = C + I$.

The money market is summarized by an equation for aggregate money demand

$$L = kY - hR, \qquad k, h > 0$$

and money supply is assumed fixed at $\bar{M}$. The money demand function and the money supply rule are the *structural equations* of the LM curve. Setting demand equal to supply and rearranging gives

$$R = \frac{kY - \bar{M}}{h}$$

as the LM equation. The graph of the LM curve traces out pairs of values for R and Y that are consistent with $L = M$, money demand = money supply.

Solving the IS and LM equations gives the reduced forms

$$Y = \frac{h(a+e) + l\bar{M}}{lk + h(1-b)}$$

$$R = \frac{k(a+e) - (1-b)\bar{M}}{lk + h(1-b)}$$

Questions can now be posed, using these equations, regarding the effects on equilibrium income and interest rate as, say, the money supply changes. These *comparative-static* methods are examined in more detail in chapter 14, but for now, we can write the reduced forms as linear functions of the money supply

$$Y = \alpha + \beta\bar{M}$$

$$R = \gamma - \lambda\bar{M}$$

where α, β, γ, and λ are constants given by

$$\alpha = \frac{h(a+e)}{lk + h(1-b)} > 0$$

$$\beta = \frac{l}{lk + h(1-b)} > 0$$

$$\gamma = \frac{k(a+e)}{lk + h(1-b)} > 0$$

$$\lambda = \frac{(1-b)}{lk + h(1-b)} > 0$$

(Before continuing, verify that each of these is positive.) Simply by inspecting the reduced forms, we see that an increase in the money supply will increase equilibrium income and will reduce the equilibrium interest rate.

For many questions of interest, this model is usually extended to include a government sector and to include international trade in goods and capital movements. The government is usually modeled as having a fixed level of expenditure, say $\bar{G}$, and as raising tax revenue by applying a proportional tax rate to income. If the (uniform) tax rate is t, then consumption takes place out of *disposable income*,

$(1-t)Y$. Question 4 in the exercises to this section provides an example. The open economy is rather more intricate, even in a simple model, since the forces that determine equilibrium will depend on the assumptions being made about the degree of capital mobility and about the exchange rate. A particularly simple version of the open-economy model is presented in example 7.9.

Example 7.8 Deriving the IS and LM Curves

From the following information about the structural equations of a closed economy, derive the IS and LM curves, and solve for equilibrium income and the interest rate:

$$\begin{aligned} C &= 50 + 0.8Y && \text{(consumption function)} \\ I &= 20 - 5R && \text{(investment function)} \\ L &= 100 - R + 0.5Y && \text{(money demand)} \\ M &= 200 && \text{(money supply)} \end{aligned}$$

Solution

Using $Y = C + I$, substitute for C and I to find

$$Y = 350 - 25R \qquad \text{(the IS equation)}$$

Using $L = M$, substitute to find

$$Y = 200 + 2R \qquad \text{(the LM equation)}$$

Solving these equations, we find that $R = 5.6$ and $Y = 211$ (approximately). ■

The following example, using a linear IS-LM model, serves to illustrate a further point about these two-variable systems. That is, if we are looking for a solution for two variables, then if a solution exists, we only require *two* equations. If we have two variables and more than two equations, then the system is **overdetermined.**

Example 7.9 IS-LM in an Open Economy

Consider a small, open economy with a fixed exchange rate. The IS and LM curves are given by

$$R = 20 - Y \qquad \text{(IS)} \tag{7.11}$$

$$R = -5 + 4Y \qquad \text{(LM)} \tag{7.12}$$

where R is the interest rate (in percent) and Y is real GDP (in \$ billions). Additionally we are told that there is perfect capital mobility so that the economy must maintain an interest rate equal to the "world interest rate," R^w, which is beyond the influence of this small economy. Currently the world interest rate is

$$R^w = 15 \tag{7.13}$$

and the balance of payments (BP) curve is horizontal at $R^w = 15$. Find the equilibrium interest rate and income.

Solution

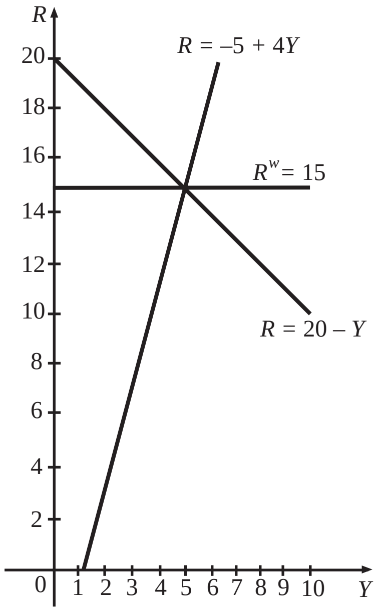


Figure 7.6 IS-LM in an open economy

This is shown in figure 7.6. For internal and external balance all three curves must intersect and we have chosen parameter values so that this is so. Internal balance is essentially achieved by the condition that R and Y satisfy both the IS and LM equations. External balance is obtained by the requirement that the R and Y combination are a point on the BP curve. Notice that, in this case, only two of the three curves are required to identify the equilibrium. We may use equations (7.11) and (7.12), or equations (7.11) and (7.13), or equations (7.12) and (7.13) and they all yield the same answer of $R^w = 15$ percent and $Y = \$5$ billion. The system is overdetermined and only two of the equations are required to find the solution.

Finally note that if the world interest rate were other than 15%, there would be no solution to the three conditions. Equations (7.11) and (7.12) would still give the internal balance equilibrium at $R = 15$ and $Y = 5$, but this would identify a point that is not on the BP curve if this is drawn at an interest rate greater than, or alternatively, less than 15%. In this case there would be a disequilibrium, requiring some economic mechanism to adjust to a situation of both internal and external balance. The adjustment mechanism depends on assumptions about the exchange rate and other considerations which we have suppressed here. ■

Where a system of two equations has more than two unknowns, then the system is said to be **underdetermined.** Consider the two equations

$$y = z - 2x$$
$$y = x - 1$$

By any of our earlier methods we obtain

$$x = \frac{1+z}{3}, \quad y = \frac{z-2}{3}$$

Equations (7.7) and (7.8) are equivalent to these equations with $z = 4$, and we can check our earlier work using the equations above. However, without the additional information about z we are not able to find numeric solutions, or rather, there are an infinite number of solutions—one for each possible value of z. In cases such as these, where the system is underdetermined, or determined only up to a variable being assigned a particular value, we say that x and y are the **basic variables** and z is the **free variable.**

EXERCISES

1. Graph the following pairs of equations and find their solutions:

(a)
$$2x = 4$$
$$3y - x = 10$$

(b)
$$x + y = 1$$
$$2x + 2y = 2$$

(c)
$$-x + 4y = 10$$
$$-x/4 + y = 5$$

(d)
$$y = 10 - x$$
$$y = 2x$$

2. Where possible, solve the following pairs of equations by substitution and by elimination:

(a)
$$x + y = 10$$
$$2x = 4$$

(b)
$$x + y = 0$$
$$x - y = 0$$

(c)
$$2x + 4y = 2$$
$$x + 2y = 1$$

(d)
$$2x + y = 8$$
$$x - z = 2$$

(e)
$$x/2 - y = \frac{3}{2}$$
$$x + 2y = 15$$

3. For which values of the constant, c, does the following system have **(a)** no solution, **(b)** one solution?

$$2x - y = 10$$
$$-cx + 2y = 5$$

4. A closed economy is described by the following simple IS-LM system:

$$\begin{aligned} C &= a + b(1-t)Y - lR && \text{(consumption)} \\ I &= \bar{I} && \text{(investment)} \\ G &= \bar{G} && \text{(government spending)} \\ L &= kY - hR && \text{(money demand)} \\ M &= \bar{M} && \text{(money supply)} \end{aligned}$$

where R and Y are the interest rate and real GDP, respectively, and $a > 0$, $0 < b < 1$, $0 < t < 1$, $k > 0$, and $h > 0$ are known constants. Find an expression for the linear IS curve and an expression for the linear LM curve. Then find algebraic solutions for R and Y.

5. In the two-market model in equations (7.9) and (7.10), what is the interpretation of the coefficient β_{21} being positive? Give an economic example of this kind of relationship.

6. There are two markets for goods that are regarded as substitutes and their supply and demand curves are given by

$$q_1^s = 2p_1, \qquad q_1^d = 20 - p_1 + p_2 \qquad \text{(good 1)}$$

$$q_2^s = -10 + 2p_2, \quad q_2^d = 40 - 2p_2 + p_1 \qquad \text{(good 2)}$$

Find the equilibrium prices and quantities of the two goods.

7. An economy has IS and LM curves given by

$$R = 25 - 2Y$$

$$R = -10 + \frac{M}{2} + Y$$

where R is the interest rate (percent), Y is GNP ($ billions), and M is the money supply.

(a) The government has a target for GNP of $7.5 billion. What level of money supply will achieve this and what is the resulting interest rate?

(b) The world interest rate is $R^w = 12$ and there is perfect capital mobility. What is the overall equilibrium? What has gone wrong?

7.2 Linear Systems in *n*-Variables

Although graphing solutions and finding solutions by simple substitution is fine for systems of equations with only two variables, we need to develop other procedures to find solutions for general systems of linear equations. These procedures often involve generalizations of some of the alternative solution methods that we have already referred to: *subtracting equations* and *multiplying equations by a constant*.

As we develop these methods, we will illustrate with 3-variable systems first and then, where necessary, show the general formulation in terms of n-variables.

First, we note immediately that theorem 7.1 generalizes to n-equation systems.

Theorem 7.2 A system of n linear equations has either no solution, exactly one solution, or infinitely many solutions.

Solution by Row Operations

The idea behind finding solutions by *row operations* is to transform a given system of equations into another with the same mathematical properties and hence the same solution. The aim is to transform a system in such a way as to produce a *simpler* system which is easier to solve. Three types of operations are permitted to transform a system:

1. Multiply an equation by a nonzero constant.
2. Add a multiple of one equation to another.
3. Interchange two equations.

In practice, not all of these operations may be required to simplify a given system, while in a complicated system it may be necessary to employ these methods repeatedly before a solution becomes apparent. Notice also that the first of these procedures includes multiplication by a reciprocal which, of course, amounts to *division* of an equation by a constant, while the second includes adding to one equation another equation which has been multiplied by −1 which amounts to *subtraction*.

Example 7.10 We will apply these rules to the following system of equations. The choice of operation at each stage is simply determined by inspection and by looking for patterns in the equations:

$$\begin{aligned} 4x - y + 2z &= 13 \\ x + 2y - 2z &= 0 \\ -x + y + z &= 5 \end{aligned}$$

Solution

Step 1 Add the second and third equations, and replace the third equation with the result (i.e., add the second equation to the third equation)

$$\begin{aligned} 4x - y + 2z &= 13 \\ x + 2y - 2z &= 0 \\ 3y - z &= 5 \end{aligned}$$

Step 2 Add the second equation to the first with the result

$$\begin{aligned} 5x + y &= 13 \\ x + 2y - 2z &= 0 \\ 3y - z &= 5 \end{aligned}$$

Step 3 Multiply the third equation by 2 and subtract from the second equation:

$$\begin{aligned} 5x + y &= 13 \\ x - 4y &= -10 \\ 3y - z &= 5 \end{aligned}$$

Step 4 Multiply the first equation by 4 and add to the second equation:

$$\begin{aligned} 21x &= 42 \\ x - 4y &= -10 \\ 3y - z &= 5 \end{aligned}$$

Step 5 Divide the first of these equations by 21:

$$\begin{aligned} x &= 2 \\ x - 4y &= -10 \\ 3y - z &= -5 \end{aligned}$$

Step 6 Subtract the second equation from the first:

$$\begin{aligned} x &= 2 \\ 4y &= 12 \\ 3y - z &= 5 \end{aligned}$$

Step 7 Divide the second equation by 4:

$$\begin{aligned} x &= 2 \\ y &= 3 \\ 3y - z &= 5 \end{aligned}$$

Step 8 Finally, subtract the third equation from three times the second equation:

$$\begin{aligned} x \qquad\qquad &= 2 \\ y \qquad &= 3 \\ z &= 4 \end{aligned}$$

■

We noted in the last section that systems of two linear equations may have no solution, exactly one solution, or infinitely many solutions. The same is true when there are $m > 2$ linear equations. The following provides an example of infinitely many solutions.

Example 7.11 Attempt to solve the system of equations:

$$\begin{aligned} -x - y + z &= -2 \\ 3x + 2y - 2z &= 7 \\ x + 3y - 3z &= 0 \end{aligned}$$

Solution

Step 1 Add the first and third equations:

$$\begin{aligned} -x - y + z &= -2 \\ 3x + 2y - 2z &= 7 \\ 2y - 2z &= -2 \end{aligned}$$

Step 2 Subtract the third equation from the second equation, and interchange the second and first equations:

$$\begin{aligned} 3x &= 9 \\ -x - y + z &= -2 \\ 2y - 2z &= -2 \end{aligned}$$

Step 3 Add three times the second equation to the first equation, and divide the first equation by 3:

$$\begin{aligned} x &= 3 \\ -3y + 3z &= 3 \\ 2y - 2z &= -2 \end{aligned}$$

■

The source of the difficulty is now apparent, since we are unable to reduce the second and third equations further to obtain unique solutions. Both equations imply that $y = z - 1$. Any values of y and z satisfying this relationship are solutions and, of course, there are infinitely many such values. So, even though we have a unique solution for $x = 3$, the system as a whole has an infinity of possible solutions. In this case, we say that the variables y and z are *not independent*. We also note that the dependence or the relationship between y and z is itself a **linear dependence,** and this is necessarily so. Through these steps, we have discovered another important property of systems of linear equations.

Definition 7.1 **Linearly dependent** equations are equations that may be derived from each other by a series of linear operations.

Theorem 7.3 If two or more *variables* in a linear system are (linearly) *dependent,* then two or more of the *equations* must be linearly dependent.

Example 7.11 illustrates this proposition since the first equation may be obtained by multiplying the second equation by -2, subtracting the third equation, and then dividing the result by 7. Thus, the first equation is a linear combination of the second and third equations.

Equations that are not linearly dependent are **linearly independent.** The equations in example 7.11 are linearly dependent.

The following example illustrates a case in which there is no solution to a system of equations:

Example 7.12 Consider the following system:

$$\begin{aligned} -x - y - z &= 0 \\ x + y + z &= 7 \\ 2x - 3y &= 0 \end{aligned}$$

Solution

Multiplying the first equation by -1 gives

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 7 \\ 2x - 3y &= 0 \end{aligned}$$

Clearly, the first and second equations imply that $0 = 7$, a contradiction. Hence there can be no values of x, y, and z that satisfy this system simultaneously. ■

Thus we have

Definition 7.2 A system of equations which yields no solution is said to be **inconsistent.**

The idea of an over- or underdetermined system of equations also can be generalized to more variables.

Definition 7.3 A system of m independent linear equations which is not inconsistent with n unknowns is **overdetermined** if $m > n$ and is **underdetermined** if $m < n$.

This leads to

Theorem 7.4 A consistent, underdetermined system of linear equations has infinitely many solutions.

Matrix Arrays

We will be discussing *matrix* manipulations in greater detail in the next two chapters. However, it is useful to introduce the idea of a matrix here, because matrices are a very convenient way of summarizing certain types of information. For our immediate purposes, a matrix is just another way of writing out the information contained in a system of linear equations such as that in example 7.10. Notice that as with the other examples, the key to finding the solutions involved identifying ways of manipulating the various *constants* so as to reduce the number of terms. For example, in the first step of example 7.10 we observed that the x in the second equation was multiplied by 1 while the x in the third equation was multiplied by -1 so that the procedure of adding these two resulting equations together produced an equation with an x multiplied by 0 (i.e., an equation with no x). In all steps the x, y, and z variables did not change position within an equation. By respecting the position of the variables and focusing only on the constants, we can summarize all the information at each step in the form of an *array* of constants. In the case of example 7.10, the array of interest is

$$\begin{bmatrix} 4 & -1 & 2 & 13 \\ 1 & 2 & -2 & 0 \\ -1 & 1 & 1 & 5 \end{bmatrix}$$

Because we are respecting the positions of the equations, we know that the first row of this array corresponds to the first equation, the second row to the second equation, and so on. Also, because we are respecting the position or the order of each variable within each equation, we know that the first column contains the constants which multiply the x variable in each equation. Since the constants to the right of the equality in each equation also play an important role in finding the solutions, these are also included in the array as the last column. Again, implicitly, we are reading an equality between the third and last columns in each row of the array.

Once we have written the information contained in the system of equations in matrix form, we can continue with the exercise of row reduction. However, now we do not need to write out the x, y, z, and $=$ at each stage.

Example 7.13 Solve the array

$$\begin{bmatrix} 4 & -1 & 2 & 13 \\ 1 & 2 & -2 & 0 \\ -1 & 1 & 1 & 5 \end{bmatrix}$$

Solution

Step 1 Add the second row to the last row:

$$\begin{bmatrix} 4 & -1 & 2 & 13 \\ 1 & 2 & -2 & 0 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

Step 2 Add the second equation to the first

$$\begin{bmatrix} 5 & 1 & 0 & 13 \\ 1 & 2 & -2 & 0 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

and so on down to

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

We have a solution when we have a diagonal of 1s in the first three columns, 0s in the off-diagonal positions, and a column of constants—the solutions—in the last column. ■

This last matrix is said to be in **reduced row-echelon form.** Moreover, for any matrix, despite the large number of row operations we could apply, we would always finish with the same row echelon form.

Theorem 7.5 Every matrix has a unique reduced row-echelon form.

Example 7.14 Solve the following equations by row operations:

$$\begin{aligned} 2x + z &= 10 \\ 2y &= z \\ z - 3y &= 6x \end{aligned}$$

Solution

Step 1 Write the equations in a way that stacks the variables in regular columns so that the variables follow the same order in each equation:

$$\begin{array}{rcrcrcr} 2x & & & + & z & = & 10 \\ & & 2y & - & z & = & 0 \\ -6x & - & 3y & + & z & = & 0 \end{array}$$

So the matrix is

$$\begin{bmatrix} 2 & 0 & 1 & 10 \\ 0 & 2 & -1 & 0 \\ -6 & -3 & 1 & 0 \end{bmatrix}$$

Step 2 Multiply the first row by 3 and add the third row:

$$\begin{bmatrix} 6 & 0 & 3 & 30 \\ 0 & 2 & -1 & 0 \\ 0 & -3 & 4 & 30 \end{bmatrix}$$

Step 3 Add the second row to the third row:

$$\begin{bmatrix} 6 & 0 & 3 & 30 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 3 & 30 \end{bmatrix}$$

Step 4 Multiply the third row by 2 and add to the second row:

$$\begin{bmatrix} 6 & 0 & 3 & 30 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 5 & 60 \end{bmatrix}$$

Step 5 Divide the third row by 5 and add to the second row:

$$\begin{bmatrix} 6 & 0 & 3 & 30 \\ 0 & 2 & 0 & 12 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

Step 6 Divide the second row by 2, multiply the third row by 3, and subtract from the first row:

$$\begin{bmatrix} 6 & 0 & 0 & -6 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

Step 7 Finally, divide the first row by 6 to obtain the reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 12 \end{bmatrix}$$

which gives the solution as $x = -1$, $y = 6$, and $z = 12$. ■

In identifying the reduced row-echelon form, we note the following features:

- Rows consisting entirely of zeros are collected at the bottom of the matrix.
- The first nonzero number in a row containing nonzero elements is 1, called the *leading 1*.
- In a row not consisting entirely of zeros, the leading 1 occurs one place to the right of the leading 1 of the preceding row.
- Each column containing a leading 1 has zeros elsewhere.

In general, suppose that we have n variables labeled $x_1, x_2, \ldots, x_n$, and m linear equations. The system may be written in equation form as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7.14}$$

and in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Notice that the constants a_{ij} have subscripts with a particular meaning. The first subscript indicates the *row* or equation number in which the constant is occurring. The second subscript indicates the *column* or variable number to which the constant is attached. This is of great importance, since deriving solutions by row operations requires that we respect the order of variables in the equation and that we keep a track of the equations.

Equilibrium in Three Markets

Extending the two-market example of the previous question to n markets is straightforward. Of course, the more markets there are, the more cumbersome the analysis becomes. In chapter 8 we will discover a neat way to write a system of demand and supply equations when there are n goods, and in chapter 9 we will learn how to solve this system for equilibrium prices using matrix algebra. The equivalent to equations (7.9) and (7.10) for the *three*-market case is

$$\begin{aligned} q_1^s &= \alpha_1 + \beta_{11} p_1 + \beta_{12} p_2 + \beta_{13} p_3 \\ q_2^s &= \alpha_2 + \beta_{21} p_1 + \beta_{22} p_2 + \beta_{23} p_3 \\ q_3^s &= \alpha_2 + \beta_{31} p_1 + \beta_{32} p_2 + \beta_{33} p_3 \end{aligned}$$

$$\begin{aligned} q_1^d &= a_1 + b_{11} p_1 + b_{12} p_2 + b_{13} p_3 \\ q_2^d &= a_2 + b_{21} p_1 + b_{22} p_2 + b_{23} p_3 \\ q_3^d &= a_3 + b_{31} p_1 + b_{32} p_2 + b_{33} p_3 \end{aligned}$$

Again, some of the b_{ij}s and β_{ij}s may be zero, thus reducing the number of interactions on the demand side and the supply side. Rather than derive the reduced form for these structural equations (which is tedious but follows the same logic as in the two-good case) we will proceed with an example and solve a particular system by finding its reduced row-echelon form.

Example 7.15 **Market Solution for Three Goods**

Consider the supply and demand functions for three goods given by

$$\begin{aligned} q_1^s &= -10 + p_1 \\ q_2^s &= 2p_2 \end{aligned}$$

$$q_3^s = -5 + 3p_3$$
$$q_1^d = 20 - p_1 - p_3$$
$$q_2^d = 40 - 2p_2 - p_3$$
$$q_3^d = 10 + p_2 - p_3 - p_1$$

Discuss the interdependencies apparent between these markets, write the equilibrium conditions, and derive the solutions for the equilibrium prices using the reduced row-echelon form.

Solution

Note first that there are no supply-side interdependencies present in these markets.

Setting supply equal to demand in each market and rearranging gives

$$\begin{array}{rcrcrcr} 2p_1 & & & + & p_3 & = & 30 \\ & & 4p_2 & + & p_3 & = & 40 \\ p_1 & - & p_2 & + & 4p_3 & = & 15 \end{array}$$

Expressing this in matrix form, we can arrive at the reduced row-echelon form through the following steps:

$$\begin{bmatrix} 2 & 0 & 1 & 30 \\ 0 & 4 & 1 & 40 \\ 1 & -1 & 4 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 1 & 30 \\ 0 & 4 & 1 & 40 \\ 0 & 2 & -7 & 0 \end{bmatrix} \quad \text{1st line } -2 \times \text{3rd line}$$

$$\begin{bmatrix} 2 & 0 & 1 & 30 \\ 0 & 4 & 1 & 40 \\ 0 & 0 & 15 & 40 \end{bmatrix} \quad \text{2nd line } -2 \times \text{3rd line}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 15 \\ 0 & 1 & 1/4 & 10 \\ 0 & 0 & 1 & 40/15 \end{bmatrix} \quad \text{1st line} \div 2\text{, 2nd line} \div 4\text{, 3rd line} \div 15$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 15 \\ 0 & 1 & 0 & 140/15 \\ 0 & 0 & 1 & 40/15 \end{bmatrix} \quad \text{2nd line } -1/4 \times \text{3rd line}$$

$$\begin{bmatrix} 1 & 0 & 0 & 205/15 \\ 0 & 1 & 0 & 140/15 \\ 0 & 0 & 1 & 40/15 \end{bmatrix} \quad \text{1st line } -1/2 \times \text{3rd line}$$

The equilibrium prices are $p_1 = 205/15$, $p_2 = 140/15$ and $p_3 = 40/15$. ■

Definition 7.4 A system of linear equations is **homogeneous** if each constant to the right of the equality is zero.

A homogeneous system therefore is generally written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array} \tag{7.15}$$

where, of course, some of the a_{ij} may also be zero. Notice immediately that $x_1 = x_2 = \cdots = x_n = 0$ is always a solution to a system of homogeneous linear equations. This solution is usually called the **trivial solution.** The following theorem states an important property of homogeneous linear equations:

Theorem 7.6 A system of homogeneous linear equations has *either* the trivial solution only, *or* an infinite number of solutions, including the trivial solution.

In particular, we know that

Theorem 7.7 A system of homogeneous equations with more unknowns than equations ($m < n$) is consistent and has infinitely many solutions.

Example 7.16 Solve the system of linear equations

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & - & x_2 & - & 2x_3 & = & 0 \\ x_1 & + & 4x_2 & & & = & 0 \end{array}$$

Solution

The last equation has solutions $x_1 = x_2 = 0$ and $x_1 = -4x_2$, while the second equation has solutions $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 + 2x_3$. The two nontrivial solutions in combination imply that

$$x_2 = -\left(\frac{2}{5}\right)x_3 \tag{7.16}$$

Now consider the first equation. It has solutions $x_1 = x_2 = x_3 = 0$ and $x_1 = -2x_2 - x_3$. This nontrivial solution, combined with the nontrivial solution for the second equation, implies that

$$x_2 = -x_3 \tag{7.17}$$

Now, equations (7.16) and (7.17) can only be satisfied simultaneously if $x_2 = x_3 = 0$. Hence we only have the trivial solution. ■

It turns out that in cases where there are solutions other than the trivial solution, there must be a linear dependence between the equations and, therefore, there must be an infinity of solutions.

Example 7.17 Find the solutions to

$$\begin{aligned} 6x_1 + 3x_2 - x_3 &= 0 \\ -4x_1 + 3x_2 + x_3 &= 0 \\ 5x_1 - x_3 &= 0 \end{aligned}$$

Solution

Clearly, $x_1 = x_2 = x_3 = 0$ is a solution. However, note that if we subtract the second equation from the first, we have

$$10x_1 - 2x_3 = 0$$

which is the third equation multiplied by 2. So, when we look at nontrivial solutions, theorem 7.6 applies and we have an infinity of nontrivial solutions. ■

EXERCISES

1. Which of the following systems are linearly dependent? For those systems that are linearly dependent, find the nature of the dependence. For the rest, find their solutions.

(a)
$$\begin{aligned} 2x + 4y - z &= 5 \\ y + z &= 2 \\ x + y + z &= 7 \end{aligned}$$

(b)
$$\begin{aligned} 4x - 2y + 2z &= 6 \\ -y + z &= 1 \\ 2x &= 2 \end{aligned}$$

(c)
$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= 2 \\ -2x_2 &= 3 \\ 2x_1 - x_2 + x_3 &= 9 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ -x_1 + x_2 - x_4 &= 1 \\ x_1 - x_2 + x_3 &= 1 \\ x_2 + x_3 &= 1 \end{aligned}$$

(e)
$$\begin{aligned} 2x_1 + x_3 - 2x_4 &= 5 \\ -x_1 + 2x_2 + x_3 &= 4 \\ x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - x_2 + 2x_3 - 5x_4 &= 4 \end{aligned}$$

2. Which of the following systems are inconsistent?

(a)
$$\begin{aligned} x + 2y + z &= 2 \\ x + y &= 3 \\ 2x + y + 2z &= -4 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 0 \\ 2x + 2y - z &= 0 \\ y + z &= 0 \end{aligned}$$

(c)
$$\begin{aligned} \frac{1}{2}x_1 - 2x_2 + x_3 &= 10 \\ x_2 - x_3 + 2x_4 &= 5 \\ -x_1 + 2x_3 &= 0 \\ x_2 &= -4 \end{aligned}$$

(d)
$$\begin{aligned} ax + by &= c \\ \alpha x + \beta y &= \gamma \end{aligned}$$

3. Write the following systems in array form and derive the reduced row-echelon form in each case. Then find the solutions to each set of equations.

(a)
$$\begin{aligned} 2x_1 - x_2 &= 0 \\ x_2 + x_3 + 2x_4 &= 100 \\ x_1 + 2x_2 + 2x_3 &= 60 \\ -x_1 + x_3 - x_4 &= -10 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 &= 20 \\ -x_1 + 2x_3 &= 10 \end{aligned}$$

4. A small, open economy with a flexible exchange rate has IS, LM, and BP curves given by

$$\begin{aligned} R &= 240 - 4Y + E \\ R &= -50 + Y \\ R &= 105 + \frac{1}{2}Y - 2E \end{aligned}$$

Solve, using row operations for the equilibrium interest rate R, output Y, and exchange rate E. (The exchange rate is defined as the price of domestic currency in terms of foreign currency.)

5. Solve the following system of excess demand functions for p_1, p_2, p_3, p_4:

$$\begin{aligned} 10 - p_1 - 2p_2 + 5p_3 - 2p_4 &= 0 \\ 5 - 2p_1 + p_2 - p_3 + 8p_4 &= 0 \\ 6 + 2p_1 - p_2 - 4p_3 - 9p_4 &= 0 \\ 20 - \frac{1}{2}p_1 - 2p_2 - 2p_3 - 2p_4 &= 0 \end{aligned}$$

CHAPTER REVIEW

Key Concepts

basic variables
free variables
homogeneous systems
inconsistent systems
linear dependence
overdetermined systems
reduced forms
reduced row-echelon form
structural equations
trivial solution
underdetermined systems

Review Questions

1. Why must a system of two linear equations have either no solution, exactly one solution, or infinitely many solutions?
2. What is meant by linear dependence in a system of equations?
3. What is meant by linear independence in a system of equations?
4. What information is conveyed by comparing the number of independent linear equations and the number of unknowns?

5. What is meant by a consistent system of linear equations?

6. Use Venn diagrams to show the relationship between consistent systems, inconsistent systems, overdetermined systems, and underdetermined systems.

7. What can be said about the appearance of an array (or matrix) that is in reduced row-echelon form?

8. What is the distinguishing feature of a homogeneous system of linear equations?

Review Exercises

1. Which of the following equations are linear in x?

(a) $y = a + b/x \quad a, b > 0$

(b) $y = a + b^x \quad a, b > 0$

(c) $y = x + e^x$

(d) $y = -x + b \quad b > 0$

(e) $y = \ln z + ax \quad a > 0$

2. Solve the following pairs of equations by substitution and by elimination.

(a)
$$\begin{aligned} y &= 24 - x \\ 2y &= 4 + 5x \end{aligned}$$

(b)
$$\begin{aligned} -y &= -8x - 4 \\ y &= 20x + 2 \end{aligned}$$

(c)
$$\begin{aligned} 0.5y + 2x &= 0 \\ -y + x &= 0 \end{aligned}$$

(d)
$$\begin{aligned} 0.5y + 2x &= 0 \\ -y - 4x &= 0 \end{aligned}$$

3. Which of the following systems are linearly dependent and which are inconsistent?

(a)
$$\begin{aligned} 2x + y - z &= 10 \\ 4y + 2z &= 4 \\ x &= 0 \end{aligned}$$

(b)
$$\begin{aligned} -y + z &= 0 \\ 4x + 2y - z/3 &= 0 \\ x + z &= 0 \end{aligned}$$

(c) $$\begin{aligned} -3x + 2y - z &= 14 \\ -x - y - z &= 0 \\ x + 10y - 3z &= 2 \end{aligned}$$

4. An economy has three markets with supply and demand functions for the three goods given by

$$\begin{aligned} q_1^s &= -20 + p_1 - 0.5p_2 \\ q_2^s &= -100 + 2p_2 \\ q_3^s &= p_3 \end{aligned}$$

$$\begin{aligned} q_1^d &= 80 - 2p_1 - p_3 \\ q_2^d &= 200 - p_2 \\ q_3^d &= 100 - 2p_3 - p_1 \end{aligned}$$

(a) Comment on the relationship between the three goods on the demand side.

(b) What is the nature of any production externality on the supply side?

(c) Solve for the equilibrium prices and quantities of the three goods.

5. An economy has an IS curve given by $R = 210 - 2Y$ and an LM curve given by $R = -M + Y/4$. The long-run equilibrium level of output must equal 100. What value of M makes the IS and LM curves intersect at $Y = 100$? What is the economic interpretation of a situation in which M exceeds this critical amount? What is the economic interpretation of a situation in which M is less than this critical amount?

Chapter 8 Matrices

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Migration
- Profit for a Multiproduct Firm

A matrix provides a very powerful way of organizing and manipulating data. In chapter 7 matrices were used to focus attention on the parameters and constants of a simultaneous-equation system. The rows of a matrix could be manipulated to find solutions to the unknown variables in the original equations using, for example, the Gauss-Jordan elimination approach. There are clearly many instances where a large amount of information can be summarized in matrix form. Moreover there are procedures or operations on matrices that allow us to discover important properties of systems of equations.

As with any area of mathematics, there are rules that guide us in manipulating matrices. These rules govern basic operations such as addition and multiplication, and more complicated operations designed to help solve systems of equations. In this chapter, we take our first steps in manipulating matrices.

8.1 General Notation

Before considering the basic rules of matrix algebra, we will examine some examples of how matrices are used to summarize information.

Example 8.1 The Prisoner's Dilemma

Game theory has been extensively used in economics to study the strategic interaction among decision makers. It can be argued that most of economic behavior can be studied using game-theoretic tools. In this example we illustrate the use of matrices as descriptions of the strategic interactions among agents that make up an economic game. Game theory in this sense is a generalization of standard one-person decision theory. In a game a rational individual, in making his/her best choice, would have to take into account the possible actions of others in the game. In other words, if we deal with a two-person game, each player would have to consider the problem faced by the other player in order to reach his/her best decision.

In this game we consider two players whose interests are only partially in conflict. Each of these players has two strategies: cooperate or defect. The two players are two suspects who have been implicated in a crime. However, the police do not have evidence against each individual but against both of them as a pair. The players (prisoners) could cooperate (strategy C) with each other and refuse to give evidence to the police against the other person or choose to defect (strategy D) and testify against the other person. These strategic interactions can be expressed in a **game matrix.** Since we are describing a game between two players, the game matrix will be 2×2. This is a table where each cell contains the payoff of each player for his/her chosen strategy. The rows of the matrix give the strategies of player A (C, D) and the columns those of player B (C, D). Each of the four cells of the matrix contain the payoffs for the combinations of the chosen strategies by each player. The game matrix describing these payoffs can be described as

$$
\begin{array}{cc}
 & \text{Player B} \\
 & \begin{array}{cc} \quad C \qquad & \qquad D \end{array} \\
\text{Player A} \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} (-1, -1) & (-4, 0) \\ (0, -4) & (-3, -3) \end{bmatrix}
\end{array}
$$

The first cell contains the payoffs for the pair of strategies where each player cooperates (C, C). In that case they would each get a sentence of one year in prison. This is the situation of a “plea-bargain,” where both players would plead guilty to a lesser charge and consequently get a lighter sentence than would have been the case if the original charge against them stood and they were convicted. If one were to defect and the other one were to cooperate, then the defector gets a pardon with a payoff of no prison term (since there is no evidence against him/her), and the player who did not confess against his/her partner would get severely punished and get a prison sentence of four years (since all the evidence about the crime is now pinned on him/her). This is the case for the cell (C, D) where player A cooperates and player B defects and for (D, C) where player A defects and player B cooperates. If they both defect, then each will get a sentence of three years each, since the police will have evidence against both individuals. The setup of the game is such that each player has an incentive to defect, irrespective of what he or she expects the other player will do. If player A believes that the other person will cooperate, then it pays off for player A to defect because he or she will go free by defecting. If player A thinks that the other person will defect, then of course it is also best to defect in order to go to prison for three years instead of four years. From a social point of view, taking into account their joint utility level, it would be best if they both cooperate, since they would each get a lighter sentence. However, if each individual acts on his/her own, then they would both defect and an equilibrium would be the pair of strategies (defect, defect) with payoffs $(-3, -3)$. ■

Example 8.2 Suppose that an automobile manufacturer produces output in five plants. The CEO wants a breakdown of the value of output produced in each plant in each of the preceding four quarters of the year, Q_1, Q_2, Q_3, and Q_4. A table providing this information (in millions of dollars) might look like this:

	Q_1	Q_2	Q_3	Q_4
Plant 1	5	3	10	12
Plant 2	6	5	9	15
Plant 3	7	5	8	14
Plant 4	17	13	22	31
Plant 5	32	17	35	44

Then enclosing the array of numbers in parentheses defines the table as a matrix:

$$A = \begin{bmatrix} 5 & 3 & 10 & 12 \\ 6 & 5 & 9 & 15 \\ 7 & 5 & 8 & 14 \\ 17 & 13 & 22 & 31 \\ 32 & 17 & 35 & 44 \end{bmatrix}$$ ■

This example illustrates that any "spreadsheet" type of data is really just an array of numbers, but notice that the position of each item is critical and it is important that we know the significance of each location in the array. To read the matrix A correctly, we must be aware that 22 represents the value of output produced by plant 4 in the third quarter.

Example 8.3 **Input-Output Matrix**

An economy consists of three industries: an agricultural industry, a mining industry, and a manufacturing industry. To produce one unit of agricultural output, the agricultural sector requires \$0.3 of its own output, \$0.2 of mining output, and \$0.4 of manufacturing output. To produce one unit of mining output, the mining sector requires \$0.5 of agricultural output, \$0.2 of its own output, and \$0.2 of manufacturing output. To produce one unit of manufacturing output requires \$0.3 of agricultural output, \$0.3 of mining output, and \$0.3 of its own output. The above information can be summarized in terms of a matrix that is known as the *input-requirements* matrix:

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$ ■

The model that is based on the input-requirements matrix is known as the *input-output model*. It looks at the economy as a system of interrelated industrial sectors. The industries are interrelated because an industry's output in general is used as an input into some other industry's production process as well as, possibly, finding its way into final demand by consumers. Therefore, in general, each industry is potentially the producer of an intermediate good that may also be used in final consumption. The problem is to find the production levels for each industry which, given a set of prices, are just sufficient to supply the total demands from industry and consumers. In chapter 9 we will investigate in more detail the input-output model and present a general way of obtaining its solution. We will also obtain the solution to the problem using the above input-requirements matrix (see example 9.20).

Definition 8.1

A **matrix** is a rectangular array of numbers enclosed in parentheses. It is conventionally denoted by a capital letter.

The numbers are the entries of the matrix. The number of rows (horizontal arrays) and the number of columns (vertical arrays) determine the dimensions of the matrix, which is also known as the *order* of the matrix. In example 8.2, A is of order 5×4 (five by four), whereas in example 8.3 and example 8.4, A is of order 3×3 (three by three). Note that when we express the dimension of a matrix, we always give the row dimension first.

In general, a matrix A of order $m \times n$ (m rows and n columns) can be explicitly written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Recall from chapter 7 that each element of the matrix in the general notation has as subscript the number of the row and column, respectively, that fully describe the position of the above element. For example, the element a_{43} describes the element that is to be found at the fourth row and the third column.

A matrix having only one row such as $(5 \;\; 3 \;\; 5 \;\; 4)$ is called a **row matrix.** A matrix having only one column such as

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

is called a **column matrix.** An alternative and very frequently used name for a row matrix is a **row vector** and for a column matrix, a **column vector.** Thus, an

n-component row vector is a matrix of order $1 \times n$, and an n-component column vector is a matrix of order $n \times 1$.

Definition 8.2 An array that consists of only one row or one column is known as a **vector.**

Vectors are special cases of matrices and obey all the rules that characterize the operations of matrices. Conventionally vectors are denoted by bold face lowercase letters, **a**, **b**, etc., in contrast to matrices denoted by capital letters.

Matrix Equality

Two matrices are equal if they have the same dimension *and* the corresponding elements are equal.

Example 8.4 Find the values of x and y if

$$\begin{bmatrix} 3 & 2 \\ x+y & 1 \end{bmatrix} = \begin{bmatrix} 3 & y \\ 2 & 1 \end{bmatrix}$$

Solution

By comparing the corresponding entries, we have

$$y = 2 \quad \text{and} \quad x + y = 2$$

Therefore $x = 0$ and $y = 2$ for the two matrices to be equal. ■

Square Matrices

Definition 8.3 A matrix that has the same number of rows and columns is called a **square matrix.**

The matrices in example 8.4 were therefore square.

Definition 8.4 Any square matrix that has only nonzero entries on the main diagonal and zeros everywhere else is known as a **diagonal matrix.**

Thus an $n \times n$ matrix A is a diagonal matrix if $a_{ij} = 0$ for all $i, j = 1, \ldots, n$ such that $i \neq j$.

A special case of a diagonal matrix is the **identity matrix,** usually represented by the letter I_n, where the subscript n denotes the order of the matrix (the number of rows and columns of the matrix). For example, I_3 is given by

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix, as we will see later when we examine matrix multiplication, plays the same role in matrix algebra as the number 1 in the algebra of real numbers.

Definition 8.5 A square matrix with all its entries being zero is known as the **null matrix.**

The null matrix plays a similar role in matrix algebra as does zero in the algebra of real numbers.

EXERCISES

1. Find the values of x and y if

$$\begin{bmatrix} 1 & 2 \\ x - y & 2 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 2 \end{bmatrix}$$

2. Find the values of x and y if

$$\begin{bmatrix} 3 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 3 & y \\ 2 & 1 \end{bmatrix}$$

3. Find the values of x and y if

$$\begin{bmatrix} 3 & 4 & x \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 4 & y \\ 2 & 5 & 7 \end{bmatrix}$$

4. Find the values of x and y if

$$\begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$$

are to be equal.

5. Find the values of y and z if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & y & z \\ 0 & y & z \\ y & 1 & z \end{bmatrix}$$

are to be equal.

8.2 Basic Matrix Operations

The rules for matrix operations ensure that we make changes to matrices that satisfy mathematical logic. Not all operations are well defined for all matrices, and in particular, the dimensions or the order of matrices will need to satisfy some basic conditions before they can be manipulated.

Addition and Subtraction of Matrices

Addition and subtraction are well defined matrix operations only if the matrices involved are of the same order. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 7 & 1 \end{bmatrix}$$

is well defined. Alternatively, we may say the two matrices on the left of the equality are **conformable** for addition. On the other hand

$$\begin{bmatrix} 3 & 4 & 1 \\ 6 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -5 & 8 \end{bmatrix}$$

is not well defined, or, not conformable for addition.

Example 8.5 **Car Production**

A car manufacturer who produces three different models in three different plants A, B, and C reaches the production levels in millions of dollars in the first and second half of the year as follows:

		First half	
	Model 1	Model 2	Model 3
Plant A	27	44	51
Plant B	35	39	62
Plant C	33	50	47

	Second half		
	Model 1	Model 2	Model 3
Plant A	25	42	48
Plant B	33	40	66
Plant C	35	48	50

Summarize this information in matrix form and find total production by each plant for the whole year.

Solution

The information above can be given in matrix form, where F represents the data for the first half and S for the second half:

$$F = \begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix}, \quad S = \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix}$$

Obtaining the total production levels for the whole year involves adding the elements in the corresponding positions

$$F + S = \begin{bmatrix} 27+25 & 44+42 & 51+48 \\ 35+33 & 39+40 & 62+66 \\ 33+35 & 50+48 & 47+50 \end{bmatrix}$$
$$= \begin{bmatrix} 52 & 86 & 99 \\ 68 & 79 & 128 \\ 68 & 98 & 97 \end{bmatrix}$$

■

In general, this leads to

Definition 8.6 The sum of two matrices is a matrix, the elements of which are the sums of the corresponding elements of the matrices.

More specifically, for two matrices of order $m \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & \cdots & a_{2n} + b_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

If we call these matrices A, B, and C respectively, then we have the **matrix equation**

$$A + B = C$$

where $c_{ij} = a_{ij} + b_{ij}$ for all i, j-pairs, with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Matrix subtraction, also, can only be performed on matrices that are of the same order, by subtracting the corresponding elements of the two matrices. In example 8.5, if we want to find the change in production levels between the first and second half of the year, we look at $F - S$:

$$F - S = \begin{bmatrix} 27 & 44 & 51 \\ 35 & 39 & 62 \\ 33 & 50 & 47 \end{bmatrix} - \begin{bmatrix} 25 & 42 & 48 \\ 33 & 40 & 66 \\ 35 & 48 & 50 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & 3 \\ 2 & -1 & -4 \\ -2 & 2 & -3 \end{bmatrix}$$

If we denote the matrix differences by D, then we have the matrix equation

$$F - S = D$$

where $d_{ij} = a_{ij} - b_{ij}$ for all i, j-pairs with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. It is clear from the definition of matrix addition and subtraction that these operations may only be performed on matrices with the same number of rows and the same number of columns.

Scalar Multiplication

In matrix algebra, real numbers are called **scalars.** Multiplying a matrix by a scalar is known as scalar multiplication.

Definition 8.7 **Scalar multiplication** is carried out by multiplying each element of the matrix by the scalar.

For example, if A is given by

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \text{then} \quad 3A = \begin{bmatrix} 6 & 3 \\ 9 & 6 \end{bmatrix}$$

In fact, since in ordinary algebra for any real number x, we have that $3x = x + x + x$, the same logic applies to matrices, where $3A = A + A + A$, as can be seen here:

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 9 & 6 \end{bmatrix}$$

The negative of a matrix A, $-A$, is the matrix obtained from A by multiplying all of its elements by -1. If we have that

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \text{then} \quad -A = \begin{bmatrix} -2 & -1 \\ -3 & -2 \end{bmatrix}$$

Just as with real numbers, we can think of matrix subtraction as being defined by scalar multiplication and addition, since

$$A - B = A + (-B)$$

Matrix Multiplication

In the light of the definitions of matrix addition and subtraction, the reader would be forgiven for guessing that matrix multiplication would be defined by the rule of multiplying the corresponding elements of two matrices of the same order. This is emphatically *not* the case. The appropriate rule is more complicated than that and at first sight not easy to rationalize. The reason it takes the form it does can be suggested by considering two simultaneous equations:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

where x_1 and x_2 are the unknowns to be determined and the a_{ij} and b_i are known constants. It is very convenient to be able to express this in matrix form as follows: Define the matrices

$$A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and write the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

For this to describe the above pair of simultaneous equations, we have to define matrix multiplication in a particular way. First, multiply the first row of A by the column vector $\mathbf{x}$. To do this, we take each element of the first row of A and multiply it by the corresponding element in the vector $\mathbf{x}$: the first element of the first row of A is multiplied by the first element in the (column) vector and the second element of the first row of A is multiplied by the second element of the (column) vector. We then add these two products to obtain

$$a_{11}x_1 + a_{12}x_2$$

Then multiply the second row of A by the column vector $\mathbf{x}$ to obtain

$$a_{21}x_2 + a_{22}x_2$$

If multiplication is defined in this way, then the very compact matrix equation does give the simultaneous equation system, since

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The rule for multiplying matrices is essentially a generalization of the above reasoning.

To multiply matrices, it is not necessary that they be of the same order. The requirement is that the number of columns of the first matrix be the same as the number of rows of the second matrix. Matrices that satisfy this requirement are said to be **conformable** for matrix multiplication.

Before we present the formal definition of matrix multiplication we will illustrate the idea by means of some examples.

Example 8.6 Total Revenue

Determine the revenue of a parking lot on a given Monday, Tuesday, and Wednesday based on the following data:

	Number of cars	Number of buses
Monday	30	5
Tuesday	25	5
Wednesday	35	15

The dollar charge per vehicle is \$4 for cars, and \$8 for buses. The daily revenue in dollars is m on Monday, t on Tuesday, and w on Wednesday. In matrix notation this information can be put as

$$\begin{aligned} \text{Monday vector} &= [30 \quad 5] \\ \text{Tuesday vector} &= [25 \quad 5] \\ \text{Wednesday vector} &= [35 \quad 15] \\ \text{Charge vector} &= \begin{bmatrix} 4 \\ 8 \end{bmatrix} \end{aligned}$$

Solution

To obtain m, we need to calculate

$$[30 \quad 5]\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 30(4) + 5(8) = 160$$

For t, we calculate

$$[25 \quad 5]\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 25(4) + 5(8) = 140$$

For w, we calculate

$$[35 \quad 15]\begin{bmatrix} 4 \\ 8 \end{bmatrix} = 35(4) + 15(8) = 260$$

The information for the total number of vehicles (TV) in the parking lot during the three days in question and the charge per vehicle (CV) are given as

$$\text{TV} = \begin{bmatrix} 30 & 5 \\ 25 & 5 \\ 35 & 15 \end{bmatrix} \quad \text{and} \quad \text{CV} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

The revenue per day is then obtained as (total number of vehicles) × (charge per vehicle) or

$$\begin{bmatrix} m \\ t \\ w \end{bmatrix} = \begin{bmatrix} 30 & 5 \\ 25 & 5 \\ 35 & 15 \end{bmatrix}\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

and so

$$\begin{bmatrix} m \\ t \\ w \end{bmatrix} = \begin{bmatrix} 160 \\ 140 \\ 260 \end{bmatrix}$$

The number 160 is the first entry of the product matrix and takes the first position (first row and first column position). It corresponds to m and it is obtained by multiplying the entries of the first row of TV by the corresponding entries of the first column of CV and then adding these products up. Of course, in our example CV has only one column. Therefore

$$30(4) + 5(8) = 160$$

Similarly the entry 140 is the entry of the product matrix for t. It is obtained by multiplying the entries of the second row of TV by the column of CV:

$$25(4) + 5(8) = 140$$

Finally, one can obtain the entry of the third row and first column of the product matrix, corresponding to w, by multiplying the third row of TV by the first column of CV:

$$35(4) + 15(8) = 260$$

■

Example 8.7 Cost of a Basket of Goods

Let the column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

denote the quantities of n goods that a consumer buys in a week. Let the row vector $\mathbf{p} = [p_1 \ \ p_2 \ \ \cdots \ \ p_n]$ denote the corresponding prices in dollars of a unit of each good (p_1 is the price of one unit of x_1, p_2 is the price of one unit of x_2, etc.). Find the consumer's weekly expenditure on these goods.

Solution

The consumer's weekly expenditure on goods is given by

$$E = \mathbf{px} = [p_1 \quad p_2 \quad \cdots \quad p_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

$$= \sum_{i=1}^{n} p_ix_i$$

■

From these examples we can move to a definition of matrix multiplication.

Definition 8.8

Two matrices A and B of dimensions $m \times n$ and $n \times q$ respectively are **conformable** to form the product matrix $AB = C$, since the number of columns of A is equal to the number of rows of B. The **product matrix** AB is of dimension $m \times q$, and its ijth element, c_{ij}, is obtained by multiplying the elements of the ith row of A by the corresponding elements of the jth column of B and adding the resulting products.

In general, the product matrix can be expressed as

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example 8.8

Multiply A and B if possible, where A and B are given below:

(i)

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

(ii)

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 2 & 4 \\ 1 & -2 \end{bmatrix}$$

Solution

For (i) we have that A is 2×3 and B is 3×2. Therefore the number of columns of A is equal to the number of rows of B. They have a common dimension 3 and are conformable for multiplication

$$\begin{bmatrix} 5 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5(4) + 1(1) + 0(0) & 5(3) + 1(1) + 0(2) \\ 2(4) + 1(1) - 1(0) & 2(3) + 1(1) - 1(2) \end{bmatrix}$$
$$= \begin{bmatrix} 21 & 16 \\ 9 & 5 \end{bmatrix}$$

In the case of (ii), A is 2×2 and B is 3×2. Therefore, since the number of columns of A is not equal to the number of rows of B, the matrices are *not* conformable for multiplication. ■

It is worth noting that for general matrices A and B of dimensions $m \times n$ and $n \times q$ respectively, whereas **premultiplying** B by A leads to a well defined product matrix AB, **postmultiplying** B by A will not result in a well defined product matrix BA (except in a special case described below). Conformability between matrices generally means either premultiplication of one matrix by another, or postmultiplication of one by another *but not* both. In fact, it is only for square matrices A and B of the same order that both product matrices AB and BA are defined.

Theorem 8.1 Both of the product matrices AB and BA are well defined only if A and B are square matrices of the same order or for A of dimension $m \times n$ with B of dimension $n \times m$.

Note that even if AB and BA are well defined, $AB \neq BA$ in general. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1) + 2(1) & 1(0) + 2(0) \\ 2(1) + 1(1) & 2(0) + 1(0) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}$$

while

$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1) + 2(0) & 1(2) + 0(1) \\ 1(1) + (2)0 & 1(2) + 0(1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

and so $AB \neq BA$.

Example 8.9 **Regional Migration**

We will consider the data of section S8.1 on the Web page, where we looked at the transition matrix between three regions. We denote the populations of the three regions in millions at some initial point in time, 0, in terms of the vector $\mathbf{x}^0$, given

$$\mathbf{x}^0 = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$

Find the populations of these regions at the beginning of the next period, $\mathbf{x}^1$.

Solution

We have to solve the equation $\mathbf{x}^1 = P\mathbf{x}^0$. The transition matrix from example 8.4 is given by

$$P = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix}$$

Then we have that

$$\begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 0.80(5) + 0.15(10) + 0.05(6) \\ 0.10(5) + 0.70(10) + 0.05(6) \\ 0.10(5) + 0.15(10) + 0.90(6) \end{bmatrix}$$
$$= \begin{bmatrix} 5.8 \\ 7.8 \\ 7.4 \end{bmatrix}$$

For example,

$$[0.8 \quad 0.15 \quad 0.05]\begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$

gives the sum of individuals initially in region 1 (0.8×5) *plus* the number of individuals initially in region 2 who move to region 1 (0.15×10) *plus* the number of individuals initially in region 3 who move to region 1 (0.05×6), giving the sum 5.8. We can see that the population distribution between the three regions changed from $\mathbf{x}^0$ to $\mathbf{x}^1$, with regions 1 and 3 gaining people, while region 2 is losing them. ■

A special case of matrix multiplication is when a square matrix is raised to a power n, that is, the matrix is multiplied by itself n times. As with the case of ordinary algebra we give the following definition.

Definition 8.9 The matrix A^n is the product matrix obtained by multiplying the square matrix A by itself n times.

Regional Migration over Time

In example 8.9 we saw how the distribution of the populations of three regions in a country changes between two periods. The matrix equation that we used to obtain the distribution of these regional populations after one period, $\mathbf{x}^1$, is given by $\mathbf{x}^1 = P\mathbf{x}^0$, where $\mathbf{x}^0$ is the distribution of the populations in the regions at time 0 and P is the transition matrix.

To determine how this distribution changes over n periods we then solve the following equation:

$$\mathbf{x}^n = P\mathbf{x}^{n-1} \tag{8.1}$$

where $\mathbf{x}^n$ describes the distribution of the regional populations at time period n and $\mathbf{x}^{n-1}$ the distribution at the previous period $n-1$. However, from the equation above it becomes clear that $\mathbf{x}^{n-1} = P\mathbf{x}^{n-2}$. Therefore, by substituting back into equation (8.1), we obtain $\mathbf{x}^n = P^2\mathbf{x}^{n-2}$. In fact, backward substitution to the vector of initial population distributions $\mathbf{x}^0$ yields

$$\mathbf{x}^n = P^n\mathbf{x}^0 \tag{8.2}$$

Equation (8.2) describes the evolution of the populations between the regions after n periods. It is based on the product matrix P^n. We can actually think of $\mathbf{x}^n = P^n\mathbf{x}^0$ as a sequence of vectors in an analogous way to the sequences studied in chapter 3. In chapter 10 we will study the behavior of such an equation in greater detail.

Example 8.10 **Regional Migration over Two Periods**

Consider the transition matrix, P, and initial distribution of population $\mathbf{x}^0$ given in example 8.9. Find the distribution of the population after two periods.

Solution

From equation (8.2) we have

$$\mathbf{x}^2 = P^2\mathbf{x}^0$$

Now

$$P^2 = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix} \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix}$$
$$= \begin{bmatrix} 0.6600 & 0.2325 & 0.0925 \\ 0.1550 & 0.5125 & 0.0850 \\ 0.1850 & 0.2550 & 0.8225 \end{bmatrix}$$

and so $P^2\mathbf{x}^0$ is

$$P^2\mathbf{x}^0 = \begin{bmatrix} 0.6600 & 0.2325 & 0.0925 \\ 0.1550 & 0.5125 & 0.0850 \\ 0.1850 & 0.2550 & 0.8225 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} 6.180 \\ 6.410 \\ 8.410 \end{bmatrix}$$

Again, we see that regions 1 and 3 are gaining people while region 2 is losing them. ■

Labor Market Transitions*

Similar methods to those used in our regional migration examples may also be used to study transitions between various labor market states. We may think of an individual at any point in time as occupying one of three distinct states. An individual is either employed (state E), unemployed (state U), or not in the labor force (nonparticipation, state N). During each period an individual may change state with some probability or remain in the current state. The transition probability matrix contains information on the average probability of remaining employed $\Pr(E, E)$, the average probability of an employed person becoming unemployed $\Pr(U, E)$, that is, moving from state E to state U, and so on. The transition probability matrix may therefore be written

$$P = \begin{bmatrix} \Pr(E, E) & \Pr(E, U) & \Pr(E, N) \\ \Pr(U, E) & \Pr(U, U) & \Pr(U, N) \\ \Pr(N, E) & \Pr(N, U) & \Pr(N, N) \end{bmatrix}$$

If $\mathbf{x}^0$ represents the initial vector of numbers of people occupying each labor market state, then the numbers employed, unemployed, and not participating after n periods is

$$\mathbf{x}^n = P^n \mathbf{x}^0$$

or, denoting the stocks of employed, unemployed, and nonparticipants at date t by E_t, U_t, and N_t respectively, we have (for $t = n$)

$$\begin{bmatrix} E_n \\ U_n \\ N_n \end{bmatrix} = \begin{bmatrix} \Pr(E, E) & \Pr(E, U) & \Pr(E, N) \\ \Pr(U, E) & \Pr(U, U) & \Pr(U, N) \\ \Pr(N, E) & \Pr(N, U) & \Pr(N, N) \end{bmatrix}^n \begin{bmatrix} E_0 \\ U_0 \\ N_0 \end{bmatrix}$$

Notice that, since these states are exhaustive and mutually exclusive, the sum of the column probabilities in P must be 1.

Example 8.11 Labor Market Conditions after One Period

Suppose the labor market transition probability matrix is

$$P = \begin{bmatrix} 0.80 & 0.1 & 0.01 \\ 0.15 & 0.6 & 0.49 \\ 0.05 & 0.3 & 0.50 \end{bmatrix}$$

and the initial distribution of individuals (in millions) is

$$\mathbf{x}^0 = \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$$

Comment on the size of the probabilities $\Pr(E, N)$ and $\Pr(N, U)$. Find the labor market status vector after one period.

Solution

The probability that a nonparticipant moves directly into employment, $\Pr(E, N)$, is small here (0.01), reflecting that some period of unemployed job search is likely to be required by most people before work is found. The probability $\Pr(N, U)$ represents the rate at which the unemployed leave the labor force. These individuals are referred to as *discouraged workers,* and this probability represents the "drop-out" rate.

After one period we have

$$\begin{aligned}\mathbf{x}^1 &= P\mathbf{x}^0 \\ &= \begin{bmatrix} 0.80 & 0.1 & 0.01 \\ 0.15 & 0.6 & 0.49 \\ 0.05 & 0.3 & 0.50 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 0.8(10) + 0.1(1) + 0.01(5) \\ 0.15(10) + 0.6(1) + 0.49(5) \\ 0.05(10) + 0.3(1) + 0.5(5) \end{bmatrix} \\ &= \begin{bmatrix} 8.15 \\ 4.55 \\ 4.10 \end{bmatrix}\end{aligned}$$

Note that the increase in unemployment comes both from a net reduction in the number of nonparticipants and a reduction in employment. A situation in which an increase in unemployment is accompanied by an increase in participation is referred to as the *added-worker effect*. ■

Finally, in this section we return to an economic problem discussed in section 7.2 that makes use of both matrix addition (subtraction) and matrix multiplication.

Equilibrium Supply and Demand in Many Markets

Recall the problem of equilibrium in three markets, discussed in section 7.2. We now have a neat way of writing out the equilibrium conditions for n markets using matrix notation. The problem of *solving* for equilibrium prices is discussed in chapter 9. For now, we are concerned with notational issues.

Denote the vector of market supplies of n goods by $\mathbf{q}^s$. Let α be an $n \times 1$ vector of constants, and β be a $n \times n$ matrix of parameters. If $\mathbf{p}$ is the $n \times 1$ vector of prices, we have

$$\mathbf{q}^s = \alpha + \beta\mathbf{p}$$

Similarly the system of demand equations may be written

$$\mathbf{q}^d = \mathbf{a} + B\mathbf{p}$$

where **a** is a $n \times 1$ vector of constants and B is a $n \times n$ matrix of parameters. Setting demand equal to supply in each market gives

$$\alpha + \beta\mathbf{p} = \mathbf{a} + B\mathbf{p}$$

or

$$\alpha - \mathbf{a} = (B - \beta)\mathbf{p}$$

In chapter 9 we will see how to solve this system for an equilibrium price vector **p**.

EXERCISES

1. For A given below obtain $3A$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. For the matrices given below obtain $A - B$ and $A + B$, where possible:

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

3. Obtain for the row vector **a** and the column vector **b**, below, the products **ab** and **ba**:

$$\mathbf{a} = [1 \quad 2 \quad 0], \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

4. Perform the following matrix multiplications to obtain AB where possible: if

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$

5. Suppose that a firm produces two types of output using three types of input. Its output quantities are given by the column vector

$$\mathbf{q} = \begin{bmatrix} 15{,}000 \\ 27{,}000 \end{bmatrix}$$

and the prices of these are given in the row vector $\mathbf{p} = [10 \;\; 12]$. The amounts of inputs it uses are given in the column vector

$$\mathbf{z} = \begin{bmatrix} 11{,}000 \\ 15{,}000 \\ 15{,}000 \end{bmatrix}$$

and the input prices are given by $\mathbf{w} = [10 \;\; 10 \;\; 8]$. Find the profit of this firm.

6. Use equation (8.2) and the data for P and $\mathbf{x}^0$ given in example 8.10 to find the regional population distribution after three time periods.

8.3 Matrix Transposition

A very useful operation in matrix algebra is that of transposition. The **transpose** of a matrix A, is the matrix in which the rows of the original matrix A become columns and the columns of A become rows. The transpose is denoted by A^T.

Definition 8.10

The **transpose matrix,** A^T, is the original matrix A with its rows and columns interchanged.

This implies that A^T will have its dimensions reversed when compared with A.

Example 8.12

Find the transpose of the 2×3 matrix A, given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix}$$

Solution

Since A is 2×3, its transpose will be 3×2. Placing the first row of A, $(1\ \ 2\ \ 3)$, as the first column of A^T and the second row of A, $(2\ \ 5\ \ 7)$, as the second column of A^T yields

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}$$ ■

Definition 8.11 A matrix A that is equal to its transpose A^T is called a **symmetric matrix.**

An example of a symmetric matrix is

$$A = \begin{bmatrix} 1 & 5 & 6 \\ 5 & 2 & 0 \\ 6 & 0 & -4 \end{bmatrix}$$

where transposing rows and columns shows that $A^T = A$.

Since equality of matrices, as we have seen earlier, implies that all the elements of the two matrices in each position have to be equal, then the orders of A and A^T have to be the same. In that case, A has to be a square matrix and so, of course, is A^T.

From now on, we will define all vectors as column vectors, taking their transposes when we want to have a row vector.

Example 8.13 **The Profit Function**

Using all the information that a firm's profit is given by $\Pi = \mathbf{pq} - \mathbf{wz}$, and expressing all the vectors as column vectors, we obtain the profit function as

$$\Pi = \mathbf{p}^T\mathbf{q} - \mathbf{w}^T\mathbf{z}$$ ■

Properties of Transposes

Below we present some useful properties of the transpose matrix by a series of theorems and examples.

Theorem 8.2 The transpose of the transpose matrix $(A^T)^T$ is the original matrix A:

$$(A^T)^T = A \tag{8.3}$$

Example 8.14 Find $(A^T)^T$, if A is given by

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 4 \end{bmatrix} \quad \text{and so} \quad (A^T)^T = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} = A$$

■

Theorem 8.3 The transpose of a sum of matrices is the sum of the transposes:

$$(A + B)^T = A^T + B^T \tag{8.4}$$

Example 8.15 Compute $(A + B)^T$, for A and B given below:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

Solution

We first compute A^T and B^T, and then $A + B$ and $(A + B)^T$.

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

while

$$\begin{aligned} A + B &= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

and

$$(A + B)^T = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

But

$$A^T + B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (A + B)^T$$

■

Somewhat less obvious is the interpretation of $(AB)^T$ or the transpose of the product matrix AB. Suppose that A is $m \times n$ and B is $n \times q$ for $m \neq q$. Then the two matrices are conformable for matrix multiplication and AB is well defined, since the number of columns of A equals the number of rows of B. On the other hand, the product matrix BA is not defined. Now, it is not possible that

$$(AB)^T = A^T B^T$$

Since AB is a matrix of dimension $m \times q$, its transpose should be of dimension $q \times m$. However, A^T is of order $(n \times m)$ and B^T is of order $(q \times n)$ so the number of columns of A^T does not equal the number of rows of B^T.

If the rule of obtaining $(AB)^T$ were to be given by equation (8.5) below, the product matrix $(AB)^T$ *would* be well defined:

$$(AB)^T = B^T A^T \tag{8.5}$$

In that case, $B^T A^T$ would result in a matrix of order $q \times m$.

Theorem 8.4 The transpose matrix of the product matrix AB, where A and B are two conformable matrices, is defined as the product of the transposes, with the order of the multiplication reversed.

The rule above extends to any product matrix made up by any number of conformable matrices, such as ABC. In the case of a product of three matrices, the transpose of the product is defined as

$$(ABC)^T = C^T (AB)^T$$
$$= C^T B^T A^T$$

EXERCISES

1. Find the transpose of the following matrices:

(a)

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

2. For the matrices below, indicate whether the operations listed under (a)–(e) are well defined. If not, explain why.

$$A = \begin{bmatrix} 7 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) $-3A$

(b) $A + E$

(c) $B - 3D$

(d) $3C - E$

(e) AC

3. Verify that for the matrices A and B below $(AB)^T = B^T A^T$.

(a)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

4. Obtain the profit function of a firm that produces three types of output using three inputs. The output vector is given by

$$\mathbf{q} = \begin{bmatrix} 2{,}000 \\ 3{,}000 \\ 6{,}000 \end{bmatrix}$$

the price per unit of output vector is given by

$$\mathbf{p} = \begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix}$$

the input vector is given by

$$\mathbf{z} = \begin{bmatrix} 2{,}000 \\ 2{,}500 \\ 2{,}000 \end{bmatrix}$$

and the price per unit of input vector is given by

$$\mathbf{w} = \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}$$

5. If the order of matrix A is 3×5 and that of the product AB is 3×7, what is the order of B?

6. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$$

Verify that $AB = AC$ even though $B \neq C$.

7. How many rows does B have if BA is a 2×6 matrix?

8.4 Some Special Matrices

A number of matrices have particular properties that are often found to be useful in studying systems of equations.

Idempotent Matrices

Definition 8.12 A square matrix A of any order is **idempotent** if

$$A = A^2 = A^3 = \cdots$$

where $A^2 = AA$, $A^3 = AAA$, etc.

Idempotent matrices play a very important role in statistical distribution theory. We will see examples of them in later chapters when we discuss quadratic forms.

Example 8.16 Verify that A below is idempotent:

$$A = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

Solution

It is only necessary to verify that $AA = A$. (Why?)

$$A = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix} \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} (1/6)(1/6) + (1/3)(1/3) + (1/6)(1/6) & -(1/6)(1/3) - (1/3)(2/3) - (1/6)(1/3) & (1/6)(1/6) + (1/3)(1/3) + (1/6)(1/6) \\ -(1/3)(1/6) - (2/3)(1/3) - (1/3)(1/6) & (1/3)(1/3) + (2/3)(2/3) + (1/3)(1/3) & -(1/6)(1/3) - (1/3)(2/3) - (1/6)(1/3) \\ (1/6)(1/6) + (1/3)(1/3) + (1/6)(1/6) & -(1/6)(1/3) - (1/3)(2/3) - (1/6)(1/3) & (1/6)(1/6) + (1/3)(1/3) + (1/6)(1/6) \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

■

Partitioned Matrices

Definition 8.13

A **partitioned matrix** contains submatrices as elements. The submatrices are obtained by partitioning the rows and columns of the original matrix.

For example,

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & 1 \\ \hline 3 & 0 & -1 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where

$$A_{11} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A_{21} = [3 \quad 0 \quad -1], \quad A_{22} = [2]$$

The rules for the addition (subtraction) and multiplication of matrices apply directly to partitioned matrices provided the submatrices are of suitable order. Let A and B be two partitioned matrices, which we write as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

provided that A and B are of the same overall dimension and that, A_{11} and B_{11}, A_{12} and B_{12}, A_{21} and B_{21}, and A_{22} and B_{22} are of the same order.

The Trace of a Matrix

The trace is defined only for square matrices.

Definition 8.14

The **trace** of a square matrix A is given by the sum of the elements of the main diagonal. In other words, if A is $n \times n$, then the trace is defined as

$$\text{trace}(A_n) = a_{11} + a_{22} + \cdots + a_{nn}$$

The trace of a matrix, if defined, has some very attractive simplifying properties. For instance,

Theorem 8.5 For two matrices A and B of dimensions $m \times n$ and $n \times m$ respectively, we have that AB is $m \times m$ and BA is $n \times n$ *and*

$$\text{trace}(AB) = \text{trace}(BA)$$

Proof

Since AB is of order $m \times m$, its ith diagonal element will be given by

$$c_{ii} = \sum_{j=1}^{n} a_{ij} b_{ji} \qquad \text{for } i = 1, \ldots, m$$

On the other hand, BA is of order $n \times n$ and its jth diagonal element will be given by

$$d_{jj} = \sum_{i=1}^{n} b_{ji} a_{ij} \qquad \text{for } j = 1, \ldots, n$$

Therefore the trace of AB and BA are given by

$$\begin{aligned} \text{trace}(AB) &= \sum_{i=1}^{m} c_{ii} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} b_{ji} \right) \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji} \end{aligned}$$

$$\begin{aligned} \text{trace}(BA) &= \sum_{j=1}^{n} d_{jj} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} b_{ji} a_{ij} \right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij} \end{aligned}$$

We can see from this result that

$$\text{trace}(AB) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ji} a_{ij} = \text{trace}(BA)$$

■

Example 8.17 For the matrices A and B below, verify that trace(AB) = trace(BA):

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}$$

In this case, A is 2×3 and B is 3×2. Therefore AB is 2×2 and BA is 3×3.

$$AB = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 12 & 1 \end{bmatrix}$$

and trace $(AB) = 11 + 1 = 12$.

$$BA = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 13 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

and trace $(BA) = 8 + 1 + 3 = 12$. ■

EXERCISES

1. Verify that the matrix I_3 below is idempotent:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Verify that matrix A below is idempotent:

$$A = \frac{1}{11} \begin{bmatrix} 6 & -2 & -5 & 1 \\ -2 & 8 & -2 & -4 \\ -5 & -2 & 6 & 1 \\ 1 & -4 & 1 & 2 \end{bmatrix}$$

3. Verify that the matrix A below is idempotent:

$$A = \begin{bmatrix} x & -x \\ x - 1 & 1 - x \end{bmatrix}$$

4. For the matrices A and B below, verify that trace(AB) = trace(BA):

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

5. For the matrix A below, obtain trace(A), trace(AA), and trace(AAA):

(a) $A = \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$

(b) $A = \frac{1}{11}\begin{bmatrix} 6 & -2 & -5 & 1 \\ -2 & 8 & -2 & -4 \\ -5 & -2 & 6 & 1 \\ 1 & -4 & 1 & 2 \end{bmatrix}$

CHAPTER REVIEW

Key Concepts

column matrix
column vector
conformable matrices
diagonal matrix
idempotent matrix
identity matrix
matrix
matrix equation
null matrix
partitioned matrix
postmultiplication
premultiplication
product matrix
row matrix
row vector
scalar
scalar multiplication
square matrix
symmetric matrix
trace
transpose matrix
vector

Review Questions

1. When are two matrices conformable for addition (subtraction)?
2. When are two matrices conformable for multiplication?
3. Why is it important to distinguish between premultiplication and postmultiplication of matrices, but not for scalars?
4. What is the transpose of a symmetric matrix?
5. What is the trace of a matrix?

Review Exercises

1. For each of the matrix operations below, identify those that result in a scalar.

   (a) AB where A is 2×1 and B is 1×2

   (b) $A^T B$ where A is 2×1 and B is 2×1

   (c) $A^T BA$ where A is 2×1 and B is 2×2

   (d) $AA^T B$ where A is 5×1 and B is 5×5

2. Show that $AB \neq BA$ where

$$A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

   Compute $(A\mathbf{x})^T$, $\mathbf{x}^T A^T$, $\mathbf{x}\mathbf{x}^T$, and $\mathbf{x}^T\mathbf{x}$. Is $A^T\mathbf{x}^T$ defined?

4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

   Compute AD and DA.

5. Let

$$A = \begin{bmatrix} 3 & -4 \\ -5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 \\ 5 & k \end{bmatrix}$$

   What values of k, if any, will make $AB = BA$?

6. Compute the quantities below using

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -8 & 4 \\ -7 & 5 \end{bmatrix}$$

   (a) A^T, B^T, $A^T + B^T$, $(A + B)^T$

   (b) AB, $(AB)^T$, $A^T B^T$, $B^T A^T$

7. Use the data of example 8.10 to find the evolution of the population after four periods.

8. Use the data in example 8.11 to find the unemployment rate after two periods. [*Hint:* The unemployment rate is the number of people unemployed as a proportion of all labor market participants.] Can the situation described by these transition probabilities evolve in the same way indefinitely?

Chapter 9 Determinants and the Inverse Matrix

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Gauss-Jordan Elimination and the Inverse Matrix

In chapter 8 we defined the operations of addition, subtraction, and multiplication of matrices. What about division? Can we define rules for dividing one matrix by another? The answer is yes, but only under certain restrictions. Division is restricted only to *square* matrices, and then only to those square matrices that satisfy a condition known as *nonsingularity*. The reason for all this can again be traced to the relation between matrix algebra and the problem of solving a system of simultaneous linear equations.

9.1 Defining the Inverse

Consider, first, the division of two numbers. If we divide b into a, we can write this as a/b, where $b \neq 0$. Alternatively, we could write $1/b = b^{-1}$ as the reciprocal or *inverse* of b, and define division as the multiplication of a and b^{-1} : $a/b = ab^{-1}$. This slightly more roundabout way of defining division is in fact the more useful one when dealing with matrices. Note finally that by the inverse of a number b, we mean the number b^{-1} that has the property

$$bb^{-1} = b^{-1}b = 1$$

For example, the inverse of the number 2 is 1/2, since $2(1/2) = 1$. This rather obvious fact is worth spelling out when we proceed to consider matrix division.

Definition 9.1 The **inverse matrix** A^{-1} of a square matrix A of order n is the matrix that satisfies the condition that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the identity matrix of order n.

Note that A^{-1} is only defined for square matrices. However, not every square matrix has an inverse.

Obtaining the inverse matrix involves obtaining the elements of A^{-1} as solutions to the equation

$$A^{-1}A = I$$

When we deal with real numbers in ordinary algebra, we know that for some real number a, the operation $a(1/a) = 1$ is valid if and only if $a \neq 0$. In the case of matrices, A^{-1} corresponds to $(1/a)$ in simple algebra, but now for A^{-1} to exist it is not sufficient simply to assume that A is different from the null matrix.

Definition 9.2 Any matrix A for which A^{-1} does not exist is known as a **singular matrix.** The matrix A for which A^{-1} exists is known as a **nonsingular matrix.**

Example 9.1 The matrix equation

$$A\mathbf{x} = \mathbf{b}$$

where A is $n \times n$, $\mathbf{x}$ is $n \times 1$, and $\mathbf{b}$ is $n \times 1$ defines a system of n simultaneous linear equations in n unknowns, $\mathbf{x}$. Solve for the vector $\mathbf{x}$.

Solution

Suppose that you were given the ordinary equation

$$ax = b$$

where a, x, and b are scalars. Then you would solve this equation simply by "dividing through by a" to obtain

$$x = b/a$$

or equivalently

$$(a^{-1}a)x = a^{-1}b$$

It then seems natural to solve the above matrix equation as follows:

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I_n\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

If A and **b** are known, this solves for the unknown vector **x**, *provided* that A^{-1} exists. ■

We see from this that existence of the inverse matrix is equivalent to being able to solve a linear system, something that we discussed in chapter 7. In fact, we will see that the conditions that have to be satisfied for the solution of a linear system also have to be satisfied if the inverse matrix is to exist. Also the Gauss-Jordan elimination method for solving a linear simultaneous system can be used to obtain the inverse matrix A^{-1}.

Obtaining the Inverse Matrix of a 2 × 2 Matrix

Let A be a general 2×2 matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and let us assume that it has an inverse matrix A^{-1} denoted by

$$A^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

From the definition of the inverse matrix we have that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We will try to solve for the unknown elements of A^{-1}, denoted by α_{11}, α_{12}, α_{21}, and α_{22}, in terms of the known elements of A, denoted by a_{11}, a_{12}, a_{21}, and a_{22}. We will first solve for α_{11} and α_{21}. In that case we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \alpha_{11} \\ \alpha_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Multiplying out yields a system of two equations in the two unknowns α_{11} and α_{21}, given by

$$a_{11}\alpha_{11} + a_{12}\alpha_{21} = 1 \tag{9.1}$$

$$a_{21}\alpha_{11} + a_{22}\alpha_{21} = 0 \tag{9.2}$$

We can rewrite equation (9.1) by taking $a_{12}\alpha_{21}$ to the right-hand side (RHS) and then dividing both sides by a_{11} to obtain

$$\alpha_{11} = \frac{1 - a_{12}\alpha_{21}}{a_{11}}$$

Substituting the above value of α_{11} into equation (9.2) yields

$$a_{21}\left(\frac{1 - a_{12}\alpha_{21}}{a_{11}}\right) + a_{22}\alpha_{21} = 0$$

Multiplying through by a_{11} yields

$$a_{21} - a_{21}a_{12}\alpha_{21} + a_{11}a_{22}\alpha_{21} = 0$$

Collecting the α_{21} terms together and rearranging yields

$$\alpha_{21} = \frac{-a_{21}}{a_{22}a_{11} - a_{21}a_{12}}$$

Returning to the expression for α_{11}, substituting α_{21} yields

$$\begin{aligned}\alpha_{11} &= \frac{1}{a_{11}} - \frac{a_{12}}{a_{11}}\left(\frac{-a_{21}}{a_{22}a_{11} - a_{21}a_{12}}\right)\\ &= \frac{a_{22}a_{11} - a_{21}a_{12} + a_{12}a_{21}}{a_{11}\left(a_{22}a_{11} - a_{21}a_{12}\right)}\\ &= \frac{a_{22}}{a_{22}a_{11} - a_{21}a_{12}}\end{aligned}$$

Similarly we can solve for α_{12} and α_{22} by looking at

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \alpha_{12} \\ \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The solutions are

$$\alpha_{12} = \frac{-a_{12}}{(a_{22}a_{11} - a_{21}a_{12})}$$
$$\alpha_{22} = \frac{a_{11}}{(a_{22}a_{11} - a_{21}a_{12})}$$

Collecting the different terms together yields the inverse matrix A^{-1} as

$$A^{-1} = \frac{1}{(a_{22}a_{11} - a_{21}a_{12})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

We can verify that $AA^{-1} = A^{-1}A = I_2$.

The Determinant of the 2 × 2 Matrix and Its Properties

Each element of A^{-1} is a function of the elements of A. It becomes apparent that for A^{-1} to exist, the quantity $a_{22}a_{11} - a_{21}a_{12}$ has to be different from zero. If it were zero, each element of A^{-1} would be undefined since $1/0$ is not defined.

Definition 9.3 The quantity $a_{22}a_{11} - a_{21}a_{12}$ is called the **determinant** of the 2×2 matrix A and is composed of all the elements of A. It is denoted by $|A|$ or det A.

Besides the algebraic definition given above, the determinant of a 2×2 matrix has a geometric interpretation. It is proportional to the area enclosed by the two column vectors that make up the 2×2 matrix in question. We provide a detailed exposition of the geometric interpretation at the end of the section.

If $|A| = 0$, then A is a singular matrix, because in this case A^{-1} does not exist. If $|A| \neq 0$, A is nonsingular. The determinant expression $a_{22}a_{11} - a_{21}a_{12}$ constitutes the denominator of each of the elements of A^{-1}. To obtain the numerators of the elements of A^{-1}, we can follow the steps outlined below.

Step 1 For each element in A, strike out the row and column containing that element and use the remaining element with a positive or negative sign in the pattern:

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

Then we obtain the matrix

$$\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

Step 2 Transpose the matrix obtained in step 1 to get

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example 9.2 Obtain the inverse of the following matrices:

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution

For (i) we have that $|A| = -1 - 6 = -7$. Since $|A| \neq 0$, the inverse matrix exists. Then $|A|$ is the denominator of each element of A^{-1}. To find the numerators of A^{-1}, we follow the steps outlined above.

(Step 1) We obtain the matrix

$$\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

which in this case is

$$\begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix}$$

(Step 2) We transpose the matrix above to get

$$\begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix}$$

and the inverse is

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{bmatrix}$$

We can verify that $A^{-1}A = I$ by looking at

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ 3/7 & -1/7 \end{bmatrix} = \begin{bmatrix} 1/7+6/7 & 2/7-2/7 \\ 3/7-3/7 & 6/7+1/7 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For (ii) we see that $|B| = 4 - 4 = 0$. Therefore B^{-1} does not exist. ■

We now present some of the properties of determinants and illustrate them with examples. The theorems relate to any $n \times n$ matrices, though our illustrations use 2×2 examples.

Theorem 9.1 The determinant of the transpose matrix of A, $|A^T|$, is the same as the determinant of A, even though A may not be symmetric.

Example 9.3 For

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

find $|A|$ and $|A^T|$.

Solution

First

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

and so $|A^T| = 4 - 6 = -2$. But $|A| = 4 - 6 = -2$. Therefore $|A| = |A^T|$. ■

Theorem 9.2 If B is obtained from A by interchanging the rows or columns of A, then $|B| = -|A|$.

Example 9.4 For

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

obtain B given in theorem 9.2.

Solution

Two methods are equivalent:

(i) interchanging the rows of A
(ii) interchanging the columns of A

For case (i) we have that

$$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$|B| = 6 - 4 = 2$. Since $|A| = -2$, $|B| = -|A|$.

For case (ii) we have that

$$B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and since $|B| = 2$, again we have that $|B| = -|A|$. ■

Theorem 9.3 If a matrix has two identical rows or columns, its determinant will be zero.

Example 9.5 Find the determinants of A and B:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Solution

We see that $|A| = 2 - 2 = 0$ (identical rows) and $|B| = 2 - 2 = 0$ (identical columns). ■

Theorem 9.4 If one of the two rows of A is a multiple of the other row, or if one of the columns of A is a multiple of the other column, then $|A| = 0$.

Example 9.6 For A and B find the determinants

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Solution

The second row of A is twice its first row. In this case, $|A| = 6 - 6 = 0$. The second column of B is twice its first column. In this case, $|B| = 6 - 6 = 0$. ■

Theorems 9.3 and 9.4 illustrate a property of the determinant of a matrix that relates to the relationship among the rows (columns) of the matrix known as *linear dependence*. Two rows (columns) are said to be linearly dependent if they are in some sense indistinguishable from each other and hence contain the same information. In chapter 10 we expand further on the notion of linear dependence in the context of vector spaces.

Theorem 9.5 If B is formed from A by adding a multiple of one row to another row, or a multiple of one column to another column, the value of the determinant remains unchanged.

Example 9.7 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Find the determinant of B formed by

(i) adding twice the second row to the first
(ii) by subtracting the second column from the first column

Solution

For case (i),

$$B = \begin{bmatrix} 7 & 10 \\ 3 & 4 \end{bmatrix}$$

Then $|B| = 28 - 30 = -2$. But we also have that $|A| = 4 - 6 = -2$. Therefore $|B| = |A|$.

For case (ii),

$$B = \begin{bmatrix} -1 & 2 \\ -1 & 4 \end{bmatrix}$$

Then $|B| = -4 + 2 = -2$, which is the same as $|A|$. ■

Theorem 9.6 The determinant of a matrix that is composed of a nonzero element in the positions above (below) the main diagonal and zero in the positions below (above) equals the product of the diagonal elements. Such a matrix is known as a **triangular matrix.**

Example 9.8 Let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Find $|A|$ and $|B|$.

Solution

We can see that $|A| = 2 - 0(2) = 2$ and $|B| = -1 - 1(0) = -1$. ■

It is easy to see that a special case of the property above is that of the diagonal matrix. For such a matrix the determinant also equals the product of its diagonal elements.

Theorem 9.7 Multiplying any row of a matrix by a scalar λ, multiplies the determinant by λ. Multiplying every element of an $n \times n$ matrix by λ, multiplies the determinant by λ^n.

Example 9.9 Let A be given by

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Verify theorem 9.7 above for matrix B formed from A by

(i) multiplying the first row of A by $\lambda = 2$
(ii) multiplying all the elements of A by 2.

Solution

For (i) we have that

$$B = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

and $|B| = 8 - 12 = -4$. Since $|A| = -2$, we have that $|B| = 2|A|$.

For (ii),

$$B = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

and $|B| = 16 - 24 = -8$. Since $|A| = -2$, we see that $|B| = 2^2(-2) = \lambda^2(-2) = -8$. ■

Theorem 9.8 The determinant of the product of two square matrices A and B of the same order, is the product of the determinants, that is, $|AB| = |A||B|$.

Example 9.10 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Find AB and verify theorem 9.8.

We have

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2+8 & 3+10 \\ 6+16 & 9+20 \end{bmatrix} = \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$$

We have that $|A| = -2$, $|B| = -2$, and $|AB| = 29(10) - 13(22) = 4$. Therefore $|AB| = |A||B|$. ■

Example 9.11 **The Linear Production Technology**

A firm produces two outputs, y_1 and y_2, with two inputs, z_1 and z_2. Let a_{ij} denote the amount of input i required to produce 1 unit of output j. The a_{ij} describe what is known as the input-requirements matrix. The matrix of these input-output coefficients is

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

Suppose that the firm produces 20 units of y_1 and 15 units of y_2. We can find the input levels of z_1 and z_2 by solving the relationship $\mathbf{z} = A\mathbf{y}$:

$$\begin{aligned} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 20 \\ 15 \end{bmatrix} = \begin{bmatrix} 75 \\ 115 \end{bmatrix} \end{aligned}$$

Suppose now that we are given the levels of input z_1 and z_2, say, 10 and 20. Find the levels of y_1 and y_2 that we will produce using these levels of inputs. This is the production function of the firm.

Solution

We need to solve $\mathbf{z} = A\mathbf{y}$ for $\mathbf{y}$ *given* $\mathbf{z}$. Put differently, we need to find $\mathbf{y} = A^{-1}\mathbf{z}$, or

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 5/13 & -1/13 \\ -2/13 & 3/13 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 30/13 \\ 40/13 \end{bmatrix} = \begin{bmatrix} 2.31 \\ 3.08 \end{bmatrix}$$

The entries of the inverse matrix, call them α_{ij}, can be interpreted as the amount of output i that is produced by *one* unit of input j. ■

We now turn to an example of multimarket equilibrium, first discussed in chapter 7. We start with a simple example, involving two markets for goods that are substitutes, and solve for equilibrium prices.

Example 9.12 The Markets for Tea and Coffee

Suppose that the market for tea is described by the demand and supply functions

$$D_t = 100 - 5p_t + 3p_c$$
$$S_t = -10 + 2p_t$$

and the market for coffee by

$$D_c = 120 - 8p_c + 2p_t$$
$$S_c = -20 + 5p_c$$

where p_t is the price of tea, p_c is the price of coffee, D_t and S_t are the quantities of tea demanded and supplied respectively, and D_c and S_c are the quantities of coffee demanded and supplied.

The first thing we notice in the specifications of the demand and supply functions is that both demands are negatively related with their own prices and positively related with the price of the other good. Also both supplies are positively related with their own prices. The fact that the demands are positively related with the price of the other good suggests that these goods are *substitutes*, since an increase in the price of one good will lead to a fall in its quantity demanded but at the same time it will lead to an increase in the demand for the other good as consumers shift into buying the relatively cheaper alternative. If the goods were *complements*, the relationship between the demand and the price of the other good would be negative.

Solve for the equilibrium prices of tea and coffee.

Solution

We write a system of equations that describes equilibrium in both markets simultaneously as follows:

$$A\mathbf{p} = \mathbf{b}$$

where A is a 2×2 matrix of coefficients, $\mathbf{p}$ is a 2×1 vector of prices, and $\mathbf{b}$ is a 2×1 vector of constants. The solution is then given by

$$\mathbf{p} = A^{-1}\mathbf{b}$$

In order to express the system of equations that describes the markets for tea and coffee into the form $A\mathbf{p} = \mathbf{b}$, we first need to obtain the equilibrium in each market, where $D_t = S_t$ and $D_c = S_c$. Then we obtain

$$\begin{aligned} 100 - 5p_t + 3p_c &= -10 + 2p_t \\ 120 - 8p_c + 2p_t &= -20 + 5p_c \end{aligned}$$

These can be rewritten more conveniently as

$$\begin{aligned} 7p_t - 3p_c &= 110 \\ -2p_t + 13p_c &= 140 \end{aligned}$$

In matrix form the equations above become

$$\begin{bmatrix} 7 & -3 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} p_t \\ p_c \end{bmatrix} = \begin{bmatrix} 110 \\ 140 \end{bmatrix}$$

The solution is then given by

$$\begin{aligned} \begin{bmatrix} p_t \\ p_c \end{bmatrix} &= \begin{bmatrix} 7 & -3 \\ -2 & 13 \end{bmatrix}^{-1} \begin{bmatrix} 110 \\ 140 \end{bmatrix} \\ &= \begin{bmatrix} 13/85 & 3/85 \\ 2/85 & 7/85 \end{bmatrix} \begin{bmatrix} 110 \\ 140 \end{bmatrix} = \begin{bmatrix} 21.76 \\ 14.12 \end{bmatrix} \end{aligned}$$

■

Another linear model introduced in chapter 7 was the simple IS-LM model of a closed economy. We will work through a model similar to that in example 7.8, but we extend the analysis slightly by introducing government activity. The government is assumed to have an expenditure program and to finance it in part by raising tax revenue. The underlying principle involved in solving the system is unchanged.

Example 9.13 The IS-LM Model of a Closed Economy

A closed economy is described by the system of equations that give the equilibrium conditions in the goods and the money markets, the IS and the LM relationships. The goods market (the IS part of the model) is described by

$$\begin{aligned} C &= 15 + 0.8(Y - T) \\ T &= -25 + 0.25Y \\ I &= 65 - R \\ G &= 94 \end{aligned}$$

where C is consumer expenditure, T is tax revenue, Y is aggregate output, I is investment expenditure, R is the interest rate, and G is government expenditures. The money market (the LM part of the model) is described by

$$\begin{aligned} L &= 5Y - 50R \\ M &= 1{,}500 \end{aligned}$$

where L is money demand and M is the fixed money supply. Find the equilibrium levels of Y and R.

Solution

We can express the system of equations above in the form

$$A\mathbf{x} = \mathbf{b}$$

where A is a 2×2 matrix of coefficients, $\mathbf{x}$ is the 2×1 vector of variables with entries Y and R, and $\mathbf{b}$ is a 2×1 vector of constants. We first have to solve for the equilibria in the goods and money markets, obtain the IS and the LM functions, and then put the two together. The IS function is obtained from $Y = C + I + G$ as follows:

$$\begin{aligned} Y &= 15 + 0.8Y - 0.8(-25 + 0.25Y) + 65 - R + 94 \\ Y(1 - 0.8 + 0.2) &= 15 + 20 + 65 + 94 - R \\ Y &= 485 - 2.5R \end{aligned}$$

The LM function is then found from $L = M$:

$$1{,}500 = 5Y - 50R \quad \text{or} \quad Y = 300 + 10R$$

Looking at the IS and LM relationships as a system of equations yields

$$\begin{aligned} Y + 2.5R &= 485 \\ Y - 10R &= 300 \end{aligned}$$

$$\text{or} \quad \begin{bmatrix} 1 & 2.5 \\ 1 & -10 \end{bmatrix} \begin{bmatrix} Y \\ R \end{bmatrix} = \begin{bmatrix} 485 \\ 300 \end{bmatrix}$$

Solving for Y and R yields

$$\begin{aligned} \begin{bmatrix} Y \\ R \end{bmatrix} &= \begin{bmatrix} 1 & 2.5 \\ 1 & -10 \end{bmatrix}^{-1} \begin{bmatrix} 485 \\ 300 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & 0.2 \\ 0.08 & -0.08 \end{bmatrix} \begin{bmatrix} 485 \\ 300 \end{bmatrix} \\ &= \begin{bmatrix} 448 \\ 14.8 \end{bmatrix} \end{aligned}$$

The equilibrium level of output in this economy is 448 and the interest rate is 14.8%. Notice that at this level of income, tax revenue is $T = -25 + 0.25(448) = 87$, while government spending is $G = 94$. The government's current deficit is therefore $G - T = 7$. ■

Finally we note

Theorem 9.9 For two square matrices, A and B

$$|A + B| \neq |A| + |B|$$

in general.

A Geometric Interpretation of the Determinant

In this section, we will consider the determinant from an alternative geometric point of view. Let us consider two vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

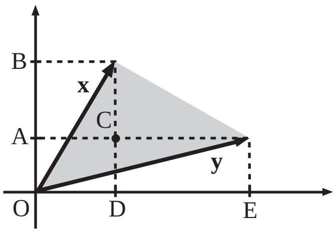


Figure 9.1 Geometric interpretation of the determinant

The components of these vectors represent the coordinates in R^2, where the first components x_1 and y_1 represent the coordinates on the horizontal axis and x_2 and y_2 represent the coordinates on the vertical axis respectively. Figure 9.1 presents the vectors as points in R^2.

The shaded area enclosed by **x**, **y**, and the origin, turns out to be half of the determinant of the matrix formed by using **x** and **y** as its columns. If we add the areas OB**x**D and OA**y**E, subtract the area OACD that has been counted twice, and finally add the area of the triangle **x**C**y**, we will obtain the area OB**xy**E. Then, to obtain the shaded area in figure 9.1, we subtract the areas of the triangles OB**x** and O**y**E from OB**xy**E.

Using the fact that the area of a rectangle is found by multiplying the base by the height, and that of a triangle by half the product of the base by the height, we calculate the shaded area in figure 9.1 to be

$$\begin{aligned} \text{Shaded area} &= x_1x_2 + y_1y_2 - y_2x_1 + \frac{1}{2}(y_1 - x_1)(x_2 - y_2) \\ &\quad - \frac{1}{2}x_1x_2 - \frac{1}{2}y_1y_2 \\ &= \frac{1}{2}y_1x_2 - \frac{1}{2}y_2x_1 \\ &= \frac{1}{2}\begin{vmatrix} y_1 & x_1 \\ y_2 & x_2 \end{vmatrix} \end{aligned}$$

This property generalizes so that the determinant of a 3×3 matrix, to be discussed in the next section, will be proportional to the volume enclosed by the three vectors that make up the columns of the matrix in question and the origin.

EXERCISES

1. Obtain the inverses of the following matrices, if they exist:

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

2. Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 37 & 2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 37 \\ 0 & 2 \end{bmatrix}$

3. Suppose that a firm, as in example 9.11, produces two outputs y_1 and y_2 with two inputs z_1 and z_2. The input requirements matrix is given by A below:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

(a) If the firm wants to produce 5 units of y_1 and 10 units of y_2, how much of z_1 and z_2 will it require?

(b) Let the prices of inputs be \$5 per unit and \$10 per unit, respectively, so that

$$\mathbf{w} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

is an input price vector. What is the interpretation of $\mathbf{w}^T A\mathbf{y}$? Is it a scalar or a matrix?

4. Suppose that the markets for coffee and sugar are characterized by the following demand and supply relationships:

$$D_c = 100 - 5p_c - p_s, \quad S_c = -20 + 2p_c$$

and

$$D_s = 80 - 4p_s - 2p_c, \quad S_s = -10 + p_s$$

where p_c is the price of coffee and p_s is the price of sugar.

(a) Set up the system in equilibrium as a matrix equation

$$A\mathbf{p} = \mathbf{b}$$

where A is a 2×2 matrix of coefficients, $\mathbf{p}$ is a 2×1 vector of prices, and $\mathbf{b}$ is a 2×1 vector of constants.

(b) Solve for the equilibrium prices of coffee and sugar.

5. Solve for the equilibrium levels of Y and R in the extended IS-LM model that allows for imports and exports. The model is given by

$$C = 15 + 0.8(Y - T)$$
$$T = -25 + 0.25Y$$
$$I = 65 - R$$

$$G = 94$$
$$X = 50 - 10 - 0.1Y$$

where C is consumer expenditure, T is tax revenue, Y is aggregate output, I is investment expenditures, R is the interest rate, G is government expenditure, and X is net exports (exports *minus* imports).

The money market (the LM part of the model) is described by

$$L = 5Y - 50R$$
$$M = 1{,}500$$

In this model, we are not imposing balance-of-payments equilibrium. Calculate the government's budget deficit (or surplus) in the equilibrium. Calculate the trade deficit (or surplus) in the equilibrium.

9.2 Obtaining the Determinant and Inverse of a 3 × 3 Matrix

In section 9.1 we obtained the determinant of a 2×2 matrix. Below we will derive the determinant of a 3×3 matrix and then obtain the determinant of a general square matrix of dimension $n \times n$.

The determinant of a 3×3 matrix A will be composed of all the elements of A. However, as we will see, the expression of the determinant of A will be reduced to particular expressions involving the determinants of certain 2×2 submatrices of A. From a computational point of view, for the calculation of $|A|$, we only have to remember how to compute the determinant of a 2×2 matrix. Let us denote the square matrix A of order 3 by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In order to obtain $|A|$, we follow the steps below:

Step 1 Define M_{ij} to be the determinant of the 2×2 submatrix obtained when the ith row and the jth column of A are deleted. M_{ij} is known as a **minor.**

Example 9.14 Find M_{11} and M_{31} from A.

Solution

M_{11} refers to the determinant of the 2×2 submatrix obtained by deleting the first row and the first column of A. This submatrix is

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

M_{31} refers to the determinant of the 2×2 submatrix obtained by deleting the third row and the first column of A. This submatrix is

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad M_{31} = a_{12}a_{23} - a_{13}a_{22}$$ ■

Step 2 Attach a sign to the minor and define

$$C_{ij} = (-1)^{i+j} M_{ij}, \qquad i = 1, 2, 3;\ j = 1, 2, 3$$

C_{ij} is known as a **cofactor.** The sign of M_{ij} does not change if $i + j$ is an even number and changes if it is an odd number. For example for M_{11}, M_{31} and M_{22}, $C_{11} = M_{11}$, $C_{31} = M_{31}$, $C_{22} = M_{22}$. However for M_{21} and M_{23}, we have that $C_{21} = -M_{21}$ and $C_{23} = -M_{23}$. The sign matrix to remember how to convert M_{ij} into C_{ij} for the 3×3 case is

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Step 3 To obtain the determinant of A, we can take any row or column of A, multiply its elements by the corresponding cofactors, and add all these products. This is known as **cofactor expansion.** For example, expanding along the first row of the 3×3 matrix A, we have that

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Similarly we could calculate $|A|$ by expanding along the first column as

$$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$

Expanding along any row or column of the matrix results in the same answer, $|A|$.

Note that if we were to multiply the elements of a row (or column) by the cofactors of another row (or column), the determinant would vanish. For example, let us look at the expression

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$$

where the elements of the first row of A are multiplied by the cofactors obtained from the second row. The expansion above would have been obtained if the determinant were calculated for a matrix where the first and second rows were identical. But theorem 9.3 suggests that if a matrix has two identical rows or columns, then its determinant will be zero. Therefore, expanding along a row (or column) using the cofactors from another row (or column) can be thought of as obtaining the determinant of a matrix with two identical rows (or columns).

Obtaining the Inverse of a 3 × 3 Matrix

Having obtained the determinant of a 3×3 matrix, it is quite straightforward to obtain its inverse matrix. Below we will present the steps that one follows to obtain the inverse of a matrix of order 3 and illustrate the method by means of an example.

Step 1 Corresponding to the elements a_{ij} of A, we obtain the cofactors C_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$. Then we form a matrix in which each element a_{ij} is replaced by the corresponding cofactors C_{ij}, given below as

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step 2 Obtain the determinant of A, $|A|$.

Step 3 Transpose the matrix obtained in the first step. The resulting matrix is known as the **adjoint** matrix of the original matrix A. The adjoint matrix is denoted by $\text{adj}(A)$ and is given by

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Step 4 Once we have obtained $\text{adj}(A)$, we divide each of its elements by $|A|$. The resulting matrix is A^{-1}. This is given by

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Example 9.15 Find the inverse of the matrix A below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution

We first obtain $|A|$. Let us obtain the minors M_{11}, M_{12}, M_{13}, M_{21}, M_{22}, M_{23}, M_{31}, M_{32}, and M_{33}, and then the corresponding cofactors,

$$M_{11} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$
$$M_{12} = \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} = 1$$
$$M_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$
$$M_{21} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$$
$$M_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -2$$
$$M_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0$$
$$M_{31} = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5$$
$$M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1$$
$$M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

so that

$$\begin{array}{lll} C_{11} = 3 & C_{12} = -1 & C_{13} = -1 \\ C_{21} = 4 & C_{22} = -2 & C_{23} = 0 \\ C_{31} = -5 & C_{32} = 1 & C_{33} = 1 \end{array}$$

We obtain the determinant of A by expanding along the second row to get $|A| = 0(4) - 1(2) - 1(0) = -2$.

We choose to expand along the second row in order to take advantage of the zeros to simplify the calculations.

The cofactor matrix is given by

$$\begin{bmatrix} 3 & -1 & -1 \\ 4 & -2 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

The adjoint matrix is given as

$$\text{adj}(A) = \begin{bmatrix} 3 & 4 & -5 \\ -1 & -2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Finally we divide each element of adj(A) by $|A|$ to obtain the inverse

$$A^{-1} = \begin{bmatrix} -3/2 & -2 & 5/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

We can verify that A^{-1} satisfies $AA^{-1} = I$, since

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3/2 & -2 & 5/2 \\ 1/2 & 1 & -1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} -3/2 + 1 + 3/2 & -2 + 2 & 5/2 - 1 - 3/2 \\ 1/2 - 1/2 & 1 & -1/2 + 1/2 \\ -3/2 + 1 + 1/2 & -2 + 2 & 5/2 - 1 - 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

■

EXERCISES

1. Compute the determinants of the following matrices:

(a) $A = \begin{bmatrix} 3 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 3 & -4 \\ 4 & 0 & 5 \\ 5 & 1 & 6 \end{bmatrix}$

(d) $D = \begin{bmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{bmatrix}$

2. Obtain the inverse of each matrix in question 1.

3. Find the determinant of B:

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix}$$

if

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

4. Find the determinant of C:

$$C = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

if

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

5. Find the determinant of D:

$$D = \begin{bmatrix} a & b & c \\ 3d + a & 3e + b & 3f + c \\ g & h & i \end{bmatrix}$$

if

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

6. Use determinants to decide if the following set of vectors is linearly independent:

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix} \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

7. Compute $|A^3|$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

9.3 The Inverse of an *n* × *n* Matrix and Its Properties

To obtain the inverse of a matrix of order n, we follow the same steps outlined in the previous section for the case of a 3×3 matrix. From a computational point of view, obtaining the cofactors becomes a fairly complicated task. For example, in the case of a matrix of order 4 given below,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

obtaining any minor, say M_{14}, would involve solving for the determinant of a 3×3 matrix, since

$$M_{14} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

To obtain the determinant of the 3×3 matrix above, we would proceed to find its corresponding minors and cofactors and then proceed through a cofactor expansion along a given row or column.

For the $n \times n$ case, finding the cofactors would involve reducing through successive substitutions these cofactors to expressions involving determinants of 2×2 matrices. Of course, from a computational point of view, these calculations will be far from trivial. Note also that the properties of the determinants that were discussed earlier for the case of a 2×2 matrix apply also for matrices of order n.

In the next section we will discuss a method that allows us to compute the inverse matrix without explicitly having to compute the adjoint matrix and the determinant. Before we do that, we will present the general definitions of the minors, cofactors, and adjoints introduced in section 9.2. We will then summarize some of the properties of the inverse matrix of order n.

Definition 9.4

Consider an $n \times n$ matrix, A, with typical element a_{ij}. The **minor** associated with each element is denoted M_{ij} and is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column of the matrix A.

Notice that we can replace each element of the original matrix A with its associated minor. The resulting matrix is called the **matrix of minors.**

Definition 9.5

Consider an $n \times n$ matrix, A, with typical element a_{ij}. The **cofactor** of element a_{ij} is the minor of that element multiplied by $(-1)^{i+j}$, and is denoted C_{ij}:

$$C_{ij} = (-1)^{i+j} M_{ij}, \qquad i, j = 1, 2, \ldots, n$$

Clearly, where $i + j$ is an even number, the cofactor and the minor associated with a_{ij} are equivalent. When $i + j$ is an odd number, the cofactor is the negative of the minor.

Definition 9.6

An $n \times n$ matrix, A, has an associated **cofactor matrix** that is also $n \times n$ and is formed by replacing each a_{ij} with its associated cofactor.

At this point we note the generalization of the method of finding the determinant of a matrix by **cofactor expansion** introduced in section 9.2 for the 3×3 case. For the $n \times n$ case we have

Theorem 9.10

The determinant of an $n \times n$ matrix A may be found by adding along *any* row *or* column the product of each element a_{ij} and its associated cofactor.

Thus we have *either*

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad \text{for any single } j = 1, \ldots, n \tag{9.3}$$

or

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad \text{for any single } i = 1, \ldots, n \tag{9.4}$$

The choice of which particular row or column to expand by may be guided by computational ease. One rule of thumb is to choose the row or column of A with the largest number of zero elements. The cofactors for these zero elements clearly do not need to be computed. The crucial thing is that only expansion along a single row or column is permitted.

Definition 9.7 The **adjoint matrix** of an $n \times n$ matrix A, denoted $\text{adj}(A)$, is the transpose of the cofactor matrix of A.

This leads to the generalization for finding the inverse of any matrix.

Theorem 9.11 The **inverse** of an $n \times n$ matrix A is the adjoint matrix of A divided by the determinant of A:

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) \tag{9.5}$$

The following theorems give some of the properties of the inverse matrix.

Theorem 9.12 $(AB)^{-1} = B^{-1}A^{-1}$ provided that A and B are of the same order and A^{-1} and B^{-1} exist.

Proof

To show that this is true, we simply have to note that

$$(AB)^{-1}AB = B^{-1}A^{-1}AB = B^{-1}IB = I \qquad \blacksquare$$

It is also worth noting that this property of inverses is similar to the property of taking the transpose of a product matrix.

Theorem 9.13 The inverse of an inverse matrix reproduces the original matrix

$$(A^{-1})^{-1} = A$$

Thus, by the definition of an inverse,

$$(A^{-1})(A^{-1})^{-1} = I$$
$$AA^{-1}(A^{-1})^{-1} = A$$
$$(A^{-1})^{-1} = A$$

Theorem 9.14 The inverse of the transpose equals the transpose of the inverse

$$(A^T)^{-1} = (A^{-1})^T$$

We have $AA^{-1} = I$, take the transpose $(A^{-1})^T A^T = I$, and postmultiply by $(A^T)^{-1}$ both sides $(A^{-1})^T A^T (A^T)^{-1} = (A^T)^{-1}$. Thus

$$(A^{-1})^T = (A^T)^{-1}$$

Theorem 9.15 The determinant of the inverse matrix is 1 over the determinant of the original matrix, $|A^{-1}| = 1/|A|$.

This property follows from theorem 9.8 of determinants where $|AB| = |A||B|$. This can be seen from the fact that

$$A^{-1}A = I$$

Then taking determinants on both sides yields

$$|A||A^{-1}| = |I|$$

But, since $|I| = 1$, we have that

$$|A||A^{-1}| = 1$$
$$|A^{-1}| = \frac{1}{|A|}$$

(Remember that a determinant is a *scalar*, not a matrix.)

Theorem 9.16 The inverse of a diagonal matrix A, with elements a_{ii}, $i = 1, \ldots n$ on the main diagonal is also a diagonal matrix with diagonal elements $1/a_{ii}$.

In other words, if A is

$$A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

then A^{-1} is

$$A^{-1} = \begin{bmatrix} 1/a_{11} & & \\ & \ddots & \\ & & 1/a_{nn} \end{bmatrix}$$

We can easily verify that $A^{-1}A = I$ in that case.

EXERCISES

1. Obtain the inverses of the following matrices by the cofactor method:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -5 & 2 & 3 \end{bmatrix}$ **(c)** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Find the inverses of the following matrices:

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$

3. Suppose that a firm produces three outputs y_1, y_2, and y_3, with three inputs z_1, z_2, and z_3. The input-requirements matrix is given by A below:

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 1 & 1 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$

If the firm wants to produce 5 units of y_1, 5 units of y_2, and 10 units of y_3, how much of z_1, z_2, and z_3 will it require?

4. Compute $|A|$ in as few steps as possible:

$$A = \begin{bmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{bmatrix}$$

5. Find the determinants of the following matrices:

(a) $A = \begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ -1 & 0 & -9 & -5 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 0 & 4 \\ -1 & 2 & 8 & 5 \\ -1 & -1 & -2 & 3 \end{bmatrix}$

6. Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

9.4 Cramer's Rule

In this section we present **Cramer's rule,** a method for solving for n unknown variables in a system of n equations that is an alternative to the inverse-matrix method. The system of equations may be written as

$$A\mathbf{x} = \mathbf{b} \tag{9.6}$$

where A is a square matrix of order n that has an inverse A^{-1}, $\mathbf{x}$ is an array of order $n \times 1$ of n unknowns, and $\mathbf{b}$ is an array of order $n \times 1$ of known elements. Premultiplying both sides of equation (9.3) by A^{-1} yields the solution for $\mathbf{x}$, given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

In other words, to solve for the n unknowns, we have to obtain the inverse A^{-1}, which is given by

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where C_{ij}, $i = 1, \ldots n$, $j = 1, \ldots n$ are the relevant cofactors. Therefore we obtain as the solution for $\mathbf{x}$:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

From the expression above the solution for the first unknown, x_1, is seen to be

$$x_1 = \frac{1}{|A|}\left(b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1}\right)$$

However, the expression in parentheses is nothing but the evaluation of the determinant of a matrix derived from A by replacing its first column by the column vector $\mathbf{b}$. More precisely

$$(b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1}) = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Similarly, for x_2, the solution is obtained by evaluating the determinant of the matrix A where its second column has been replaced by $\mathbf{b}$. Then x_2 is simply the ratio of the determinant of A with its second column replaced by $\mathbf{b}$, to the determinant of A. In fact, x_j, the jth unknown, is calculated as the ratio of the determinant of A with its jth column replaced by $\mathbf{b}$, to the determinant of A.

Example 9.16 Solve the system for the unknowns x_1, x_2, and x_3, using Cramer's rule:

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= 15 \\ x_1 - 3x_2 + 2x_3 &= -5 \\ 6x_1 + 5x_2 + x_3 &= 28 \end{aligned}$$

Solution

We rewrite the system above in the form of equation (9.6) as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & -3 & 2 \\ 6 & 5 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

The solution for x_1 is given by $x_1 = |A_1|/|A|$, where A_1 is the matrix A with its first column replaced by $\mathbf{b}$:

$$A_1 = \begin{bmatrix} 15 & 4 & -1 \\ -5 & -3 & 2 \\ 28 & 5 & 1 \end{bmatrix}$$

We first obtain $|A|$. This is given by the cofactor expansion method along the first column as

$$\begin{aligned}|A| &= 2\begin{vmatrix}-3 & 2\\ 5 & 1\end{vmatrix} - 1\begin{vmatrix}4 & -1\\ 5 & 1\end{vmatrix} + 6\begin{vmatrix}4 & -1\\ -3 & 2\end{vmatrix}\\ &= 2(-13) - 9 + 6(5) = -5\end{aligned}$$

Then we obtain the determinant of A_1 by expanding again along the first column of A_1:

$$\begin{aligned}|A_1| &= 15\begin{vmatrix}-3 & 2\\ 5 & 1\end{vmatrix} + 5\begin{vmatrix}4 & -1\\ 5 & 1\end{vmatrix} + 28\begin{vmatrix}4 & -1\\ -3 & 2\end{vmatrix}\\ &= 15(-13) + 5(9) + 28(5) = -10\end{aligned}$$

Therefore

$$x_1 = \frac{|A_1|}{|A|} = \frac{-10}{(-5)} = 2$$

Similarly, for x_2, we have to evaluate the ratio of $|A_2|/|A|$, where A_2 is A with its second column replaced by **b**:

$$A_2 = \begin{bmatrix}2 & 15 & -1\\ 1 & -5 & 2\\ 6 & 28 & 1\end{bmatrix}$$

The determinant of A_2 is given by

$$\begin{aligned}|A_2| &= -15\begin{vmatrix}1 & 2\\ 6 & 1\end{vmatrix} - 5\begin{vmatrix}2 & -1\\ 6 & 1\end{vmatrix} - 28\begin{vmatrix}2 & -1\\ 1 & 2\end{vmatrix}\\ &= -15(-11) - 5(8) - 28(5) = -15\end{aligned}$$

and

$$x_2 = \frac{|A_2|}{|A|} = \frac{-15}{(-5)} = 3$$

Finally, $x_3 = |A_3|/|A|$, where A_3 is

$$A_3 = \begin{bmatrix}2 & 4 & 15\\ 1 & -3 & -5\\ 6 & 5 & 28\end{bmatrix}$$

The determinant $|A_3|$ is found as

$$|A_3| = 15\begin{vmatrix} 1 & -3 \\ 6 & 5 \end{vmatrix} + 5\begin{vmatrix} 2 & 4 \\ 6 & 5 \end{vmatrix} + 28\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}$$
$$= 15(23) + 5(-14) + 28(-10) = -5$$

Therefore

$$x_3 = \frac{|A_3|}{|A|} = \frac{-5}{(-5)} = 1$$

The solutions for the unknowns x_1, x_2, and x_3 are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

■

The IS-LM Model Again

Let us look at the following IS-LM model, first considered in chapter 7. The goods market is described by

$$C = a + b(1-t)Y$$
$$I = e - lR$$
$$G = \bar{G}$$

The money market is described by

$$L = kY - hR$$
$$M = \bar{M}$$

The economy in equilibrium is then characterized by

$$Y = C + I + \bar{G}$$
$$C = a + b(1-t)Y$$
$$I = e - lR$$
$$\bar{M} = kY - hR$$

There are four endogenous variables in the system, Y, C, I, and R and four exogenous variables, $\bar{G}$, a, e, and $\bar{M}$. The system can be written in the form

$$A\mathbf{x} = \mathbf{b}$$

where A is a 4×4 matrix of parameters, $\mathbf{x}$ is a 4×1 vector of the endogenous variables, and $\mathbf{b}$ is a vector of constants and exogenous variables.

Suppose that we are interested in determining R. We will do that by Cramer's rule. The system is given by

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -b(1-t) & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ k & 0 & 0 & -h \end{bmatrix} \begin{bmatrix} Y \\ C \\ I \\ R \end{bmatrix} = \begin{bmatrix} \bar{G} \\ a \\ e \\ \bar{M} \end{bmatrix}$$

Solving for R yields $R = |A_4|/|A|$, where A_4 is obtained by replacing the fourth column of A by $\mathbf{b}$:

$$A_4 = \begin{bmatrix} 1 & -1 & -1 & \bar{G} \\ -b(1-t) & 1 & 0 & a \\ 0 & 0 & 1 & e \\ k & 0 & 0 & \bar{M} \end{bmatrix}$$

We obtain $|A|$ by expanding along the third row of A:

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 1 & -1 & 0 \\ -b(1-t) & 1 & 0 \\ k & 0 & -h \end{vmatrix} - l \begin{vmatrix} 1 & -1 & -1 \\ -b(1-t) & 1 & 0 \\ k & 0 & 0 \end{vmatrix} \\ &= -h \begin{vmatrix} 1 & -1 \\ -b(1-t) & 1 \end{vmatrix} - l \left(-1 \begin{vmatrix} -b(1-t) & 1 \\ k & 0 \end{vmatrix} \right) \\ &= -h[1 - b(1-t)] - lk \end{aligned}$$

Then $|A_4|$ is obtained by expanding along the third row of A_4:

$$\begin{aligned} |A_4| &= 1 \begin{vmatrix} 1 & -1 & \bar{G} \\ -b(1-t) & 1 & a \\ k & 0 & \bar{M} \end{vmatrix} - e \begin{vmatrix} 1 & -1 & -1 \\ -b(1-t) & 1 & 0 \\ k & 0 & 0 \end{vmatrix} \\ &= k \begin{vmatrix} -1 & \bar{G} \\ 1 & a \end{vmatrix} + \bar{M} \begin{vmatrix} 1 & -1 \\ -b(1-t) & 1 \end{vmatrix} - e \left(k \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} \right) \\ &= -k(a + \bar{G}) + \bar{M}[1 - b(1-t)] - ke \end{aligned}$$

Therefore R is given by

$$R = \frac{k(a + e + \bar{G}) - \bar{M}[1 - b(1-t)]}{h[1 - b(1-t)] + lk}$$

This solution for R is the *reduced-form* equation for R. (Compare your answer to question 4 in the exercises to section 7.1.) It expresses the endogenous variable R as a function of exogenous variables and parameters only.

Example 9.17 Consider the following closed economy IS-LM model:

$$\begin{aligned} C &= 15 + 0.8(Y - T) \\ T &= -25 + 0.25Y \\ I &= 65 - R \\ G &= \bar{G} \\ L &= 5Y - 50R \\ M &= 1{,}500 \end{aligned}$$

Solve for the equilibrium level of income in terms of government spending, $\bar{G}$. At what level of public spending does the government balance its budget?

Solution

The IS curve is found from $Y = C + I + \bar{G}$:

$$0.4Y + R = \bar{G} + 100$$

and the LM curve from $L = M$:

$$Y - 10R = 300$$

These can be written in the form

$$\begin{bmatrix} 0.4 & 1 \\ 1 & -10 \end{bmatrix}\begin{bmatrix} Y \\ R \end{bmatrix} = \begin{bmatrix} \bar{G} + 100 \\ 300 \end{bmatrix}$$

Now, $|A| = 0.4(-10) - 1(1) = -5$ and to find Y we need $|A_1|$ which is

$$\begin{vmatrix} \bar{G} + 100 & 1 \\ 300 & -10 \end{vmatrix} = (\bar{G} + 100)(-10) - 300(1) = -10\bar{G} - 1{,}300$$

and so

$$Y = \frac{-10\bar{G} - 1{,}300}{-5} = 2\bar{G} + 260$$

Income is just a simple linear function of government spending. In this case the government expenditure multiplier is 2—each dollar increase in government spending increases income by 2 dollars. Tax revenue is

$$\begin{aligned} T &= -25 + 0.25Y \\ &= -25 + 0.25[2\bar{G} + 260] \\ &= 40 + 0.5\bar{G} \end{aligned}$$

This tells us that, while an increase in $\bar{G}$ generates more income (through the multiplier), each dollar increase in government spending only generates an extra 50 cents in tax revenue. The government deficit is $\bar{G} - T$, and the budget is balanced when this is zero, or when $\bar{G} = T$. This implies that

$$\bar{G} = 40 + 0.5\bar{G} \quad \Rightarrow \quad \bar{G} = 80$$ ■

The "Open" Leontief Input-Output Model

This model looks at the economy as a number of interrelated industrial sectors. The industries are interrelated because an industry's output, in general, is used as an input into some other industries' production processes as well as possibly finding its way into final demand by consumers. Therefore, in general, each industry is potentially the producer of an intermediate good that may also be used in final consumption. The problem is to find the production level for each industry that is just sufficient to supply the demands from industry and consumers alike.

To model such a system, we start by expressing all outputs and demands in money terms. Since prices are assumed to be fixed, we can always recover the implied *physical* quantities by dividing through by the appropriate price per unit. There are assumed to be n goods produced by n industries, with the money value of output of the ith industry given by x_i. The production vector for the economy in money terms is therefore given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{x} \geq 0$$

The final demand by consumers in money terms for the output of industry i is fixed at d_i, and so the final demand vector as a whole is

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}, \qquad \mathbf{d} \geq 0$$

Finally we need to specify the *input requirements* of each industry. Denote by a_{ij} the amount of the money value of the output of industry i needed to produce one unit of output in industry j. This is a fixed technological requirement and the full, economywide array of input requirements is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Note that this is necessarily a square matrix, though, of course, some of the a_{ij} may be zero, reflecting the fact that industry j may not use any of the output of industry i as input. We refer to A as the matrix of *production coefficients*. Notice also that a_{ii} may be positive for some or all industries, meaning that some (or all) industries require some of their own output to be used in their production process.

The total money value of the output of industry i required by all industries is

$$\sum_{j=1}^{n} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

where $a_{ij}x_j$ is the money value of the output of industry i required to produce the x_j units of output of industry j. In total, the demands made on the output of all industries can be expressed as a column array

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

which is simply

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Each row here is the total demand on the output of industry i made by the entire production sector. Of course, in general there will also be a final demand from the consumption sector for the output of industry i, d_i. This economy demand for the output of industry i in the economy as a whole is

$$\sum_{j=1}^{n} a_{ij}x_j + d_i$$

and for supply to equal demand in sector i, we must have

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + d_i$$

If all demands in the economy are to be supplied, this must hold for all n industries, and so

$$\mathbf{x} = A\mathbf{x} + \mathbf{d} \tag{9.7}$$

We can now pose the problem set by this model. Given the matrix of production coefficients A (summarizing the input requirements of industry) and a final demand vector $\mathbf{d}$, what is the vector of outputs $\mathbf{x}$, that will just satisfy equation (9.7)?

We can rearrange equation (9.7) to give

$$\mathbf{x} - A\mathbf{x} = \mathbf{d}$$

or

$$(I - A)\mathbf{x} = \mathbf{d}$$

and so, if $(I - A)^{-1}$ exists, we may write the solution as

$$\mathbf{x} = (I - A)^{-1}\mathbf{d}$$

Theorem 9.17 If $(I - A)^{-1}$ has only nonnegative entries, then for any $\mathbf{d} \geq \mathbf{0}$, there is a unique nonnegative solution for $\mathbf{x}$.

Example 9.18 **A Three-Sector Input-Output Model**

We will use the input requirements matrix that was first presented in example 8.3. In this case the economy consists of three industries: an agricultural industry, a mining industry, and a manufacturing industry. To produce one unit of agricultural output, the agricultural sector requires \$0.3 of its own output, \$0.2 of mining output and \$0.4 of manufacturing output. To produce one unit of mining output, the mining sector requires \$0.5 of agricultural output, \$0.2 of its own output, and \$0.2 of manufacturing output. To produce one unit of manufacturing output requires \$0.3 of agricultural output, \$0.3 of mining output, and \$0.3 of its own output. Final demands by consumers amount to \$20,000, \$10,000, and \$40,000 for goods 1, 2, and 3, respectively. Find the equilibrium quantities of output for the three sectors.

We have

$$A = \begin{bmatrix} 0.3 & 0.5 & 0.3 \\ 0.2 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 20{,}000 \\ 10{,}000 \\ 40{,}000 \end{bmatrix}$$

First, construct $(I - A)$:

$$I - A = \begin{bmatrix} 0.7 & -0.5 & -0.3 \\ -0.2 & 0.8 & -0.3 \\ -0.4 & -0.2 & 0.7 \end{bmatrix}$$

Now, find the inverse of the above by the usual methods described earlier:

$$(I - A)^{-1} = \begin{bmatrix} 4.4643 & 3.6607 & 3.4821 \\ 2.3214 & 3.3036 & 2.4107 \\ 3.2143 & 3.0357 & 4.1071 \end{bmatrix}$$

So

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 4.4643 & 3.6607 & 3.4821 \\ 2.3214 & 3.3036 & 2.4107 \\ 3.2143 & 3.0357 & 4.1071 \end{bmatrix} \begin{bmatrix} 20{,}000 \\ 10{,}000 \\ 40{,}000 \end{bmatrix} \\ &= \begin{bmatrix} 89{,}286 + 36{,}607 + 139{,}284 \\ 46{,}424 + 33{,}036 + 96{,}428 \\ 64{,}286 + 30{,}357 + 164{,}284 \end{bmatrix} \\ &= \begin{bmatrix} 265{,}177 \\ 175{,}892 \\ 258{,}927 \end{bmatrix} \end{aligned}$$

The agricultural industry should produce \$265,177 worth of output, the mining industry \$175,892 worth of output, and the manufacturing industry \$258,927 worth of output. ■

Example 9.19 **A "Closed" Leontief Model**

Take the economy of example 9.18. Suppose that there are two *primary inputs* (i.e., inputs that cannot be increased in the short run), labor and capital. The vectors

$$\mathbf{e} = \begin{bmatrix} 2.2 \\ 3.0 \\ 0.8 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 1.5 \\ 2.6 \\ 3.8 \end{bmatrix}$$

give the amounts of labor and capital required to produce, respectively, one unit of agricultural, mining, and manufacturing output.

(i) Find the primary input requirements of the economy when it wishes to produce the final demand vector

$$\mathbf{d} = \begin{bmatrix} 20{,}000 \\ 10{,}000 \\ 40{,}000 \end{bmatrix}$$

(ii) Is this final demand vector *feasible* for this economy if it has available a maximum of 1,200,000 units of labor and 1,700,000 units of capital?

(iii) Find the set of final demand vectors that are feasible for the economy given the primary input availability in (ii).

Solution

To answer (i), we simply note that to produce the given demand vector, we require a total output vector

$$\mathbf{x} = (I - A)^{-1}\mathbf{d} = \begin{bmatrix} 265{,}177 \\ 175{,}892 \\ 258{,}927 \end{bmatrix}$$

as was computed in example 9.18. It follows that the total labor requirement of the economy is

$$L_D = \mathbf{e}^T\mathbf{x} = [2.2 \quad 3.0 \quad 0.8] \begin{bmatrix} 265{,}177 \\ 175{,}892 \\ 258{,}927 \end{bmatrix} = 1{,}318{,}207$$

Similarly the total capital requirement is

$$K_D = \mathbf{k}^T\mathbf{x} = [1.5 \quad 2.6 \quad 3.8] \begin{bmatrix} 265{,}177 \\ 175{,}892 \\ 258{,}927 \end{bmatrix} = 1{,}839{,}007$$

Then, to answer (ii), we note that if the economy has 1,200,000 units of labor and 1,700,000 units of capital, the given final demand vector is *not* feasible, since it requires more labor and capital than are available to the economy.

We can answer (iii) by first expressing the problem algebraically. In general, the required amount of labor is

$$L_D = \mathbf{e}^T\mathbf{x} = \mathbf{e}^T(I - A)^{-1}\mathbf{d}$$

and that of capital is

$$K_D = \mathbf{k}^T\mathbf{x} = \mathbf{k}^T(I - A)^{-1}\mathbf{d}$$

For a final demand vector to be feasible, we must have the following weak inequalities simultaneously satisfied:

$$L_D \leq 1{,}200{,}000 \quad \text{or} \quad \mathbf{e}^T(I - A)^{-1}\mathbf{d} \leq 1{,}200{,}000$$

$$K_D \leq 1{,}700{,}000 \quad \text{or} \quad \mathbf{k}^T(I - A)^{-1}\mathbf{d} \leq 1{,}700{,}000$$

These inequalities then define the feasible set of demand vectors. To obtain more insight into this, let us compute the 1×3 vectors $\mathbf{e}^T(I - A)^{-1}$ and $\mathbf{k}^T(I - A)^{-1}$:

$$\begin{aligned}\mathbf{e}^T(I - A)^{-1} &= [2.2 \quad 3.0 \quad 0.8]\begin{bmatrix} 4.4643 & 3.6607 & 3.4821 \\ 2.3214 & 3.3036 & 2.4107 \\ 3.2143 & 3.0357 & 4.1071 \end{bmatrix} \\ &= [19.3571 \quad 20.6499 \quad 18.1884]\end{aligned}$$

$$\begin{aligned}\mathbf{k}^T(I - A)^{-1} &= [1.5 \quad 2.6 \quad 3.8]\begin{bmatrix} 4.4643 & 3.6607 & 3.4821 \\ 2.3214 & 3.3036 & 2.4107 \\ 3.2143 & 3.0357 & 4.1071 \end{bmatrix} \\ &= [24.9464 \quad 23.1875 \quad 27.0980]\end{aligned}$$

(Note that we have found the inverse matrix $(I - A)^{-1}$ in example 9.18.)

This then allows us to write the feasibility constraints as

$$[19.3571 \quad 20.6499 \quad 18.1884]\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \leq 1{,}200{,}000$$

$$[24.9464 \quad 23.1875 \quad 27.0980]\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \leq 1{,}700{,}000$$

or

$$19.3571d_1 + 20.6499d_2 + 18.1884d_3 \leq 1{,}200{,}000$$
$$24.9464d_1 + 23.1875d_2 + 27.0980d_3 \leq 1{,}700{,}000$$

If we take these constraints as equalities, each defines a plane in three dimensions, which we graph in figure 9.2.

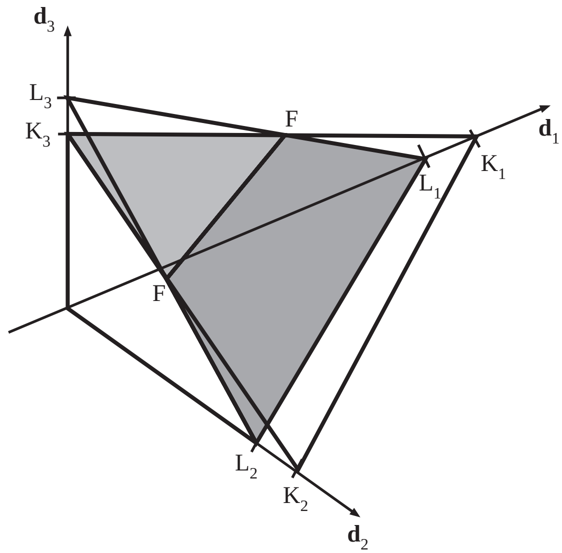


Figure 9.2 Feasibility constraints for example 9.19

The end-points of each plane are found in turn as the points $(d_1, 0, 0)$, $(0, d_2, 0)$, and $(0, 0, d_3)$. Then, the feasible set of final demand vectors for this economy is given by the set of points in the shaded area and along the portions of the planes bounding it. We see that the point

$$\begin{bmatrix} 20{,}000 \\ 10{,}000 \\ 40{,}000 \end{bmatrix}$$

is not feasible, because it lies above the labor plane. Note finally that *only* the final demand vectors along the segment FF at the intersection of the planes achieve exactly full employment of both capital and labor in this economy. All other feasible final demand vectors imply unemployment of at least one primary input. ■

Equilibrium in *n*-Markets

Section 8.2 ended with a quite general matrix formulation of equilibrium in n markets. We now have a method of solving such a system. Recall that equilibrium requires that

$$\alpha - a = (B - \beta)\mathbf{p}$$

where α and a are $n \times 1$ vectors of constants, p is an $n \times 1$ vector of (equilibrium) prices, and B and β are $n \times n$ matrices of demand and supply parameters, respectively.

In view of the definition of the inverse, the solution for the equilibrium vector of prices is

$$\mathbf{p} = (B - \beta)^{-1}(\alpha - a)$$

which gives a system of reduced-form equations for the prices. The following 3×3 example illustrates and shows how to solve for a particular price (rather than the entire vector of prices) using Cramer's rule.

Example 9.20 Consider the markets for coffee, tea, and sugar. These goods are related in demand, since the first two are often substitutes for each other while the third is often complementary with each of the other two goods. Ignoring any supply side links (which are, in any case, unlikely), we have as an example:

$$\begin{aligned} q_t^d &= 100 - 5p_t + 3p_c - p_s \\ q_t^s &= -10 + 2p_t \end{aligned}$$

$$\begin{aligned} q_c^d &= 120 - 8p_c + 2p_t - 2p_s \\ q_c^s &= -20 + 5p_c \end{aligned}$$

$$\begin{aligned} q_s^d &= 300 - 10p_t - 5p_c - p_s \\ q_s^s &= 15p_s \end{aligned}$$

Setting these pairs of equations equal to each other and arranging in matrix form gives the system for equilibrium prices:

$$\begin{bmatrix} 110 \\ 140 \\ 300 \end{bmatrix} = \begin{bmatrix} 7 & -3 & 1 \\ -2 & 13 & 2 \\ 10 & 5 & 16 \end{bmatrix} \begin{bmatrix} p_t \\ p_c \\ p_s \end{bmatrix}$$

We can solve for any one price, say p_c, using Cramer's rule. The determinant of the square matrix in this equation system is

$$7\begin{vmatrix} 13 & 2 \\ 5 & 16 \end{vmatrix} - (-3)\begin{vmatrix} -2 & 2 \\ 10 & 16 \end{vmatrix} + 1\begin{vmatrix} -2 & 13 \\ 10 & 5 \end{vmatrix}$$

where we have done a cofactor expansion along the first row. Solving gives

$$7(208 - 10) + 3(-32 - 20) + (-10 - 130) = 1{,}090$$

Since we are solving for p_c, we need $|A_2|$, in the notation developed earlier, where

$$\begin{vmatrix} 7 & 110 & 1 \\ -2 & 140 & 2 \\ 10 & 300 & 16 \end{vmatrix}$$

Solving as before (i.e., expanding along the first row of A_2) gives

$$7(2{,}240 - 600) - 110(-32 - 20) + (-600 - 1{,}400) = 15{,}200$$

and so $p_c = 15{,}200/1{,}090 \doteq 14$. ■

In the following exercises, we ask you to solve for the entire vector of prices in this last example using matrix inversion. It appears that selectively solving for particular prices of interest using Cramer's rule is computationally easier than solving for the entire vector of equilibrium prices.

EXERCISES

1. Use Cramer's rule to compute the solution of the following system:

$$\begin{aligned} -2x_1 + x_2 \quad &= 7 \\ -3x_1 \quad + x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

2. Use Cramer's rule to compute the solution of the following system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ -x_1 \quad + 2x_3 &= 7 \\ 3x_1 + x_2 + 3x_3 &= -3 \end{aligned}$$

3. Solve the following system for the unknowns x_1, x_2, and x_3:

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= 3 \\ x_1 - 3x_2 + 2x_3 &= -1 \\ 6x_1 + 5x_2 + x_3 &= 5 \end{aligned}$$

4. Find the equilibrium quantities of output for the three sectors of example 9.18 with the following new sets of data:

(a) $A = \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.2 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 40{,}000 \\ 10{,}000 \\ 20{,}000 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0.2 & 0.2 & 0.3 \\ 0.3 & 0.5 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 10{,}000 \\ 20{,}000 \\ 40{,}000 \end{bmatrix}$

5. Using the IS-LM model of section 9.4, obtain the reduced-form equations for Y and C.

6. Solve for the entire vector of equilibrium prices in example 9.20, using matrix inversion.

CHAPTER REVIEW

Key Concepts

adjoint matrix
cofactor
cofactor expansion
cofactor matrix
Cramer's rule
determinant
inverse matrix
Leontief model
matrix of minors
minor
nonsingular matrix
singular matrix
triangular matrix

Review Questions

1. What is the scalar equivalent of the matrix operation of multiplication by the inverse of a matrix?
2. What is the determinant of a matrix?
3. What are the main properties of determinants?
4. What is a cofactor matrix?
5. How is the adjoint matrix related to the cofactor matrix?

6. If you had a system of 10 equations with 10 endogenous variables to be determined, but you were only interested in the values of two of the variables, would you solve the system by (a) matrix inversion (b) Cramer's rule. Or, is it difficult to choose between the methods? Explain.

Review Exercises

1. Does the following matrix have an inverse?

$$A = \begin{bmatrix} 12 & 13 & 14 \\ 15 & 16 & 17 \\ 18 & 19 & 20 \end{bmatrix}$$

2. Verify that $\det A = \det B + \det C$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1 + v_1 \\ a_{21} & a_{22} & u_2 + v_2 \\ a_{31} & a_{32} & u_3 + v_3 \end{bmatrix} \quad B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix}$$

$$C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

What do you conclude about the statement that in general for all matrices B and C, $\det(B + C) = \det B + \det C$?

3. Use the cofactor expansion method to compute $|A|$, where

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

4. Suppose that a firm produces three outputs, y_1, y_2, and y_3, with three inputs, z_1, z_2, and z_3. The input-requirement matrix is given by

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

If the firm wants to produce 10 units of y_1, 20 units of y_2, and 10 units of y_3, how much of z_1, z_2, and z_3 will it require?

5. Use Cramer's rule to compute the solutions to

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ -3x_1 + x_3 &= -8 \\ x_2 + 2x_3 &= -2 \end{aligned}$$

6. Let A and B be square matrices. Show that even though AB may not be equal to BA, it is always true that $|AB| = |BA|$.

7. Let A be a square matrix such that $A^T A = I$. Show that $|A| = \pm 1$.

8. Let A and B be square matrices, with B invertible. Show that $|BAB^{-1}| = |A|$.

9. Suppose that A is a square matrix such that $|A^3| = 0$. Show that A is not invertible.

Chapter 10 Some Advanced Topics in Linear Algebra

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Statistical Distribution of Quadratic Forms
- The Classical Least-Squares Model: Example
- The Generalized Least-Squares Transformation: Example

In this chapter we consider three important advanced topics in matrix algebra: vector spaces, eigenvalues, and quadratic forms. All play important roles in a variety of contexts in economic theory and in econometrics. Vector spaces enable us to talk about distance between points, and linear dependence between vectors. They are therefore closely linked to the study of systems of linear equations of chapter 7. Eigenvalues play an important role in determining the stability properties of dynamic, linear systems and so this topic is of use in chapters 18, 20, 21, 23, and 24. Quadratic forms have applications in econometrics, and to the study of second-order derivatives in multivariate calculus discussed in chapter 11.

10.1 Vector Spaces

A vector can be thought of as a special kind of matrix, with n rows and just one column (see definition 8.2). Thus we can write the vector $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

For obvious reasons, we refer to this as a *column vector* and its $1 \times n$ transpose

$$\mathbf{v}^T = [v_1 \quad v_2 \quad \cdots \quad v_n]$$

as a *row vector*. The rules for adding, subtracting, and multiplying vectors then follow from those defined in chapter 8 for matrices in general. Thus addition (subtraction) is defined for two vectors with the same dimension,

$$\mathbf{w} \pm \mathbf{v} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \pm \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} w_1 \pm v_1 \\ w_2 \pm v_2 \\ \vdots \\ w_n \pm v_n \end{bmatrix}$$

while the product of two vectors is now referred to as the **inner product,** denoted $\mathbf{w} \cdot \mathbf{v}$, and is defined for two vectors of the same dimension

$$\mathbf{w} \cdot \mathbf{v} \equiv \mathbf{w}^T \mathbf{v} = [w_1 \quad w_2 \quad \cdots \quad w_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} w_i v_i$$

Note that $\mathbf{w}^T \mathbf{v} = \mathbf{v}^T \mathbf{w}$. Example 8.7 goes through the steps involved in vector addition (subtraction) and multiplication.

However, there is an alternative approach that begins with the idea of a vector, defines the above operations without reference to matrices in general, and then goes on to develop a number of important concepts concerning vectors. An important aspect of this approach is the close link with coordinate geometry, which is harder to achieve in the case of matrices in general. The link between vectors and matrices is reestablished by regarding a matrix as a collection of vectors. In this section we set out the main elements of this approach.

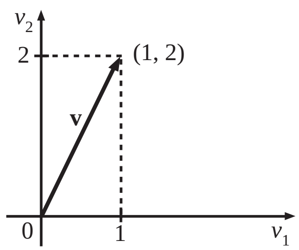


Figure 10.1 Vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Define a (real) vector as an array of (real) numbers and define the operations of addition, subtraction, and multiplication as above. The link with geometry is established by noting that we can associate with any two-component vector, $\mathbf{v}$, a point in $\mathbb{R}^2$, with v_1 for the x-coordinate and v_2 the y-coordinate, as figure 10.1 illustrates for

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{10.1}$$

This vector is shown in figure 10.1. The vector is pictured as an arrow from the origin to a point with coordinates given by the two elements of $\mathbf{v}$. The first element gives the coordinate on the horizontal axis and the second gives the coordinate on the vertical axis.

Using Pythagoras's theorem, we can define the *length* of this vector as the square root of the sum of squares of the coordinates, since the length of the vector is the length of the hypotenuse of the triangle formed by connecting the end point of the vector to the horizontal (or vertical) axis. So the length of the vector (10.1) is $\|\mathbf{v}\| = \sqrt{1^2 + 2^2}$. In general, for an n-dimensional vector, we have

Definition 10.1

The **length** of an n-dimensional vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \tag{10.2}$$

The length of a vector is also often called its **Euclidean norm** because the concept of distance underlying $\|\mathbf{v}\|$ is an important element of Euclidean geometry. Note that the length of the vector $\mathbf{v}$ may be defined equivalently as the square root of the inner product of $\mathbf{v}$ with itself: $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T\mathbf{v}}$.

Example 10.1

Find the length of

(i) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

Solution

For (i) we have $\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$.
For (ii) we have $\|\mathbf{w}\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$. ■

We now look at the graphical interpretation of the operation of **vector addition.**

If we add to the vector (10.1) the vector

$$\mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

we obtain

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

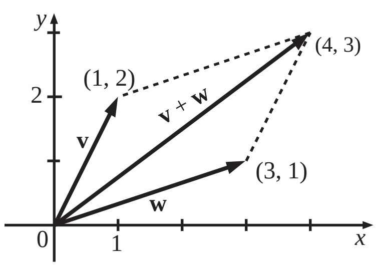


Figure 10.2 Vector addition

Graphically, this is shown in figure 10.2. Diagrammatically, summing two vectors involves constructing a parallelogram, and it is clear from this example that $\mathbf{v} + \mathbf{w}$ and $\mathbf{w} + \mathbf{v}$ are equivalent operations.

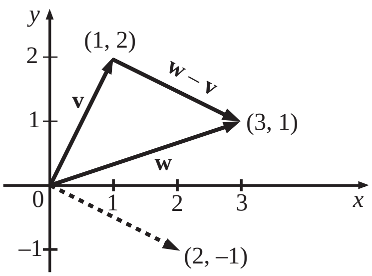


Figure 10.3 Vector subtraction

Vector subtraction is similarly illustrated by

$$\mathbf{w} - \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Graphically, this is shown in figure 10.3. The resulting vector is the one connecting the two end points of the original vectors.

Figure 10.3 also illustrates another feature of vectors. Since vectors are defined by their *length* and *direction*, vectors that are parallel and of the same length are equivalent. Thus the vector obtained by subtracting **v** from **w** is equivalent to the dotted vector drawn from the origin to the point $(2, -1)$ in figure 10.3.

Scalar multiplication of a vector is done by multiplying all the entries of the vector by the scalar. If the scalar is 2, for instance, we have

$$2\mathbf{v} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Graphically scalar multiplication changes the *length* of a vector but not its *direction*. If the scalar is a fraction, then the resulting vector is shorter than the original. If the scalar is greater than 1, the resulting vector is an extended version of the original. Figure 10.4 illustrates. Vector **v** is multiplied by a fraction k to obtain vector $k\mathbf{v}$,

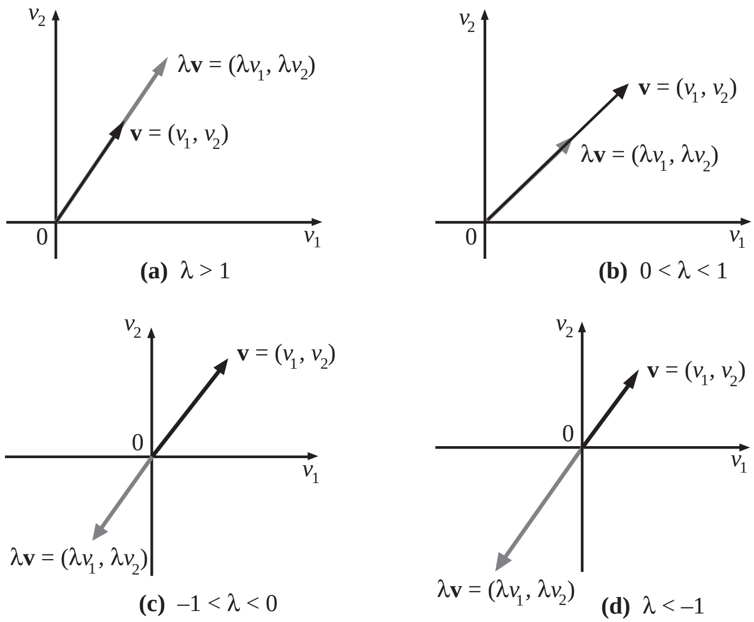


Figure 10.4 Scalar multiplication

while vector **u** is multiplied by $\lambda > 1$ to obtain $\lambda\mathbf{u}$. If a vector is multiplied by a negative scalar, then the direction of the original vector is reversed and its length changes according to the absolute value of the scalar. Notice that the vectors **v** and $-\mathbf{v}$ have the same length but point in opposite directions.

Combining the operation of scalar multiplication and vector addition allows the representation of any vector as a linear combination of vectors **v** and **w**. If **u** represents such an arbitrary vector, we write

$$\mathbf{u} = \lambda_1\mathbf{v} + \lambda_2\mathbf{w} \tag{10.3}$$

where λ_1 and λ_2 are scalars.

Example 10.2 Find the scalars λ_1 and λ_2 that are attached to

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

to yield

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using equation (10.3), we are looking for λ_1 and λ_2 satisfying

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

or carrying out the scalar multiplication and vector addition

$$\lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 + 3\lambda_2 \\ 2\lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

From the second equation $\lambda_2 = -2\lambda_1$, and substitution back into the first equation yields

$$3(-2\lambda_1) + \lambda_1 = 1 \quad \Rightarrow \quad \lambda_1 = \frac{-1}{5} \quad \text{and} \quad \lambda_2 = \frac{2}{5}$$

This is illustrated in figure 10.5. ■

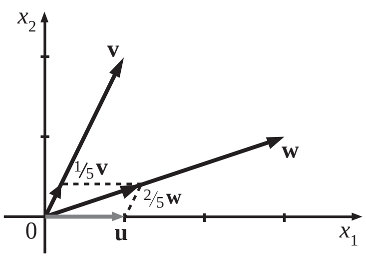


Figure 10.5 Linear combination of two vectors in example 10.2

Example 10.3 Find λ_1 and λ_2 that are attached to

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

to give

$$\mathbf{u} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Solution

Again, using equation (10.3), we have

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and so

$$\lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 + 3\lambda_2 \\ 2\lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

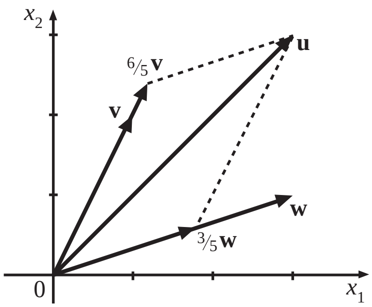


Figure 10.6 Linear combination of two vectors in example 10.3

From the first equation we obtain $\lambda_1 = 3 - 3\lambda_2$, and substitution into the second equation yields

$$2(3 - 3\lambda_2) + \lambda_2 = 3 \Rightarrow \lambda_2 = \frac{3}{5} \quad \text{and} \quad \lambda_1 = \frac{6}{5}$$

This is illustrated in figure 10.6. ■

Given the relationship between vectors and geometry, two vectors of different lengths may point in different directions or in the same direction. Consider first those that point in the same direction. These vectors must simply be scalar multiples of each other. That is, the direction of the vector **v** must be the same as the direction of the vector $\lambda\mathbf{v}$ for any $\lambda > 0$. This family of vectors is said to be **linearly dependent:**

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{to give} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

that is, $\mathbf{u} = (-1/5)\mathbf{v} + (2/5)\mathbf{w}$. The requirement that vectors point in different directions is known as **linear independence.**

Definition 10.2 Two vectors in $\mathbb{R}^2$, $\mathbf{v}$ and $\mathbf{w}$, are **linearly independent** if the scalars λ_1 and λ_2 satisfying

$$\lambda_1\mathbf{v} + \lambda_2\mathbf{w} = \mathbf{0}$$

are zero. Here $\mathbf{0}$ is the null vector.

If the λ_is are nonzero, then the pair of vectors $\mathbf{v}$ and $\mathbf{w}$ would point in the same direction and they would be **linearly dependent.** To see this, note that a nontrivial solution to the equation $\lambda_1\mathbf{v} + \lambda_2\mathbf{w} = \mathbf{0}$ would imply that

$$\mathbf{v} = -\frac{\lambda_2}{\lambda_1}\mathbf{w}$$

which states that $\mathbf{v}$ is a scalar multiple of $\mathbf{w}$.

Example 10.4 Establish whether the following vectors in $\mathbb{R}^2$ are linearly independent:

(i) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(ii) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution

In case (i),

$$\lambda_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 1$$

Therefore $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent.
In case (ii),

$$\lambda_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \lambda_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The only solution to this equation is $\lambda_1 = \lambda_2 = 0$, and the vectors are linearly independent. ■

The idea of linear independence generalizes, as summarized in the following:

Theorem 10.1 Let $\mathcal{V}$ be a set of vectors in $\mathbb{R}^m$,

$$\mathcal{V} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$$

Then, if $n > m$, the vectors in $\mathcal{V}$ are linearly dependent and

$$\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \cdots + \lambda_n\mathbf{v} = \mathbf{0}$$

has only the trivial solution.

Proof

If we write out this homogeneous system, it looks similar to the system in equation (7.15) in chapter 7 with the λ_i as the unknowns. Theorem 10.1 then follows directly from theorem 7.7. ■

Now consider two linearly independent vectors in $\mathbb{R}^2$, **v** and **w**. If we let **u** be any other vector in $\mathbb{R}^2$, then we can always find two numbers λ_1 and λ_2 such that

$$\lambda_1\mathbf{v} + \lambda_2\mathbf{w} = \mathbf{u}$$

In other words

Theorem 10.2 Any vector in $\mathbb{R}^2$ can be expressed as a linear combination of two independent vectors in $\mathbb{R}^2$.

Proof

For given, arbitrary vectors **v**, **u**, and **w** in $\mathbb{R}^2$, we have a system of two equations in the two unknowns λ_1 and λ_2. That is,

$$\lambda_1\mathbf{v} + \lambda_2\mathbf{w} = \mathbf{u}$$

$$\lambda_1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \lambda_2\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\lambda_1 v_1 + \lambda_2 w_1 = u_1$$

$$\lambda_1 v_2 + \lambda_2 w_2 = u_2$$

We know from chapter 7 that this system always has a solution for the λ_i given by

$$\lambda_1 = \frac{u_1 w_2 - w_1 u_2}{w_2 v_1 - w_1 v_2}, \quad \lambda_2 = \frac{u_2 v_1 - v_2 u_1}{w_2 v_1 - w_1 v_2}$$

■

We now build on this idea by introducing the idea of a *vector space*.

Basis for a Vector Space

We can identify the set of all two-component vectors with $\mathbb{R}^2$, the set of all three-component vectors with $\mathbb{R}^3$, and so on. We also note that (i) if $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then $\mathbf{v} + \mathbf{u}$ is a vector in $\mathbb{R}^n$ and (ii) if $\mathbf{v}$ is a vector in $\mathbb{R}^n$, then $\lambda\mathbf{v}$ is a vector in $\mathbb{R}^n$ for any scalar λ. We then say that $\mathbb{R}^n$ is **closed** under the operations of addition and scalar multiplication. A set of mathematical objects for which addition and scalar multiplication can be defined and which is closed under these operations is called a **vector space.** Then, $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^n$ are clearly vector spaces. It would be wrong to conclude that these are the *only* vector spaces, although this is perhaps encouraged by the terminology. For example, the set of real-valued, continuous functions defined on the domain [0, 1] is a vector space, since the sum of two such functions is a continuous function on this domain, as is the product of any such function and a scalar.

Example 10.5 Let

$$\mathcal{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } xy \geq 0 \right\}$$

denote the union of the first and third quadrants in the xy-plane.

(i) If $\mathbf{u}$ is in $\mathcal{V}$ and λ is any scalar, is $\lambda\mathbf{u}$ in $\mathcal{V}$?
(ii) Find specific vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$ such that $\mathbf{u} + \mathbf{v}$ is *not* in $\mathcal{V}$. This would imply that $\mathcal{V}$ is not a vector space.

Solution

(i) We can choose

$$\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

and $\lambda = -1$. Then

$$\lambda \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which belongs to $\mathcal{V}$.

(ii) We choose

$$\mathbf{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

both belonging to $\mathcal{V}$. However

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

does *not* belong to $\mathcal{V}$. Hence $\mathcal{V}$ is not a vector space. ■

We have just seen that every vector in $\mathbb{R}^2$ can be derived as a linear combination of two linearly independent vectors in $\mathbb{R}^2$. Any such pair of linearly independent vectors is then said to form a **basis** for $\mathbb{R}^2$.

Definition 10.3 A **basis** is a set of linearly independent vectors that generates all vectors in the space.

In the case of $\mathbb{R}^2$, the basis consists of any two linearly independent vectors.

Example 10.6 Consider vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for a basis of $\mathbb{R}^2$. Show that the following vectors can be expressed as linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$:

(i) $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (ii) $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (iii) $\mathbf{y} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Solution

$$\begin{bmatrix}2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}-3\\1\end{bmatrix} = -3\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$$ ■

Vector Orthogonality

We have seen how linearly independent vectors point in different directions. Now consider the vectors in figure 10.2 given by

$$\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\2\end{bmatrix}$$

Let A denote the angle between $\mathbf{v}$ and the horizontal axis, and let B denote the angle between $\mathbf{w}$ and the horizontal axis. The angle between the two vectors is given by $\phi = B - A$. Since the lengths of $\mathbf{v}$ and $\mathbf{w}$ are given by $\sqrt{v_1^2 + v_2^2}$ and $\sqrt{w_1^2 + w_2^2}$, we can use the definition of cos and sin to obtain

$$\cos A = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos B = \frac{w_1}{\|\mathbf{w}\|}, \quad \sin A = \frac{v_2}{\|\mathbf{v}\|}, \quad \sin B = \frac{w_2}{\|\mathbf{w}\|}$$

A trigonometric equality tells us that

$$\cos(B - A) = \cos A \cos B + \sin A \sin B$$

and so

$$\cos(B - A) = \cos\phi = \frac{v_1 w_1}{\|\mathbf{v}\|\,\|\mathbf{w}\|} + \frac{v_2 w_2}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$
$$= \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

In the case that $\phi = 0^\circ$, then $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, so they can be written as $\mathbf{v} = \lambda\mathbf{w}$. In that case

$$\mathbf{v}^T\mathbf{w} = \lambda\mathbf{w}^T\mathbf{w} = \lambda\|\mathbf{w}\|^2$$

Also $\|\mathbf{v}\| = \lambda\|\mathbf{w}\|$. Hence

$$\cos\phi = \frac{\lambda\|\mathbf{w}\|^2}{\lambda\|\mathbf{w}\|^2} = 1$$

If $\mathbf{v}$ and $\mathbf{w}$ are at right angles to each other so that $\phi = 90°$, then $\cos\phi = 0$. Conversely, when $\mathbf{v}^T\mathbf{w} = 0$, $\phi = 90°$.

Definition 10.4 Two vectors, **v** and **w,** are **orthogonal** if and only if

$$\mathbf{v}^T\mathbf{w} = 0 \tag{10.4}$$

Vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in example 10.6 constitute a basis that is known as an **orthonormal basis,** since it consists of vectors that are orthogonal to each other and are also of unit length. In the case of the vector space of all three-dimensional, real-valued vectors, $\mathbb{R}^3$, we can denote the orthonormal basis as

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any vector, $\mathbf{v}$, in $\mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ with coordinates given by the entries of $\mathbf{v}$.

Example 10.7 Express the following vector in terms of the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$:

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Solution

$$\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

■

The following application combines the operation of vector subtraction and the idea of vector orthogonality.

The Consumer's Budget Line

In the standard theory of consumer behavior, the consumer faces a budget line of the form

$$p_1x_1 + p_2x_2 = m$$

where the p_i are the (exogenous) prices of the two goods, the x_i are the quantities consumed, and m is the given income available to be spent on the two goods. Find a vector representation of the budget line and show that price vector is orthogonal to the budget line.

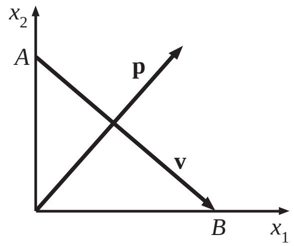


Figure 10.7 Orthogonality between budget line and price vector

Solution

Figure 10.7 helps with the construction of the answer to this question. The vector associated with point A is $\begin{bmatrix} 0 \\ m/p_2 \end{bmatrix}$, while the vector associated with point B is $\begin{bmatrix} m/p_1 \\ 0 \end{bmatrix}$. From vector subtraction we know that the vector connecting point A with point B is

$$\begin{bmatrix} 0 \\ m/p_2 \end{bmatrix} - \begin{bmatrix} m/p_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -m/p_1 \\ m/p_2 \end{bmatrix} \equiv \mathbf{v}$$

This is the vector representation of the budget line AB. The price vector is simply $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, and so, applying definition 10.4, we construct $\mathbf{p}^T\mathbf{v}$ to find

$$\mathbf{p}^T\mathbf{v} = [p_1 \quad p_2] \begin{bmatrix} -m/p_1 \\ m/p_2 \end{bmatrix} = -m + m = 0$$

Hence the price vector is orthogonal to the budget line. ■

The generalization to the n-dimensional space $\mathbb{R}^n$ is straightforward. In this case, $\mathbb{R}^n$ is the collection of all n-tuples which is closed under the operations of vector addition and scalar multiplication. The basis for $\mathbb{R}^n$ consists of the maximum set of linearly independent vectors that generate other vectors in the same space. This number in the case of $\mathbb{R}^n$ is n. When the number of the basis vectors is finite, then the vector space is said to be a **finite-dimensional** space.

Definition 10.5 The number of vectors that belong to the basis of a finite vector space is known as the **dimension** of the space.

For instance, in $\mathbb{R}^3$ the dimension is 3 since the basis consists of three linearly independent vectors. In $\mathbb{R}^m$ the dimension is m, since the basis consists of m linearly independent vectors.

Rank of a Matrix

The concept of linear independence or dependence of vectors is closely linked to the concept of nonsingularity or singularity of matrices discussed in chapter 9. We now investigate further the nonsingularity (singularity) properties of matrices by introducing the concept of the **rank** of a matrix.

Consider any arbitrary matrix of order $m \times n$. It consists of n column vectors with m elements each, and m row vectors with n elements each. Therefore the n column, m-element vectors belong to $\mathbb{R}^m$, whereas the m row n-element vectors belong to $\mathbb{R}^n$. Denote by c the maximum number of linearly independent columns of A so that $c \leq n$. If c is strictly less than n, then there will be more than one subset of the n column vectors that consists of linearly independent vectors. For example, if $n = 5$, then it may be that columns 1, 2, and 3 form a linearly independent subset, but so may columns 1, 4, and 5. However, all five column vectors taken together will be linearly dependent. In this example, $c = 3$. If we take the set of c linearly independent columns and discard the remaining $n - c$ columns, we can form a matrix B with c linearly independent columns of dimension $m \times c$. We proceed by denoting by r the number of linearly independent rows of A. This number r must also be the number of linearly independent rows of B. Since each row of B has c elements it follows that

$$r \leq c \tag{10.5}$$

since any vector in $\mathbb{R}^c$ may be expressed as a linear combination of c linearly independent vectors that form a basis.

Reversing the argument, we can form a matrix C of order $r \times n$ by retaining the set of r linearly independent rows and discarding the remaining $m - r$. Matrix C will be of order $r \times n$. The maximum number of linearly independent columns of A, namely c, will be the same as that of C. It follows that since C has r-element column vectors,

$$c \leq r \tag{10.6}$$

Using equations (10.5) and (10.6), we see that $r = c$. Hence, for the $m \times n$ matrix A, the maximum number of linearly independent rows is equal to the number of linearly independent columns.

Definition 10.6 The maximum number of linearly independent columns equals the number of linearly independent rows. This number is known as the **rank** of the matrix A.

In the case where A is a square matrix of order n, if the rank of A is n, then the matrix is nonsingular.

Definition 10.7 A $n \times n$ matrix A is nonsingular if and only if

(i) $\det(A) \neq 0$
(ii) A^{-1} exists
(iii) $\text{rank}(A) = n$

This definition makes clear that for a matrix to be invertible it must have linearly independent rows and columns.

Example 10.8 Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 4 \end{bmatrix}$$

Solution

Columns 1 and 4 are linearly dependent, since column 1 is column 4 multiplied by 2. Columns 1, 2, and 3, and columns 2, 3, and 4 constitute two different sets of three linearly independent columns of A. Therefore $c = 3$. Also the rows of A constitute a set of linearly independent vectors since there are no nonzero λs such that

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}^T + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}^T + \lambda_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $r = 3$. Since $c = r = 3$, the rank of A is 3. ■

EXERCISES

1. Find the lengths of the following vectors:

$$\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

2. Find the inner product of the following pairs of vectors:

(a) $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $\mathbf{y} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Show that the following sets of vectors are linearly independent:

(a) $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$

4. Express the following vectors in terms of the basis vectors given by sets (a) and (b) in question 3.

(a) $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ **(b)** $\mathbf{z} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

5. Check whether the following pairs of vectors are orthogonal:

(a) $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(b) $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

(c) $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

6. Express the following vectors in terms of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$(\mathbf{a}) \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (\mathbf{b}) \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \qquad (\mathbf{c}) \quad \mathbf{z} = \begin{bmatrix} 1/2 \\ 0 \\ -1 \end{bmatrix}$$

7. Suppose $\mathcal{V}$ is the positive quadrant in xy-plane defined as

$$\mathcal{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } x \geq 0 \;\; y \geq 0 \right\}$$

(a) If two vectors $\mathbf{u}$ and $\mathbf{v}$ are in $\mathcal{V}$, is $\mathbf{u} + \mathbf{v}$ in $\mathcal{V}$?

(b) If $\mathbf{u}$ is in $\mathcal{V}$, is $\lambda\mathbf{u}$ in $\mathcal{V}$, where λ is any scalar?

On the basis of (a) and (b), is $\mathcal{V}$ a vector space?

8. If A is a 6×9 matrix, what is the maximum number of linearly independent columns that A may have?

9. Find the rank of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

10.2 The Eigenvalue Problem

In previous chapters we examined the solution of a system of n linear equations formulated as

$$A\mathbf{x} = \mathbf{b}$$

where A is an $n \times n$ matrix of coefficients, $\mathbf{x}$ is an $n \times 1$ vector of unknowns and $\mathbf{b}$ is an $n \times 1$ vector of constants.

In this section we investigate the solution to an alternative problem formulated as

$$A\mathbf{q} = \lambda\mathbf{q} \tag{10.7}$$

where A is a known square matrix of order $n \times n$, $\mathbf{q}$ is an unknown n-element column vector, and λ is an unknown scalar. This problem, known as the **eigenvalue problem,** arises in many situations in economics and econometrics as will become apparent by the examples that follow. In contrast to the problem of solving for the unknown vector **x** in a system of equations, we now have to solve for two unknowns, **q** and λ, a vector and a scalar.

Definition 10.8

In the eigenvalue problem formulated by equation (10.7), A is a known square matrix of order $n \times n$, **q** is a $n \times 1$ unknown vector known as the **eigenvector,** or **characteristic vector,** or **latent vector,** and λ is an unknown scalar known as the **eigenvalue,** or **characteristic root,** or **latent root.**

To fix these ideas, let us look at a special case of equation (10.7), where A is a 2×2 matrix and **q** is a 2×1 vector. In this case we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \lambda \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

This can be written explicitly as a system of two equations

$$a_{11}q_1 + a_{12}q_2 = \lambda q_1$$
$$a_{21}q_1 + a_{22}q_2 = \lambda q_2$$

or

$$(a_{11} - \lambda)q_1 + a_{12}q_2 = 0$$
$$a_{21}q_1 + (a_{22} - \lambda)q_2 = 0$$

In matrix form these equations become

$$(A - \lambda I)\mathbf{q} = \mathbf{0} \tag{10.8}$$

where I is the identity matrix. For equation (10.8) to hold when $(A - \lambda I)$ is nonsingular, we require that **q** is the zero vector. However, for a nontrivial solution, that is, a solution where $\mathbf{q} \neq \mathbf{0}$, we need $(A - \lambda I)$ to be singular. The singularity of $(A - \lambda I)$ would imply that

$$|A - \lambda I| = 0 \tag{10.9}$$

Equation (10.9) is known as the **characteristic equation** or **characteristic polynomial** for matrix A. It gives a polynomial of degree n in λ, where n is the order of the matrix A, with its n roots being given by $\lambda_1, \ldots, \lambda_n$. Each $\lambda_i,\ i = 1, \ldots, n$ can then be substituted into equation (10.8) to obtain the corresponding eigenvector $\mathbf{q}_i$.

Example 10.9 Find the characteristic equation for the 2×2 matrix A.

Solution

We can derive the characteristic equation by looking at

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{aligned} \begin{vmatrix} (a_{11} - \lambda) & a_{12} \\ a_{21} & (a_{22} - \lambda) \end{vmatrix} &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= a_{11}a_{22} - \lambda a_{22} - \lambda a_{11} + \lambda^2 - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0 \end{aligned}$$

The roots of the quadratic equation above are the values of λ, λ_1, and λ_2 that satisfy the quadratic equation. In general, the roots of the quadratic equation $a\lambda^2 + b\lambda + c = 0$, are given by the formulas

$$\lambda_1 = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}$$

$$\lambda_2 = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

In our case, $a = 1$, $b = -(a_{11} + a_{22})$, and $c = a_{11}a_{22} - a_{12}a_{21}$. Then

$$\lambda_1 = \frac{1}{2}\left((a_{11} + a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\right)$$

$$\lambda_2 = \frac{1}{2}\left((a_{11} + a_{22}) - \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}\right)$$

In the special case in which the A matrix is symmetric, so that $a_{12} = a_{21}$, the roots become

$$\lambda_1 = \frac{1}{2}\left((a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}\right)$$

$$\lambda_2 = \frac{1}{2}\left((a_{11} + a_{22}) - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}\right)$$

■

Example 10.10 For the matrix A given below, find the roots of the characteristic equation:

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution

Now

$$(A - \lambda I) = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

and

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(1-\lambda) - 4 \\ &= 4 - \lambda - 4\lambda + \lambda^2 - 4 \\ &= \lambda^2 - 5\lambda = 0 \end{aligned}$$

The roots are $\lambda_1 = 5$ and $\lambda_2 = 0$. ■

The Diagonalization of a Square Matrix

Once we obtain the eigenvalues of a matrix A as solutions to the characteristic equation, we proceed to obtain the corresponding eigenvectors. This leads us to a very important result whereby matrix A is transformed to a diagonal matrix. This result is known as the **spectral decomposition** of a square matrix.

Example 10.11 For the matrix A of example 10.10, find the eigenvectors corresponding to the characteristic roots $\lambda_1 = 5$ and $\lambda_2 = 0$.

Solution

For $\lambda_1 = 5$, substituting into equation (10.8) yields

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-q_1 + 2q_2 = 0$$

$$2q_1 - 4q_2 = 0$$

Therefore $q_1 = 2q_2$. Since one of the elements of the eigenvector is arbitrary, there will be an infinite number of eigenvectors that would satisfy equation (10.8). In order to find a *unique* eigenvector, we can choose the vector **q** whose length is unity. This implies the condition that

$$q_1^2 + q_2^2 = 1$$

This condition is known as the **Euclidean distance condition** or **normalization.** If we represent **q** in $\mathbb{R}^2$ as an arrow from the origin to the point given by the coordinates q_1 and q_2, then the length of this arrow will be unity. Using the fact that $q_1 = 2q_2$, we obtain

$$4q_2^2 + q_2^2 = 1 \Rightarrow q_2^2 = \frac{1}{5} \Rightarrow q_2 = \pm\frac{1}{\sqrt{5}}$$

Choosing $q_2 = 1/\sqrt{5}$, we obtain $q_1 = 2/\sqrt{5}$. Therefore corresponding to λ_1, the eigenvector is found to be

$$\mathbf{q}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

Similarly the eigenvector corresponding to $\lambda_2 = 0$, is given by

$$\mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$ ■

The eigenvectors obtained in example 10.11 can be used to illustrate some of the properties of eigenvectors of *symmetric* matrices, since the matrix A in example 10.11 is symmetric. In fact, all the results that we are going to discuss in the remainder of the chapter will be illustrated for the case of symmetric matrices.

Recall from definition 10.4 that two vectors **x** and **z** are orthogonal if $\mathbf{x}^T\mathbf{z} = 0$. The eigenvectors $\mathbf{q}_1$ and $\mathbf{q}_2$ from example 10.11 can be seen to be orthogonal, since

$$\mathbf{q}_1^T\mathbf{q}_2 = [2/\sqrt{5} \quad 1/\sqrt{5}]\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} = 2/5 - 2/5 = 0$$

We also have that $\mathbf{q}_1^T\mathbf{q}_1 = 1$ and $\mathbf{q}_2^T\mathbf{q}_2 = 1$, since $\mathbf{q}_1$ and $\mathbf{q}_2$ have been chosen to be of unit length. If we put the two vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ to be the columns of a 2×2 matrix Q, then Q will satisfy

$$Q^T Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}^T \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 4/5 + 1/5 & 1/5 - 2/5 \\ 2/5 - 2/5 & 1/5 + 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition 10.9 A matrix Q that has the property that

$$Q^T Q = Q Q^T = I$$

is known as an **orthogonal matrix.** An orthogonal matrix is a matrix for which its inverse equals its transpose.

Theorem 10.3 For the problem in equation (10.8), where A is a symmetric matrix, the eigenvectors that correspond to distinct eigenvalues are pairwise orthogonal and if put together into a matrix, they form an orthogonal matrix.

Proof

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ denote the eigenvectors corresponding to λ_1 and λ_2. Then

$$A\mathbf{q}_1 = \lambda_1 \mathbf{q}_1 \quad \Rightarrow \quad \mathbf{q}_2^T A \mathbf{q}_1 = \lambda_1 \mathbf{q}_2^T \mathbf{q}_1$$

and

$$A\mathbf{q}_2 = \lambda_2 \mathbf{q}_2 \quad \Rightarrow \quad \mathbf{q}_1^T A \mathbf{q}_2 = \lambda_2 \mathbf{q}_1^T \mathbf{q}_2$$

Since A is symmetric we have that

$$\mathbf{q}_1^T A \mathbf{q}_2 = \mathbf{q}_2^T A \mathbf{q}_1$$

Then

$$\lambda_1 \mathbf{q}_2^T \mathbf{q}_1 = \lambda_2 \mathbf{q}_1^T \mathbf{q}_2$$

If $\lambda_1 \neq \lambda_2$, the last relationship above implies that

$$\mathbf{q}_1^T \mathbf{q}_2 = \mathbf{q}_2^T \mathbf{q}_1 = 0 \qquad \blacksquare$$

Theorem 10.4 If an eigenvalue λ is repeated r times, there will be r orthogonal vectors corresponding to this root.

Example 10.12 Consider the diagonal matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the characteristic roots and vectors of A.

Solution

The characteristic equation is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda)$$

The roots of the equation above are easily seen to be $\lambda_1 = 1$ (repeated twice) and $\lambda_2 = 3$. For $\lambda_2 = 3$, $(A - \lambda_2 I)\mathbf{q}_2 = \mathbf{0}$ gives

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Corresponding to that root we obtain the eigenvector

$$\mathbf{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Notice that $\mathbf{q}_2$ is of unit length. For the multiple root λ_1 we get

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\mathbf{q}_1 = \begin{bmatrix} q_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ q_3 \end{bmatrix} \Rightarrow$$

$$= q_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + q_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The multiple root has two orthogonal vectors associated with it

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

call them $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. It is easy to verify that $\mathbf{e}_1^T \mathbf{e}_1 = 1$, $\mathbf{e}_2^T \mathbf{e}_2 = 1$, and $\mathbf{e}_1^T \mathbf{e}_2 = 0$. ■

Theorem 10.5 Let the $n \times n$ symmetric matrix A have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, possibly not all distinct. Then, by theorems 10.3 and 10.4, there will be a set of n orthogonal eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ such that

$$\mathbf{q}_i^T \mathbf{q}_j = 0, \qquad i \neq j;\ i, j = 1, 2, \ldots, n \tag{10.10}$$

Any eigenvector is arbitrary up to a scale factor, since

$$A\mathbf{q}_i = \lambda \mathbf{q}_i \Rightarrow A(c\mathbf{q}_i) = \lambda c \mathbf{q}_i$$

where c is any constant. We remove this arbitrariness by defining the eigenvectors to be of unit length. Then

$$\mathbf{q}_i^T \mathbf{q}_i = 1, \qquad i = 1, \ldots, n \tag{10.11}$$

Together equations (10.10) and (10.11) define an orthonormal set of vectors. This can be expressed in the following statement

$$\mathbf{q}_i^T \mathbf{q}_j = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{10.12}$$

where δ_{ij} is known as the **Kronecker delta.** If we put all the eigenvectors as columns of a matrix Q, then equation (10.12) can be written in the form

$$Q^T Q = I$$

that is, Q is an orthogonal matrix. Since the inverse matrix is unique, we also have that

$$QQ^T = I$$

The above implies that both the columns and rows of Q are orthogonal.

Example 10.13 Show that the matrix A below is orthogonal.

$$A = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{10} & 2/\sqrt{15} \\ 0 & -2/\sqrt{10} & 3/\sqrt{15} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{5} & \sqrt{2}/\sqrt{15} \end{bmatrix}$$

Solution

We have to show that $A^T A = I$:

$$\begin{bmatrix} -1/\sqrt{3} & 0 & \sqrt{2}/\sqrt{3} \\ 2/\sqrt{10} & -2/\sqrt{10} & 1/\sqrt{5} \\ 2/\sqrt{15} & 3/\sqrt{15} & \sqrt{2}/\sqrt{15} \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{10} & 2/\sqrt{15} \\ 0 & -2/\sqrt{10} & 3/\sqrt{15} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{5} & \sqrt{2}/\sqrt{15} \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 + 2/3 & -2/\sqrt{30} + \sqrt{2}/\sqrt{15} & -2/\sqrt{45} + 2/\sqrt{45} \\ -2/\sqrt{30} + \sqrt{2}\sqrt{15} & 4/10 + 4/10 + 1/5 & 4/\sqrt{150} - 6/\sqrt{150} + \sqrt{2}/\sqrt{75} \\ -2/\sqrt{45} + 2/\sqrt{45} & 4/\sqrt{150} - 6/\sqrt{150} + \sqrt{2}/\sqrt{75} & 4/15 + 9/15 + 2/15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

Theorem 10.6 The orthogonal matrix of eigenvectors **diagonalizes** the symmetric matrix A:

$$Q^T A Q = \Lambda$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Proof

For λ_i and $\mathbf{q}_i$, we have that

$$A\mathbf{q}_i = \lambda_i \mathbf{q}_i$$

Premultiplying by $\mathbf{q}_j^T$ gives

$$\mathbf{q}_j^T A \mathbf{q}_i = \lambda_i \mathbf{q}_j^T \mathbf{q}_i = \lambda_i \delta_{ji}$$

using equation (10.12). ■

Theorem 10.7 If A is a nonsymmetric matrix and the eigenvectors can be arranged into a nonsingular matrix Q, then

$$Q^{-1}AQ = \Lambda$$

Note that in the case where A is not symmetric, Q is no longer an orthogonal matrix.

Example 10.14 Diagonalize matrix A of example 10.10.

Solution

From example 10.10 we have that

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

and $\lambda_1 = 5$ and $\lambda_2 = 0$. We have also obtained in example 10.11 the eigenvectors corresponding to λ_1 and λ_2 to be

$$\mathbf{q}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Putting $\mathbf{q}_1$ and $\mathbf{q}_2$ as columns of matrix Q leads to

$$Q^T AQ = \Lambda$$

since

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$ ■

Theorem 10.8 The sum of the eigenvalues for a symmetric matrix A is equal to the sum of the elements of the main diagonal, that is, the trace of matrix A.

Proof

Since $Q^T A Q = \Lambda$, using the properties of traces (see chapter 8), we have that

$$\begin{aligned}\text{trace}(\Lambda) &= \text{trace}(Q^T A Q)\\ &= \text{trace}(A Q Q^T)\\ &= \text{trace}(A)\end{aligned}$$

The above implies that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}$$

■

Theorem 10.9 The product of the eigenvalues of a symmetric matrix equals the determinant of the matrix.

$$|A| = \lambda_1 \lambda_2 \dots \lambda_n$$

Proof

When Q is orthogonal we have that $|Q| = \pm 1$. This is because $Q^T Q = I$. Taking the determinant on both sides yields

$$|Q^T Q| = |Q^T \| Q| = 1$$

using theorem 9.8 of determinants (see chapter 9). Since $|Q^T| = |Q|$, we have that

$$|Q|^2 = 1 \quad \Rightarrow \quad |Q| = \pm 1$$

To prove theorem 10.9, we have that $Q^T A Q = \Lambda$ so $|Q^T A Q| = |\Lambda|$. Using theorem 9.8 of determinants, we obtain

$$|Q^T \| A \| Q| = |\Lambda|.$$

Using the result $|Q|^2 = 1$, we finally get $|A| = \lambda_1 \lambda_2 \dots \lambda_n$. ■

Theorem 10.10 The eigenvalues of A^2 are the squares of the eigenvalues of A, but the eigenvectors of both matrices are the same.

$$A^2\mathbf{q} = \lambda^2\mathbf{q}$$

Proof

Premultiplying by A gives

$$A^2\mathbf{q} = \lambda A\mathbf{q} = \lambda^2\mathbf{q} \qquad \blacksquare$$

Theorem 10.10 generalizes to the following:

Theorem 10.11 The eigenvalues of A^n are the same as the eigenvalues of A, raised to the nth power, but the eigenvectors of both matrices A and A^n are the same.

Example 10.15 **Regional Migration over Time**

Suppose that we have a transition matrix that describes the population movements between regions (see examples 8.9 and 8.10). Then the population at time t is described by

$$\mathbf{x}^t = P\mathbf{x}^{t-1}$$

where $\mathbf{x}^t$ represents the population at time t, $\mathbf{x}^{t-1}$ the population at time $t-1$, and P the transition matrix. If we start from an initial conditions vector $\mathbf{x}^0$, then, as explained in example 8.10, we obtain

$$\mathbf{x}^t = P^t\mathbf{x}^0$$

Notice that P is not necessarily symmetric. We can diagonalize P assuming that its eigenvectors form a nonsingular matrix as follows:

$$\begin{aligned} Q^{-1}PQ &= \Lambda \quad \Rightarrow \\ P &= Q\Lambda Q^{-1} \quad \Rightarrow \\ P^2 &= Q\Lambda Q^{-1}Q\Lambda Q^{-1} = Q\Lambda^2 Q^{-1} \quad \Rightarrow \\ P^t &= Q\Lambda^t Q^{-1} \end{aligned}$$

The elements of $\mathbf{x}^t$ are linear combinations of the tth power of the eigenvalues of P. For the process to be *stable* we need that

$$-1 \leq \lambda_i \leq 1, \qquad i = 1, \ldots, n \tag{10.13}$$

for all the λ_is, since the higher powers of the λ_is will mean smaller values for the diagonals of Λ^t and the matrix $P^t = Q\Lambda^t Q^{-1}$ will not increase with large values of t. In other words, the population at time t will be given by

$$\mathbf{x}^t = Q\Lambda^t Q^{-1}\mathbf{x}^0 \tag{10.14}$$

and since $Q\Lambda^t Q^{-1}$ will be stable the population will also be stable. In example 8.9, we looked at the transition matrix between three regions, given by P:

$$P = \begin{bmatrix} 0.80 & 0.15 & 0.05 \\ 0.10 & 0.70 & 0.05 \\ 0.10 & 0.15 & 0.90 \end{bmatrix}$$

The population of the three regions in millions at time $t = 0$ was given by $\mathbf{x}^0$:

$$\mathbf{x}^0 = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$$

Suppose that we are interested in finding the evolution of the populations in these three regions after 10 time periods. The eigenvalues of P can be found after solving the characterstic polynomial to be $\lambda_1 = 1$, $\lambda_2 = 0.613397$, and $\lambda_3 = 0.786603$. Corresponding to these eigenvalues there are the eigenvectors

$$\mathbf{q}_1 = \begin{bmatrix} 0.557086 \\ 0.371391 \\ 1.114170 \end{bmatrix}$$

$$\mathbf{q}_2 = \begin{bmatrix} -0.495053 \\ 0.676254 \\ -0.181202 \end{bmatrix}$$

$$\mathbf{q}_3 = \begin{bmatrix} -0.570945 \\ -0.208980 \\ 0.779926 \end{bmatrix}$$

The eigenvectors above can be arranged into a matrix Q, the inverse of which is found to be

$$Q^{-1} = \begin{bmatrix} 0.489560 & 0.489560 & 0.489560 \\ -0.522497 & 1.070620 & -0.095624 \\ -0.820760 & -0.450629 & 0.560590 \end{bmatrix}$$

It can be seen that the eigenvalues satisfy the stability equation (10.14), since each one has an absolute value that is less than or equal to 1. The population at time 10 can be found solving equation (10.14). In that case

$$\begin{aligned} \mathbf{x}^{10} &= Q\Lambda^{10}Q^{-1}\mathbf{x}^0 \\ &= \begin{bmatrix} 0.557086 & -0.495053 & -0.570945 \\ 0.371391 & 0.676254 & -0.208980 \\ 1.114170 & -0.181202 & 0.779926 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.0075408 & 0 \\ 0 & 0 & 0.0906893 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0.489560 & 0.489560 & 0.489560 \\ -0.522497 & 1.070620 & -0.095624 \\ -0.820760 & -0.450629 & 0.560590 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 5.97 \\ 3.96 \\ 11.07 \end{bmatrix} \end{aligned}$$

After 10 periods, the populations of regions 1 and 3 have increased, whereas the population of region 2 has decreased. ■

Theorem 10.12 The eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A, but the eigenvectors of both matrices are the same.

Proof

Premultiply $A\mathbf{q} = \lambda\mathbf{q}$ by A^{-1} to obtain

$$\mathbf{q} = A^{-1}\lambda\mathbf{q} \Rightarrow A^{-1}\mathbf{q} = \left(\frac{1}{\lambda}\right)\mathbf{q}$$

The above establishes the result. ■

Theorem 10.13 Each eigenvalue of an idempotent matrix is either zero or one.

Proof

By theorem 10.10, $A^2\mathbf{q} = \lambda^2\mathbf{q}$. Since A is idempotent, $A^2 = A$. Then $A\mathbf{q} = A^2\mathbf{q} = \lambda^2\mathbf{q}$. Therefore $A\mathbf{q} = \lambda^2\mathbf{q}$ and

$$A^2\mathbf{q} = \lambda\mathbf{q} \ \Rightarrow\ \lambda^2\mathbf{q} - \lambda\mathbf{q} = (\lambda^2 - \lambda)\mathbf{q} = \mathbf{0} \ \Rightarrow\ \lambda(\lambda - 1)\mathbf{q} = \mathbf{0}$$

Since $\mathbf{q} \neq \mathbf{0}$, the above establishes the result. ■

EXERCISES

1. For the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Write the characteristic equation and find the characteristic roots.

(b) Find the eigenvectors corresponding to the characteristic equation.

(c) Diagonalize A.

2. Repeat the question 1 for A given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

3. Suppose that we have the matrix $P = X(X^T X)^{-1} X^T$.

(a) Show that P is idempotent.

(b) If

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}$$

obtain the eigenvalues of P.

4. Suppose that matrix A below has one eigenvalue, 2, with multiplicity 2. Find $|A|$.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

5. Show that the product of two orthogonal matrices of the same order is also an orthogonal matrix.

10.3 Quadratic Forms

Quadratic forms are special matrix functions that are very important for the derivation of second-order conditions of maxima and minima in multivariate calculus, (see chapter 11). In econometrics most of the estimators of model parameters can be formulated as solutions to the minimization of particular quadratic forms. Also, in hypothesis testing, all of the classical statistical testing procedures can be seen to depend on the distribution properties of certain quadratic forms.

Let us start by looking at a 2×2 matrix A and a 2×1 vector $\mathbf{x}$. In this case

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The *scalar* expression $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is said to be a **quadratic form** and its value depends on the choice of the vector $\mathbf{x}$, for a given matrix A. In the case above it is given as

$$q(\mathbf{x}) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_2x_1 + a_{22}x_2^2$$

Definition 10.10

Given an $n \times n$ matrix A and a $n \times 1$ vector $\mathbf{x}$, we define the **quadratic form** to be the scalar function $q(\mathbf{x})$ such that

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$$

If A is not symmetric, then we can always define a symmetric matrix A^* that yields the same quadratic form as A.

Theorem 10.14 For a general nonsymmetric matrix A, we define the symmetric matrix A^*, such that its elements are given by

$$a_{ij}^* = a_{ji}^* = \left(\frac{1}{2}\right)(a_{ij} + a_{ji})$$

Then the quadratic form defined by A equals the quadratic form defined by A^*.

Example 10.16 Verify the equality of the quadratic forms for A and A^* for the 2×2 case.

Solution

If we denote the quadratic form corresponding to a 2×2 nonsymmetric matrix A by $q(\mathbf{x})$ and the one corresponding to A^* by $q^*(\mathbf{x})$, we have

$$\begin{aligned} q(\mathbf{x}) &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \\ &= a_{11}x_1^2 + 2(1/2)(a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \\ &= a_{11}x_1^2 + 2a_{12}^*x_1x_2 + a_{22}x_2^2 \\ &= q^*(\mathbf{x}) \end{aligned}$$

■

Without any loss of generality, from now onwards we will concentrate on the case that the matrix A is symmetric.

Example 10.17 Derive the quadratic form for the 3×3 case.

Solution

Write out explicitly $q(\mathbf{x})$, where A is a 3×3 symmetric matrix and $\mathbf{x}$ is 3×1. In this case

$$\begin{aligned} q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= a_{11}x_1^2 + 2a_{12}x_2x_1 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 \end{aligned}$$

■

Definition 10.11

- If $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$, $q(\mathbf{x})$ is said to be **positive definite** and A is said to be a positive definite matrix.
- If $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \neq 0$, then $q(\mathbf{x})$ is said to be **positive semidefinite** and A is a positive semidefinite matrix.
- If $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$, then $q(\mathbf{x})$ is said to be **negative definite** and A is a negative definite matrix.
- If $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x} \neq 0$, then $q(\mathbf{x})$ is said to be **negative semidefinite** and A is a negative semidefinite matrix.

If a quadratic form is positive for some $\mathbf{x}$ and negative for some other $\mathbf{x}$, it is said to be **indefinite,** as is the matrix A that defines it.

Definition 10.12

Given an $n \times n$ matrix A, we define a **principal submatrix** of order k $(1 \leq k \leq n)$ to be a submatrix that is obtained by removing $n - k$ rows and columns of A.

Clearly, for any order $k < n$, there is more than one principal submatrix. For example, if $k = 1$, all the entries on the main diagonal constitute principal submatrices of order 1. The **leading principal submatrix** of order k is the submatrix obtained by removing the last $n - k$ rows and columns.

Definition 10.13

The determinants of the principal submatrices are known as **minors** and those of the leading principal submatrices as **leading principal minors.**

Example 10.18 Find the principal submatrices of a 3×3 matrix.

Solution

Let the 3×3 matrix A be given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For $k = 1$, the principal submatrices are all the scalar entries of the main diagonal, that is, a_{11}, a_{22}, a_{33}. The leading principal submatrix is a_{11}. For $k = 2$, the principal submatrices are

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

the first of these being the leading principal submatrix. ■

Theorem 10.15 A necessary and sufficient condition for the real symmetric matrix A to be positive definite is that *all* its eigenvalues be positive.

Proof

From theorem 10.7 we have that

$$Q^T A Q = \Lambda$$

where Q is the orthogonal matrix of eigenvectors such that $Q^T Q = I$ and Λ is the diagonal matrix of eigenvalues. Let us start with an arbitrary vector $\mathbf{x}$. Then we can define $\mathbf{x} = Q\mathbf{y}$. It also follows that

$$\begin{aligned} \mathbf{x}^T A \mathbf{x} &= \mathbf{y}^T Q^T A Q \mathbf{y} \\ &= \mathbf{y}^T \Lambda \mathbf{y} \\ &= \sum_{i=1}^{n} \lambda_i y_i^2 \end{aligned}$$

Then $\mathbf{x}^T A \mathbf{x} > 0$ if and only if $\lambda_i > 0$ for all i. If one of the eigenvalues were negative, say $\lambda_1 < 0$, then we could choose $\mathbf{x}$ such that

$$y_1 = 1, \quad y_2 = 0, \ldots, y_n = 0$$

In this case $\mathbf{x}^T A \mathbf{x} = y_1 < 0$, and $\mathbf{x}^T A \mathbf{x}$ would not be positive. Therefore we need all the eigenvalues to be positive. ■

Theorem 10.16 A necessary and sufficient condition for a symmetric matrix A to be positive semidefinite is that its eigenvalues be greater than *or* equal to zero.

Theorem 10.17 A necessary and sufficient condition for a symmetric matrix A to be negative definite is that *all* its eigenvalues be negative.

Theorem 10.18 A necessary and sufficient condition for a symmetric matrix A to be negative semidefinite is that its eigenvalues be less than *or* equal to zero.

Theorem 10.19 A necessary and sufficient condition for a symmetric matrix A to be positive definite is that the determinant of every leading principal submatrix be positive. The leading principal submatrices of A are a set of submatrices

$$A_1 = a_{11}, \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \ldots \quad A_n = A$$

Proof

When A is positive definite, $\mathbf{x}^T A\mathbf{x} > 0$ for every nonzero $\mathbf{x}$. Let us consider an $\mathbf{x}$ vector whose first k elements are nonzero and whose last $n - k$ elements are zero:

$$\mathbf{x}^T = \left(\mathbf{x}_k^T, \mathbf{0}^T\right)$$

where $\mathbf{x}_k$ is the subvector of $\mathbf{x}$ with the nonzero elements. Then we have

$$\mathbf{x}^T A\mathbf{x} = \left(\mathbf{x}_k^T, \mathbf{0}^T\right)\begin{bmatrix} A_{k,k} & A_{n-k,k} \\ A_{k,n-k} & A_{n-k,n-k} \end{bmatrix}\begin{bmatrix} \mathbf{x}_k \\ \mathbf{0} \end{bmatrix} = \mathbf{x}_k^T A_{k,k}\mathbf{x}_k$$

where $A_{k,k}$ refers to the first $k \times k$ partition of A and $A_{k,n-k}$, $A_{n-k,k}$, and $A_{n-k,n-k}$ to the remaining partitions of A, which will be eliminated by the zero subvector of $\mathbf{x}$. Now, since

$$\mathbf{x}^T A\mathbf{x} > 0$$

we have that

$$\mathbf{x}_k^T A_{k,k}\mathbf{x}_k > 0$$

But then, by theorem 10.15, all the characteristic roots of $A_{k,k}$ will be positive. This implies that

$$|A_{k,k}| > 0$$

To cover all possible cases of suitable choice of **x** vectors gives the necessary and sufficient condition for A to be positive definite, since

$$|A_1| > 0, \quad |A_2| > 0, \ldots, |A| > 0 \qquad \blacksquare$$

Theorem 10.20 A necessary and sufficient condition for a symmetric matrix A to be positive semidefinite and not positive definite is that some of its principal minors be zero and the rest positive.

Theorem 10.21 A necessary and sufficient condition for a symmetric matrix A to be negative definite is that its principal minors alternate in sign *starting* with *negative*.

Theorem 10.22 A necessary and sufficient condition for a symmetric matrix A to be negative semidefinite and not negative definite is that some of its principal minors be zero and the rest alternate in sign *starting* with *negative*.

Example 10.19 What is the "definiteness" of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Since the eigenvalues of this matrix are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 3$, then by theorem 10.15, the quadratic form is positive. This can be written as

$$\begin{aligned} q(\mathbf{x}) &= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= x_1^2 + 3x_2^2 + x_3^2 \end{aligned}$$

It can be seen that unless $\mathbf{x}$ is the null vector, $q(\mathbf{x})$ above will be positive for any values that x_1, x_2, and x_3 may take. Note that all the principal minors are positive. This matrix is positive definite. ■

Example 10.20 What is the "definiteness" of the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution

The eigenvalues are given by $\lambda_1 = 5$ and $\lambda_2 = 0$. By theorem 10.16, the quadratic form will be positive semidefinite. This can be written as

$$\begin{aligned} q(\mathbf{x}) &= [x_1 \quad x_2] \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 4x_1^2 + 4x_1x_2 + x_2^2 \end{aligned}$$

It can be seen that $q(\mathbf{x})$ is greater than or equal to zero for different choices of $\mathbf{x}$. For example, if $\mathbf{x} = [-1 \ \ 2]$, then $q(\mathbf{x})$ will be zero. Note that the first principal minor is positive and the second, the determinant of A, is zero. ■

Example 10.21 What is the "definiteness" of the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solution

There is an eigenvalue given by $\lambda = -1$, repeated twice. By theorem 10.17, the quadratic form is negative definite. This is given by

$$\begin{aligned} q(\mathbf{x}) &= [x_1 \quad x_2] \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= -x_1^2 - x_2^2 \end{aligned}$$

We have that $\mathbf{x}$ above is negative unless $\mathbf{x}$ is the null vector. The principal minors are -1 and 1, alternating in sign starting with a negative value. ■

Example 10.22 What is the “definiteness” of the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Solution

The eigenvalues are found to be $\lambda_1 = 0$ and $\lambda_2 = -2$. By theorem 10.18, we expect the quadratic form to be negative semidefinite. This is written as

$$\begin{aligned} q(\mathbf{x}) &= [x_1 \quad x_2] \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= -x_1^2 + 2x_1x_2 - x_2^2. \end{aligned}$$

This is seen to be either negative or zero depending on the choice of x_1 and x_2. Note also that the principal minors are −1 and 0. Therefore, by theorem 10.22, the quadratic form is negative semidefinite but not negative definite. ■

Theorem 10.23 If A is symmetric and positive definite, one can find a nonsingular matrix P such that

$$A = PP^T$$

Proof

Since A is a symmetric matrix we have that

$$Q^T AQ = \Lambda$$

which becomes

$$A = Q\Lambda Q^T$$

where Q and Λ are the orthogonal matrices of eigenvectors and the diagonal matrix of eigenvalues, respectively.

Since A is positive definite, all its eigenvalues are positive by theorem 10.15. Then we can define $\Lambda^{1/2}$ as

$$\Lambda^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_n} \end{bmatrix}$$

It follows that $\Lambda = \Lambda^{1/2}\Lambda^{1/2}$, which in turn implies that

$$\begin{aligned} A &= Q\Lambda^{1/2}\Lambda^{1/2}Q^T \\ &= (Q\Lambda^{1/2})(Q\Lambda^{1/2})^T \\ &= PP^T \end{aligned}$$

where $P = Q\Lambda^{1/2}$ is nonsingular since it is the product of two nonsingular matrices. ■

EXERCISES

1. State whether the following matrices are positive definite, negative definite, or indefinite:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

2. Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute $\mathbf{x}^T A\mathbf{x}$ for the following matrices:

(a) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

3. For $\mathbf{x} \in R^3$ let

$$g(\mathbf{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_1x_3$$

Write this quadratic form as $\mathbf{x}^T A\mathbf{x}$.

4. Examine whether the following quadratic forms are positive definite, negative definite, or indefinite:

(a) $6x_1^2 + 25x_2^2 + 9x_3^2 - 60x_2x_3 + 40x_1x_3 - 6x_1x_2$

(b) $9x_2^2 + 9x_3^2 + 10x_2x_3 + x_3 + 6x_1x_2$

5. Is

$$g(\mathbf{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$$

positive definite, negative definite, or indefinite?

CHAPTER REVIEW

Key Concepts

basis
characteristic equation
characteristic polynomial
characteristic root
characteristic vector
closed
diagonalize
dimension
eigenvalue
eigenvalue problem
eigenvector
Euclidean distance
Euclidean norm
finite-dimensional
indefinite
inner product
Kronecker delta
latent root
latent vector
leading principal minor
leading principal submatrix
linear dependence
linear independence
minor
negative definite
negative semidefinite
normalization
orthogonal
orthogonal matrix
orthonormal basis
positive definite
positive semidefinite
principal submatrix
quadratic form
rank
scalar multiplication
spectral decomposition
vector addition
vector space
vector subtraction

Review Questions

1. How does the idea of *distance* between vectors relate to the Euclidean norm?
2. What is the relationship between the inner product of a vector and the Euclidean norm?
3. What is the effect on a vector of multiplying it by a negative fraction?
4. How does the idea of linear dependence as presented in this chapter relate to the idea of linear dependence discussed in chapter 7?
5. How would you decide if two vectors were orthogonal?
6. What does orthogonality mean in terms of the geometry of vectors?
7. What is the rank of a matrix?

8. What is an eigenvalue?
9. What is a quadratic form?
10. Find two ways of determining whether a matrix A is negative definite.
11. Find two ways of determining whether a matrix A is positive semidefinite.
12. What does it mean to say that a matrix is indefinite?

Review Exercises

1. Establish whether the following pairs of vectors are linearly independent:

   (a) $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

   (b) $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

   (c) $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

2. If A is a 7×9 matrix with three linearly independent rows, what is the rank of A?

3. **(a)** If A is a 9×7 matrix, what is the largest possible rank of A?

   (b) If A is a 7×9 matrix, what is the largest possible rank of A? Explain.

4. Find the rank of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

5. Provide two different bases for $\mathbb{R}^3$ such that the vectors of the bases are not orthogonal to each other and are not of unit length.

6. Let $\mathcal{V}$ be the set of points inside and on the unit circle in xy-plane defined by

$$\mathcal{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{such that} \quad x^2 + y^2 \leq 1 \right\}$$

   Showby means of a specific example that $\mathcal{V}$ is not a vector space.

7. Find the eigenvalues of

(a) $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ **(b)** $B = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

8. Find the eigenvalues of

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

9. Obtain the orthogonal decomposition of

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$$

10. Find the matrix of the the quadratic forms below. Assume that $\mathbf{x}$ is in $\mathbb{R}^3$.

(a) $8x_1^2 + 7x_2^2 - 3x_3^2 - 6x_1x_2 + 4x_1x_3 - 2x_2x_3$

(b) $4x_1x_2 + 6x_1x_3 - 8x_2x_3$

Part IV Multivariate Calculus

Chapter 11 Calculus for Functions of *n*-Variables

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- More Properties of Homogeneous Functions
- Homotheticity
- Finding First-Order Partial Derivatives: Example
- Finding Marginal-Product Functions: Example
- Finding Marginal Rate of Technical Substitution: Example
- Practice Exercises

We have already discussed at length the basic principles of calculus for functions of one variable, $y = f(x)$ with $x \in \mathbb{R}$. Continuity was presented in chapter 4, and the derivative was presented in chapter 5. Economic analysis, however, often demands consideration of functions of more than one variable. For example, it is often important to model how the level of output produced by a firm depends on several inputs rather than just one. In this chapter we consider the fundamental relationships of differential calculus for functions of more than one variable. Fortunately, much of what was learned in chapters 4 and 5 carries over in a straightforward manner. For example, the rules of differentiation for functions of many variables are straightforward extensions of the rules for differentiation of functions of a single variable. However, we must be careful in interpreting these new results.

11.1 Partial Differentiation

We first deal with notation. We use $(x_1, x_2, \ldots, x_n)$ or $\mathbf{x}$ to denote a point in $\mathbb{R}^n$, the domain of the function, and $y = f(x_1, x_2, \ldots, x_n)$ or $y = f(\mathbf{x})$ to denote a function. For simplicity, we will often focus on the case of $n = 2$ variables. The extension to the general case of n variables is usually straightforward, although sometimes cumbersome to write down.

Continuity of functions of one variable was discussed at length in chapter 4. The intuition obtained there about continuity carries over to functions of more than one variable. A function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is continuous at a point $\mathbf{x} = \mathbf{a}$, provided that it does not *jump* or have a *break* as the value of $\mathbf{x}$ approaches the point $\mathbf{a}$. For

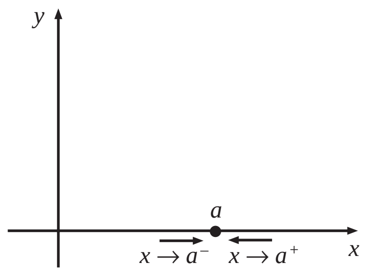


Figure 11.1 Two paths to approach a point $x = a$ in $\mathbb{R}$

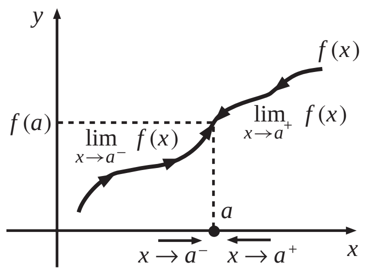


Figure 11.2 For a continuous function, both the left-hand and right-hand limits are equal to the function value at each point $x = a$, $x \in \mathbb{R}$

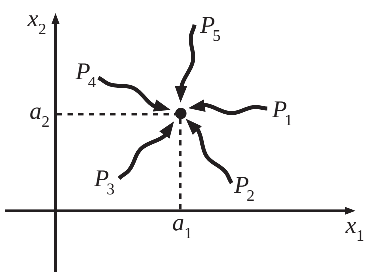


Figure 11.3 Some of the infinite number of paths to approach a point in $\mathbb{R}^2$

functions of one variable there are only two directions from which a point $x = a$ can be approached, from the left or from the right, as indicated in figure 11.1. One can then define the left-hand and right-hand limits of the function at $x = a$, depicted by $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, and then define the function to be continuous at $x = a$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$, as is illustrated by the example in figure 11.2 and stated formally in definition 4.3.

The added complication in defining continuity for functions of more than one variable is that there are more than two directions from which any point $\mathbf{x} = \mathbf{a}$ can be approached, and so we cannot simply refer to the left- and right-hand limits of a function at a point. In fact there are an infinite number of directions or paths to be considered when thinking about approaching some point in $\mathbb{R}^n$, $n > 1$. This is illustrated in figure 11.3, where the paths marked $P_1, P_2, \ldots, P_5$ are clearly just a few examples.

In section 4.1 we also considered an alternative definition of continuity. In definition 4.4, which states that a function is continuous at a point $x = a$ if, within a *small* neighborhood of this point (i.e., for points *close to* $x = a$), the function values $f(x)$ are *close to* the value $f(a)$. This definition of continuity extends to functions defined on $\mathbb{R}^n$ in a straightforward manner. For $x \in \mathbb{R}$, closeness is determined by the (absolute) distance $|x - a|$. To extend the concept of continuity to functions on $\mathbf{x} \in \mathbb{R}^n$, we can simply use the Euclidean distance between the points, $\|\mathbf{x} - \mathbf{a}\|$, introduced in chapters 2 and 10. For example, in $\mathbb{R}^2$ we have

$$\|\mathbf{x} - \mathbf{a}\| = \|(x_1, x_2) - (a_1, a_2)\| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}$$

We do not study this definition of continuity further here, though question 12 at the end of this section illustrates how to use this definition for functions on $\mathbb{R}^n$.

Extending the idea of the derivative for functions of one variable to functions defined on $\mathbb{R}^n$ is more straightforward than is continuity. Recall from chapter 5 that the derivative of a function $y = f(x)$ with domain $x \in \mathbb{R}$ is the rate at which y changes as x changes $(\Delta y/\Delta x)$ as we let the change in x become arbitrarily small $(\Delta x \to 0)$. If y depends on more than one variable, as in $y = f(x_1, x_2)$, we can define the rate at which y changes with respect to changes in each of the variables x_1 and x_2, *taken separately*, in the same way. For example, we can find the ratio $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$ while holding x_2 constant. The result of this operation is called the partial derivative of the function $y = f(x_1, x_2)$ with respect to the variable x_1, and it is written

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \quad \text{or} \quad \frac{\partial y}{\partial x_1} \quad \text{or} \quad f_1(x_1, x_2) \quad \text{or simply } f_1$$

where

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}$$

The partial derivative of the function $f(x_1, x_2)$ with respect to the variable x_2 is written

$$\frac{\partial f(x_1, x_2)}{\partial x_2} \quad \text{or} \quad \frac{\partial y}{\partial x_2} \quad \text{or} \quad f_2(x_1, x_2) \quad \text{or simply } f_2$$

where

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2}$$

The reason for calling these expressions *partial* derivatives and using the notation "∂" is that we are changing only one of x_1 or x_2 at a time, even though y depends on both of these variables. It is important to remember that since y is a function of both x_1 and x_2, the ratio $\Delta y/\Delta x_1$ will in general depend on the level of x_2. Similarly the ratio $\Delta y/\Delta x_2$ will depend on the level of x_1. The notation $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ reminds us that the rate of change of y with respect to x_1 or x_2 is itself a function which in general depends on the values of *both* x_1 *and* x_2.

The idea of the derivative being the slope of the tangent to the curve at some point in the one-variable case carries over to the case where $\mathbf{x} \in \mathbb{R}^2$. However, one must take care in drawing and interpreting the relevant diagram.

Notice in figure 11.4 (a) that with x_2 fixed, there are now only two directions, rather than an arbitrary number of directions, from which to approach any given point. This being the case, the partial derivative $\partial y/\partial x_1$ behaves just like the derivative of a function of one variable since only one variable, x_1, is changing. This similarity with the one-variable case explains why the process of partial differentiation is a straightforward extension of the derivative for functions of one variable.

The idea of the partial derivative generalizes readily to functions of n variables.

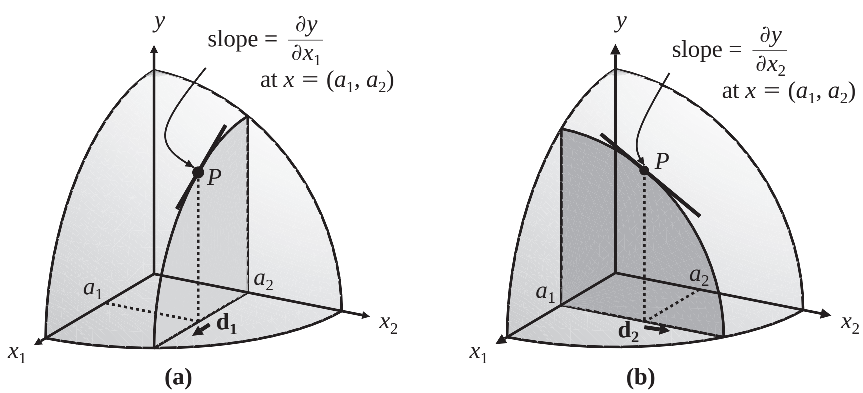


Figure 11.4 The partial derivatives for a function on $\mathbb{R}^2$

Definition 11.1 The **partial derivative** of a function $y = f(x_1, x_2, \ldots, x_n)$ with respect to the variable x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i}$$

The notations $\partial y/\partial x_i$ or $f_i(\mathbf{x})$ or simply f_i are used interchangeably. Notice that in defining the partial derivative $f_i(\mathbf{x})$ all other variables, x_j, $j \neq i$, are *held constant*.

As in the case of the derivative of a function of one variable, we can use definition 11.1 to compute the partial derivatives of a specific function. The following example illustrates.

Example 11.1 Derive and interpret the partial derivatives of the revenue function for a multi-product, competitive firm.

Solution

Suppose that we let x_1 and x_2 represent the quantities of two products sold by a competitive firm with p_1 and p_2 being the price of each, respectively. Total revenue for the firm is $R(x_1, x_2) = p_1x_1 + p_2x_2$. According to definition 11.1, the partial derivative, $\partial R(x_1, x_2)/\partial x_1$ is

$$\begin{aligned}
\frac{\partial R(x_1, x_2)}{\partial x_1} &= \lim_{\Delta x_1 \to 0} \frac{R(x_1 + \Delta x_1, x_2) - R(x_1, x_2)}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{[p_1(x_1 + \Delta x_1) + p_2x_2] - [p_1x_1 + p_2x_2]}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{p_1x_1 + p_1\Delta x_1 + p_2x_2 - p_1x_1 - p_2x_2}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{p_1\Delta x_1}{\Delta x_1} \\
&= p_1
\end{aligned}$$

The derivation of $R_2(x_1, x_2)$ is left as one of the exercises at the end of this section. ■

The result $R_1(x_1, x_2) = p_1$ accords with simple intuition. The amount by which revenue increases as one more unit of good 1 is sold with the output of the other good left unchanged is simply the price of good 1. Notice that this derivative is just a constant. This result is the same as for the case of a linear function of one variable. The independence of R_1 with respect to the value of x_1 or x_2 carries over for any *linear* function. It is of course not true for functions in general, as the following example illustrates.

Example 11.2 In this example the derivative $\partial y/\partial x_1$ of the function $y = x_1^2 x_2$ depends on the value of both x_1 and x_2. According to definition 11.1, the partial derivative, $\partial f(x_1, x_2)/\partial x_1$, is

$$\begin{aligned}
\frac{\partial f(x_1, x_2)}{\partial x_1} &= \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{(x_1 + \Delta x_1)^2 x_2 - x_1^2 x_2}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{\left(x_1^2 + 2x_1 \Delta x_1 + (\Delta x_1)^2\right) x_2 - x_1^2 x_2}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} \frac{(2x_1 \Delta x_1 + (\Delta x_1)^2) x_2}{\Delta x_1} \\
&= \lim_{\Delta x_1 \to 0} (2x_1 + \Delta x_1) x_2 \\
&= 2x_1 x_2
\end{aligned}$$

■

Rather than derive them from first principles (i.e., by using the definition of the derivative), we can use rules of differentiation to find partial derivatives just as we did for functions of one variable in chapter 5. Since we hold all variables except x_i fixed when finding $\partial f/\partial x_i$, we can explicitly treat all parts of the function $f(\mathbf{x})$ that do not depend on x_i as a constant, c, and then use the rules of differentiation for functions of one variable. For the function of example 11.1, $R(x_1, x_2) = p_1 x_1 + p_2 x_2$; this means setting $p_2 x_2 = c$, where c is some constant. Then noting that

$$R(x_1, x_2) = p_1 x_1 + c$$

we have

$$R_1(x_1, x_2) = \frac{d[p_1 x_1 + c]}{dx_1} = p_1$$

Similarly, for the function of example 11.2, $y = f(x_1, x_2) = x_1^2 x_2$, the variable x_2 is held fixed when computing $\partial f/\partial x_1$. If we explicitly set $x_2 = c$, c a constant, then the function becomes

$$y = cx_1^2$$

and so

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{d\left[cx_1^2\right]}{dx_1} = 2cx_1$$

which, upon substituting back for $c = x_2$, gives the result

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 2x_2x_1$$

Now let us see how the procedure above carries over to functions of any number of variables. If we have $y = f(x_1, x_2, \ldots, x_n)$, then to find the derivative of f with respect to one of the variables, x_i, we factor out all parts of the expression that are not dependent on x_i and treat these as constant values. For example, given the function

$$y = 5x_1^2x_2^4x_3^6$$

we find the partial derivative $\partial y/\partial x_2$ by first setting

$$c = 5x_1^2x_3^6$$

which implies that

$$y = cx_2^4$$

It follows from the rules of differentiation given in chapter 5 that

$$\frac{\partial y}{\partial x_2} = 4cx_2^3$$

which, upon substituting back for $c = 5x_1^2x_3^6$, gives the result that

$$\frac{\partial y}{\partial x_2} = 4\left(5x_1^2x_3^6\right)x_2^3 = 20x_1^2x_2^3x_3^6$$

After a little practice you don't need to make these substitutions explicitly.

We can see from these examples that the partial derivative $\partial f/\partial x_i$ will in general depend on the values of all variables, x_j, even for $j \neq i$. There is a class of functions, however, that has the property that the partial derivative with respect to x_i is independent of all the other variables x_j, $j \neq i$. We define this class by

Definition 11.2 A function $y = f(x_1, x_2, \ldots, x_n)$ which can be written in the form

$$f(x_1, x_2, \ldots, x_n) = g^1(x_1) + g^2(x_2) + \cdots + g^n(x_n)$$
$$= \sum_{k=1}^{n} g^k(x_k)$$

is **additively separable.**

Example 11.3 The functions

$$f(x_1, x_2) = x_1^3 + 5x_2$$

and

$$g(x_1, x_2, x_3) = 5x_1^2 - 6x_2 + e^{x_3}$$

are examples of additively separable functions. Notice that the partial derivative with respect to any of the variables does not depend on the level of any of the other variables, since

$$f_1 = 3x_1^2$$
$$f_2 = 5$$

$$g_1 = 10x_1$$
$$g_2 = -6$$
$$g_3 = e^{x_3}$$

Notice also that additive separability includes the case where variables enter the function with a *negative* sign, as illustrated by the term $-6x_2$ in the function g. ■

Marginal-Product Functions

If $y = f(x_1, x_2, \ldots, x_n)$ represents a production function, with x_i being the level of input i and y being the level of output, then the partial derivative $\partial y/\partial x_i$ is the marginal product of input i. This is the rate at which output increases as a result of increasing input i when there is no change in the level of other inputs. Since partial derivatives are defined in the limit as $\Delta x_i \to 0$, it follows that $\partial y/\partial x_i$ represents (approximately) the change in output resulting from a one-unit increase in the input i only if the choice of units is sufficiently *small*.

Since the partial derivative $\partial y/\partial x_i$ will in general depend on the value of the other variables x_j, $j = 1, 2, \ldots, n$, the marginal-product function of input i will be a function of all the inputs. Noting these relationships can help us understand the technological assumptions that are implied by using a particular production function. This point is illustrated in the following example.

Example 11.4 Find and interpret the partial derivatives of the production function

$$y = 10x_1^{1/2}x_2^{1/2}$$

Solution

$$f_1 = \frac{\partial y}{\partial x_1} = \frac{1}{2}\left(10x_1^{-1/2}x_2^{1/2}\right) = 5x_1^{-1/2}x_2^{1/2} = \frac{5x_2^{1/2}}{x_1^{1/2}}$$

$$f_2 = \frac{\partial y}{\partial x_2} = \frac{1}{2}\left(10x_1^{1/2}x_2^{-1/2}\right) = 5x_1^{1/2}x_2^{-1/2} = \frac{5x_1^{1/2}}{x_2^{1/2}}$$

If we think of x_1 as the input labor and x_2 as the input capital, then we can see from f_1, the marginal product of labor function, that higher values of capital lead to a bigger increase in output being generated by a given increase in labor. In figure 11.5 we illustrate how the relationship between x_1 and y depends on the value of x_2 by considering two specific values of x_2, ($x_2 = 4$ and $x_2 = 9$), and then in figure 11.6 we see how the value of x_2 affects the derivative function f_1. (*Note:* For $x_2 = 4$, $f_1 = 10/x_1^{1/2}$, while for $x_2 = 9$, $f_1 = 15/x_1^{1/2}$.) These graphs also emphasize the fact that for a fixed amount of capital available, the marginal product of labor falls as more labor is used. Thus this production function conforms to the law of diminishing marginal productivity of an input (see chapter 5, example 5.6). An analogous interpretation follows for the marginal product of input 2. ■

The production function used in example 11.4 is a specific case of the Cobb-Douglas production function. We explore a more general form in the examples below, where we will denote inputs as L (labor) and K (capital) when there are only two inputs.

Example 11.5 Find and interpret the partial derivatives for the Cobb-Douglas production function with two inputs:

$$y = f(K, L) = AK^{\alpha}L^{\beta} \qquad A > 0,\ 0 < \alpha, \beta < 1$$

where A, α, and β are technological parameters.

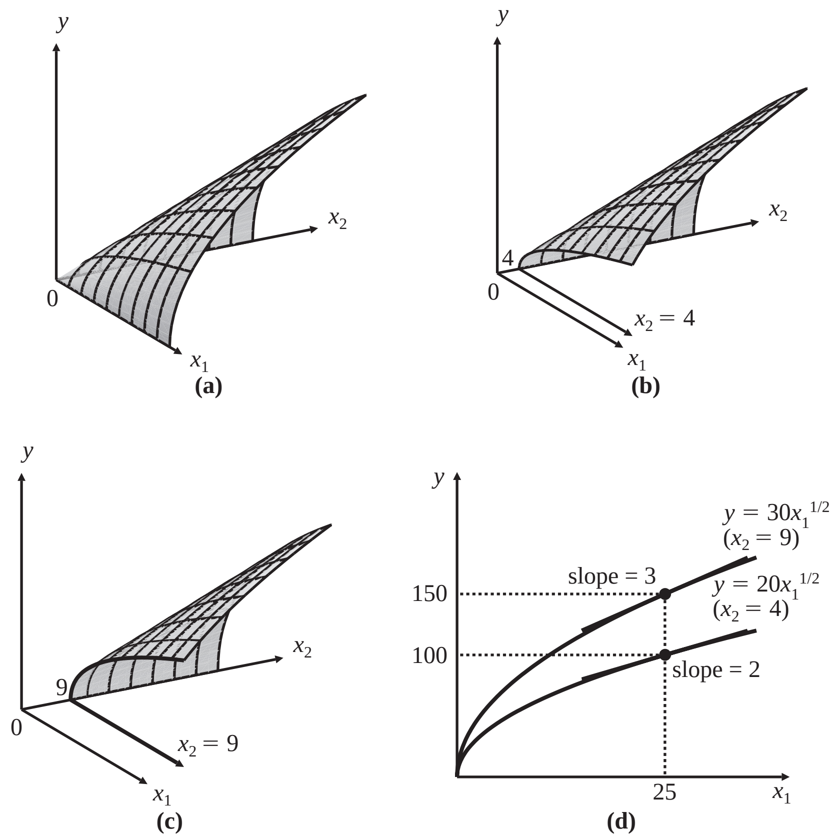


Figure 11.5 Sections of the function $y = 10x_1^{1/2}x_2^{1/2}$. Note that (d) suppresses x_2.

Solution

The marginal-product functions are

$$f_K = \alpha A K^{\alpha-1} L^{\beta}$$
$$f_L = \beta A K^{\alpha} L^{\beta-1}$$

Given the assumptions on α and β, it is clear that both inputs satisfy the law of diminishing marginal productivity, while the marginal product of each input is positively related to the the level of the other input. In this sense the inputs are complementary to each other (e.g., increasing K leads to a higher marginal product of L). ■

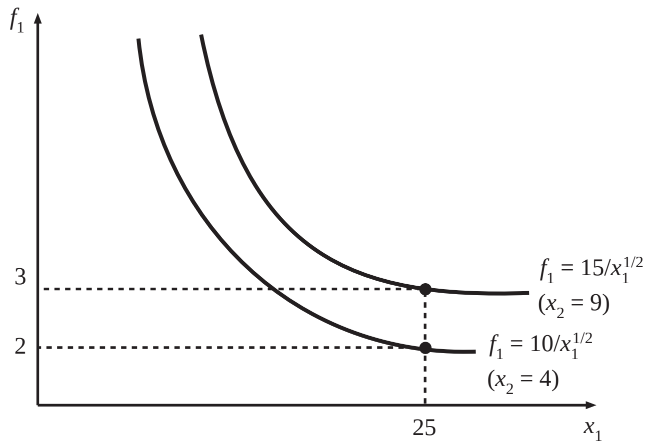


Figure 11.6 Effect of x_2 on the derivative function $f_1 = 5x_2^{1/2}/x_1^{1/2}$

Example 11.6 Find the marginal-product functions for the Cobb-Douglas production function with three inputs:

$$y = Ax_1^{\alpha}x_2^{\beta}x_3^{\gamma}, \qquad A > 0; 0 < \alpha, \beta, \gamma < 1$$

Solution

The marginal products are

$$\frac{\partial y}{\partial x_1} = \alpha A x_1^{\alpha-1}x_2^{\beta}x_3^{\gamma} = \alpha A\frac{x_2^{\beta}x_3^{\gamma}}{x_1^{1-\alpha}}$$

$$\frac{\partial y}{\partial x_2} = \beta A x_1^{\alpha}x_2^{\beta-1}x_3^{\gamma} = \beta A\frac{x_1^{\alpha}x_3^{\gamma}}{x_2^{1-\beta}}$$

$$\frac{\partial y}{\partial x_3} = \gamma A x_1^{\alpha}x_2^{\beta}x_3^{\gamma-1} = \gamma A\frac{x_1^{\alpha}x_2^{\beta}}{x_3^{1-\gamma}}$$

Again, in each case we see that the marginal product of each input *decreases* as the level of that input increases but *increases* as either of the other two inputs increases. ■

It is easy to see that these examples point to the general case for n inputs, and so the general Cobb-Douglas production function is given by

$$y = f(x_1, x_2, \ldots, x_n) = Ax_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$$

where $A > 0$ and $0 < \alpha_i < 1$ for every $i = 1, 2, \ldots, n$. Now, for any input x_j, $j = 1, 2, \ldots, n$, we have

$$f_j \equiv \frac{\partial y}{\partial x_j} = \alpha_j A x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{j-1}^{\alpha_{j-1}} x_j^{\alpha_j - 1} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$$
$$= \alpha_j A \frac{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{j-1}^{\alpha_{j-1}} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}}{x_j^{1-\alpha_j}}$$

where again each input has diminishing marginal productivity, and all inputs are complementary to each other.

Example 11.7 Find the marginal-product functions for the *constant elasticity of substitution* (CES) production function with two inputs:

$$y = A\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-1/r}$$

where $A > 0$, $0 < \delta < 1$, $r > -1$.

Solution

The marginal products are

$$\frac{\partial y}{\partial x_1} = A\left(-\frac{1}{r}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}\left(-r\delta x_1^{-r-1}\right)\right)$$
$$= \delta A x_1^{-r-1}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}$$

$$\frac{\partial y}{\partial x_2} = A\left(-\frac{1}{r}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}\left(-r(1-\delta)x_2^{-r-1}\right)\right)$$
$$= (1-\delta)A x_2^{-r-1}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}$$

These can be written in a more convenient form. Multiplying $\partial y/\partial x_1$ by A^r/A^r, which leaves it unchanged, and noting that $x_1^{-r-1} = 1/x_1^{r+1}$, we get

$$\frac{\partial y}{\partial x_1} = \frac{\delta A^{r+1}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}}{A^r\left(x_1^{r+1}\right)}$$

Noting that

$$y^{r+1} = \left(A\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-1/r}\right)^{r+1}$$
$$= A^{r+1}\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-(1/r)-1}$$

we can write

$$\frac{\partial y}{\partial x_1} = \frac{\delta y^{r+1}}{A^r x_1^{r+1}} = \frac{\delta}{A^r}\left(\frac{y}{x_1}\right)^{r+1}$$

Following similar steps, we can write

$$\frac{\partial y}{\partial x_2} = \frac{(1-\delta) y^{r+1}}{A^r x_2^{r+1}} = \frac{(1-\delta)}{A^r}\left(\frac{y}{x_2}\right)^{r+1}$$

■

Although the rules of differentiation studied in chapter 5 for functions of one variable can generally be applied to functions of many variables in a straightforward manner, it is worth reconsidering the chain rule here. Suppose that $y = f(x_1(t), x_2(t))$; that is, both variables x_1 and x_2 depend on time which we denote by the variable t. Since y is affected by t through its effect on both variables x_1 *and* x_2, we must take this influence into account when applying the chain rule to find the derivative of $f(x_1(t), x_2(t))$ with respect to t, and so we get

$$\frac{dy}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt}$$

To see how this rule actually works consider the example below:

$$y = f(x_1, x_2) = 3x_1 + 5x_2, \quad \text{with } x_1 = t^2, \quad x_2 = 4t^3$$

The chain rule gives the result

$$\frac{dy}{dt} = 3\frac{d[t^2]}{dt} + 5\frac{d[4t^3]}{dt} = 3[2t] + 5[12t^2] = 6t + 60t^2$$

while direct substitution gives

$$y = 3[t^2] + 5[4t^3] = 3t^2 + 20t^3$$

implying that

$$\frac{dy}{dt} = 6t + 60t^2$$

which is the same result as found using the chain rule.

If we had a function of three variables, with only two of them depending on some common variable t, for example, $y = g(x_1(t), x_2(t), x_3)$, then we would write

$$\frac{\partial y}{\partial t} = \frac{\partial g}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial g}{\partial x_2}\frac{dx_2}{dt}$$

Notice that in this expression we use ∂ to denote the derivative of the function with respect to t. The reason for doing so is that although we have evaluated each impact of changing t on the function value y by considering its effect on both variables x_1 and x_2, we have held the variable x_3 fixed. Thus we consider $\partial y/\partial t$ to be a partial derivative rather than a total derivative, as would be indicated by using dy/dt. This distinction is fairly subtle and is not always made.

It is important to emphasize that although as a mathematical exercise one can always compute a partial derivative of a function by implicitly holding *all the other variables* fixed, one must sometimes be cautious when considering the economic meaning of the result. The following discussion illustrates.

Suppose that one wishes to determine the impact of capital accumulation and technical change on the level of productivity in an economy. Let the value of capital stock at time t be represented by the function $K(t)$. Suppose that both the amount of capital and the efficiency with which capital is used over time affect the level of gross national product (Y), and hence write

$$Y = f(K, t), \quad \text{where } K = K(t)$$

Suppose that the level of capital is fixed artificially at $K(t) = \bar{K}$, but suppose that this capital is used more efficiently over time. It follows that the expression

$$\frac{\partial[f(\bar{K}, t)]}{\partial t} = f_t > 0$$

measures the impact of time passing on the level of output, *assuming* that the capital stock does not increase in size. If, however, capital stock is growing over time, then it does not make economic sense to ignore this impact on Y. The *total* effect of increasing t on output would then be

$$\begin{aligned}\frac{d[f(K(t), t)]}{dt} &= \frac{\partial[f(K(t), t)]}{\partial K}\frac{dK(t)}{dt} + f_t \\ &= f_K K'(t) + f_t\end{aligned}$$

The first term in this expression indicates the impact that an increase in capital stock has on output as time passes while the second term indicates the impact that time passing has on output through the more efficient use of any existing level of capital.

The relevance of the distinction between the total and partial derivative for functions of more than one variable often arises in equations which represent the *reduced form* of some economic model. That is, there are a number of relationships that are expressed in the equation $f(K(t), t)$ which could be considered separately. We will return to this issue in chapter 14.

EXERCISES

1. Find the partial derivatives of the function

$$y = 3x_1 + 5x_2$$

using definition 11.1 (see example 11.1).

2. Find the partial derivatives of the function

$$y = ax_1 + bx_2$$

where a and b are any constants, using definition 11.1 (see example 11.1).

3. For the revenue function of example 11.1, $R(x_1, x_2) = p_1x_1 + p_2x_2$, find the partial derivative $\partial R(x_1, x_2)/\partial x_2$ by using definition 11.1. Give an intuitive explanation of your result.

4. Discuss why it is the case that the partial derivatives in questions 1, 2, and 3 are constant functions.

5. For the function of example 11.2, $y = x_1^2x_2$, find the partial derivative $\partial y/\partial x_2$ by using definition 11.1.

6. For the function

$$y = x_1x_2$$

find the partial derivatives by using definition 11.1.

7. Find the marginal-product functions for the Cobb-Douglas production function

$$y = 10x_1^{1/2}x_2^{1/3}x_3^{1/4}$$

8. Find the marginal-product functions for the Cobb-Douglas production function

$$y = Ax_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, \qquad A > 0,\ 0 < \alpha_i < 1 \quad \text{for } i = 1, 2, 3, 4$$

9. Find the marginal-product functions for the CES (constant elasticity of substitution) production function

$$y = 12\left[0.4x_1^{-1/2} + 0.6x_2^{-1/2}\right]^{-2}$$

10. Find the marginal-product functions for the CES production function

$$y = A\left[w_1 x_1^{-r} + w_2 x_2^{-r} + w_3 x_3^{-r}\right]^{-1/r}$$

$A > 0$, $r > -1$, $0 < w_i < 1$ for $i = 1, 2, 3$ and $w_1 + w_2 + w_3 = 1$.

11. Suppose both that the amount of capital at time t, $K = K(t)$, and that the efficiency with which it is used affect GNP according to the function

$$Y = f(K, t) = 0.2(1 + t)^{1/2} K, \quad \text{where } K = K_0 e^{0.05t}$$

Find and give the economic intuition of the derivative

$$\frac{dY}{dt} = f_t + f_K \frac{dK}{dt}$$

12. In general, a function $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is said to be continuous at the point $\mathbf{x} = \mathbf{a}$ if there is some $\delta > 0$ (possibly very small) such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $\|\mathbf{x} - \mathbf{a}\| < \delta$ for any $\epsilon > 0$, where $\|.\|$ is the Euclidean distance.

Show that according to this definition, the linear function $f(x_1, x_2) = c_0 + c_1 x_1 + c_2 x_2$, $c_1, c_2 \neq 0$, is continuous at any point $(a_1, a_2) \in \mathbb{R}^2$.

11.2 Second-Order Partial Derivatives

In chapter 5 we saw that since the derivative of a function $f(x)$ is itself a function, sometimes written $f'(x)$, we can find the derivative of the derivative function, $df'(x)/dx$, which is called the second derivative and is often written $f''(x)$. The second derivative is itself also a function and so we can continue the process to find the third derivative and so on to any number of higher-order derivatives, $f^{(n)}(x)$ for $n = 1, 2, 3, \ldots$. The same can be done for functions of more than one variable, although we must recognize that there will be more than one derivative of the first order, second order, and so on. That is, a function of n variables, $y = f(x_1, x_2, \ldots, x_n)$, has n first-order partial derivatives $f_1, f_2, \ldots, f_n$. Each of these can be differentiated with respect to each of the n variables, and so it follows that a function of n variables has n sets of n second-order partial derivatives, n^2 in all, each denoted

$$f_{ij} \equiv \frac{\partial f_i(x_1, x_2, \ldots, x_n)}{\partial x_j}, \qquad i, j = 1, 2, \ldots, n$$

where f_{ij} is found by first differentiating the function $f(\mathbf{x})$ with respect to the variable x_i and then differentiating the result, $f_i(\mathbf{x})$, with respect to the variable x_j. In this way we can use f_{ij} to represent all of the second-order partial derivatives.

We can see from this discussion that even if we restrict our attention to only the first- and second-order derivatives of functions of more than one variable, it is important to have a simple technique for keeping track of, or cataloging, all the various derivatives. This is done using vector and matrix notation. It is standard notation to arrange the first-order partial derivatives in a column or row vector and refer to it as the **gradient vector** using the following notation:

$$\nabla f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad \text{or} \quad \nabla f^T = [f_1 \quad f_2 \quad \ldots \quad f_n]$$

The reason for calling this a gradient vector is that each element, f_i, indicates the rate of change in the function value with respect to the variable x_i. This is analogous to signs indicating the *grade* or *steepness* that we find when driving a vehicle through hills or mountains.

Example 11.8 Find the gradient vector for the function $f(x_1, x_2) = 5 - 2x_1 + 3x_2$.

Solution

The first derivatives of the function are

$$f_1 = -2, \quad f_2 = 3$$

and so the gradient vector is

$$\nabla f = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

The meaning of this gradient vector is that the rate of change of the function value is -2 to 1 for a change in the variable x_1 and $+3$ to 1 for a change in the variable x_2, as indicated in figure 11.7. ■

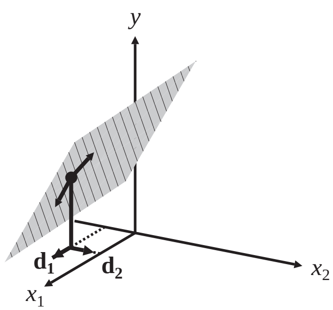


Figure 11.7 Graph of the plane $y = 5 - 2x_1 + 3x_2$ in example 11.8

Example 11.9 Find the gradient vector for the function $f(x_1, x_2, x_3) = x_1^\alpha x_2^\beta x_3^\gamma$.

Solution

The first-order derivatives of the function are

$$f_1 = \alpha x_1^{\alpha-1} x_2^\beta x_3^\gamma$$
$$f_2 = \beta x_1^\alpha x_2^{\beta-1} x_3^\gamma$$
$$f_3 = \gamma x_1^\alpha x_2^\beta x_3^{\gamma-1}$$

and so the gradient vector is

$$\nabla f = \begin{bmatrix} \alpha x_1^{\alpha-1} x_2^\beta x_3^\gamma \\ \beta x_1^\alpha x_2^{\beta-1} x_3^\gamma \\ \gamma x_1^\alpha x_2^\beta x_3^{\gamma-1} \end{bmatrix}$$

■

To keep track of second-order derivatives, it is best to use a matrix. Take as an example a function of two variables, $y = f(x_1, x_2)$. There are $2^2 = 4$ second-order partial derivatives:

$$f_{11} \equiv \frac{\partial f_1(x_1, x_2)}{\partial x_1}, \quad f_{12} \equiv \frac{\partial f_1(x_1, x_2)}{\partial x_2}$$
$$f_{21} \equiv \frac{\partial f_2(x_1, x_2)}{\partial x_1}, \quad f_{22} \equiv \frac{\partial f_2(x_1, x_2)}{\partial x_2}$$

A function of three variables, $y = f(x_1, x_2, x_3)$, has $3^2 = 9$ second-order partial derivatives, and so on. These then can be represented conveniently in matrix notation as follows:

$$\nabla_2 F \equiv \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \quad \text{for } n = 2$$

$$\nabla_2 F \equiv \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \quad \text{for } n = 3$$

and

$$\nabla_2 F \equiv \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \quad \text{for general } n$$

For the n variable case we can more simply write the matrix of second-order partial derivatives as

$$\nabla_2 F \equiv [f_{ij}]$$

with the i, jth element of the matrix $\nabla_2 F$ representing the result of first differentiating the function $f(\mathbf{x})$ with respect to the ith variable and then differentiating this derivative function with respect to the jth variable.

A remark about notation is in order here. The sign ∇ indicates the operation of differentiation and with the subscript 2, ∇_2 indicates that the operation relates to *second-order* partial derivatives. We use the capital letter, F, because we are referring to a matrix. The matrix of the second-order partials of f is called the **Hessian matrix.**

Example 11.10 Find and arrange in vector/matrix notation the first- and second-order partial derivatives of the function $f(x_1, x_2) = x_1^2 x_2$.

Solution

The first-order partials are

$$f_1 = 2x_1 x_2, \quad f_2 = x_1^2$$

while the second-order partials are

$$f_{11} = 2x_2, \quad f_{12} = 2x_1$$
$$f_{21} = 2x_1, \quad f_{22} = 0$$

Arranging these in vector and matrix notation gives

$$\nabla f = \begin{bmatrix} 2x_1 x_2 \\ x_1^2 \end{bmatrix}, \quad \nabla_2 F = \begin{bmatrix} 2x_2 & 2x_1 \\ 2x_1 & 0 \end{bmatrix}$$ ■

For functions of two variables, some intuition about the meaning of second-order partial derivatives can be obtained from the graph of the function used in example 11.10. The derivative $f_{11} = \partial f_1/\partial x_1$ gives the rate at which the derivative f_1 changes as x_1 changes with the value of x_2 fixed. Notice in this case that $f_{11} = 2x_2$ is positive for $x_2 > 0$ and negative for $x_2 < 0$. So, *in the x_1 direction* (i.e., as x_1 changes but x_2 is held fixed), the function behaves as a one-dimensional convex function whenever $x_2 > 0$ and as a one-dimensional concave function whenever $x_2 < 0$ (recall section 5.5). This is illustrated in figure 11.8.

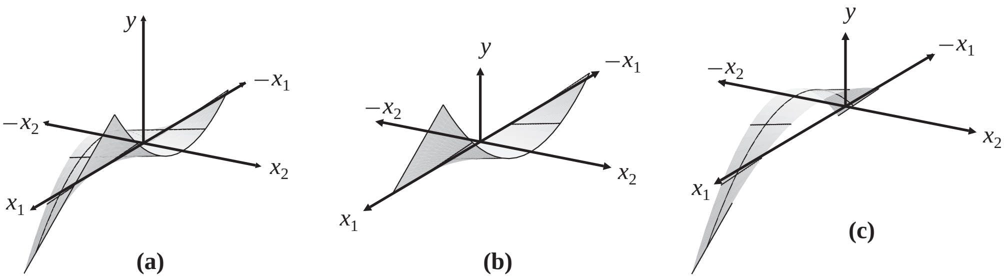


Figure 11.8 Function $f(x_1, x_2) = x_1^2 x_2$. In (b), $x_2 > 0$; in (c), $x_2 < 0$.

The second-order partial derivatives f_{ij} with $i \neq j$ are called **cross-partial derivatives.** These indicate the rate at which the first-order derivative f_i changes as the value of the variable x_j changes. Continuing with example 11.10, we see that $f_{12} = 2x_1$. Since $f_{12} > 0$ when $x_1 > 0$, this implies that f_1 gets larger in value with an increase in x_2. This is illustrated in figure 11.9, where we see that f_1 is greater at point $v = (a, d)$ than at point $u = (a, b)$, since $d > b$.

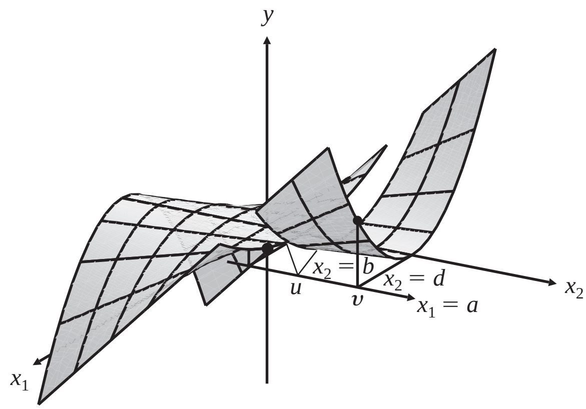


Figure 11.9 Function $y = x_1^2 x_2$ with a “slice" removed

Notice that for this function $f_{12} = f_{21} = 2x_1$. That the two cross-partial derivatives are equal is not a coincidence. As the following theorem indicates, the order of differentiation is irrelevant in the determination of cross-partial derivatives.

Theorem 11.1 **(Young's theorem)** For a function $y = f(x_1, x_2, \ldots, x_n)$, with continuous first- and second-order partial derivatives, the order of differentiation in computing the cross-partials is irrelevant. That is, $f_{ij} = f_{ji}$ for any pair i, j; $i, j = 1, 2, \ldots n$; $i \neq j$. (Of course, this statement holds trivially when $i = j$ as well.)

Example 11.11 Illustrate Young's theorem (theorem 11.1) for the function

$$f(x_1, x_2, x_3) = x_1^2 e^{3x_2 + x_1 x_3} + \frac{2x_2^3}{x_1}$$

Solution

The theorem in this case tells us to expect

$$f_{12} = f_{21} \quad f_{13} = f_{31} \quad f_{23} = f_{32}$$

We will show the last of these and leave the others as an exercise. We have

$$f_2 = (x_1^2) 3e^{3x_2 + x_1 x_3} + \frac{6x_2^2}{x_1}$$

$$f_{23} = \frac{\partial f_2}{\partial x_3} = (3x_1^2) x_1 e^{3x_2 + x_1 x_3} = 3x_1^3 e^{3x_2 + x_1 x_3}$$

$$f_3 = (x_1^2) x_1 e^{3x_2 + x_1 x_3}$$

$$f_{32} = \frac{\partial f_3}{\partial x_2} = 3x_1^3 e^{3x_2 + x_1 x_3}$$

and so $f_{23} = f_{32}$. ■

For additively separable functions, the cross-partial derivatives are zero. In general, this can be shown as follows. An additively separable function may be written in the form

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} g^i(x_i)$$

We have

$$\nabla f = \begin{bmatrix} \partial[g^1(x_1)]/\partial x_1 \\ \partial[g^2(x_2)]/\partial x_2 \\ \vdots \\ \partial[g^n(x_n)]/\partial x_n \end{bmatrix}$$

and, since $\partial[\partial g^i(x_i)/\partial x_i]/\partial x_j = 0$ for $i \neq j$, the Hessian matrix is a diagonal matrix

$$\nabla_2 F = \begin{bmatrix} \partial^2 g^1(x_1)/\partial x_1^2 & 0 & \cdots & 0 \\ 0 & \partial^2 g^2(x_2)/\partial x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \partial^2 g^n(x_n)/\partial x_n^2 \end{bmatrix}$$

Since much of the calculus used in economics is made much more straightforward by results such as Young's theorem, we will make much use of functions which possess continuous first- and second-order partial derivatives and refer to these as C^2 functions. Throughout the remainder of the chapter, functions are assumed to be C^2 unless otherwise stated.

The exercise of determining the economic interpretation of first- and second-order partial derivatives is illustrated in the following example:

Example 11.12 Find and interpret the second-order partial derivatives of the Cobb-Douglas production function with two inputs.

Solution

The general form of the Cobb-Douglas production function with two inputs is

$$y = f(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}, \qquad x_1, x_2 > 0$$

where x_1 and x_2 are input levels, y is the output level, and $\alpha,\ \beta,\ A > 0$, are technological parameters. We usually add the restrictions that $\alpha < 1$ and $\beta < 1$, for reasons that are described below.

The (first-order) partial derivatives of this function are

$$f_1 = \alpha A x_1^{\alpha-1} x_2^{\beta}$$

which is the marginal product of input 1, and

$$f_2 = \beta A x_1^{\alpha} x_2^{\beta-1}$$

which is the marginal product of input 2. The conditions $\alpha > 0$ and $\beta > 0$ ensure that the marginal products are positive, which implies that adding more of either input leads to a greater level of output, as one would expect.

The second-order partial derivatives are

$$\frac{\partial f_1}{\partial x_1} \quad \text{or} \quad f_{11} = \alpha(\alpha - 1)Ax_1^{\alpha-2} x_2^{\beta}$$

$$\frac{\partial f_1}{\partial x_2} \quad \text{or} \quad f_{12} = \alpha\beta A x_1^{\alpha-1} x_2^{\beta-1}$$

$$\frac{\partial f_2}{\partial x_1} \quad \text{or} \quad f_{21} = \alpha\beta A x_1^{\alpha-1} x_2^{\beta-1}$$

$$\frac{\partial f_2}{\partial x_2} \quad \text{or} \quad f_{22} = \beta(\beta-1) A x_1^{\alpha} x_2^{\beta-2}$$

Given the assumptions made about the parameters, $0 < \alpha, \beta < 1$, $A > 0$, it follows that f_{11} and f_{22} are negative (since $\alpha < 1 \Rightarrow \alpha - 1 < 0$ and $\beta < 1 \Rightarrow \beta - 1 < 0$), which implies diminishing marginal productivity of each input. (See example 5.15 for a discussion of this phenomenon in the context of a single input.) f_{12} and f_{21} are positive. The cross-partial derivative f_{12} is the rate at which the marginal product of input 1 changes as more of input 2 is added. $f_{12} > 0$ implies that as more of input 2 is added (e.g., capital), an additional unit of input 1 (e.g., labor) becomes more productive; that is, the marginal productivity of one input is enhanced by having more of the other input available. ■

EXERCISES

1. For the linear function $f(x_1, x_2) = a_1x_1 + a_2x_2$, where a_1 and a_2 are constants, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation. Give an intuitive account of your result.

2. For the linear function $f(x_1, x_2) = a_1x_1 + a_2x_2 + a_3x_3$, where a_1, a_2 and a_3 are constants, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation. Give an intuitive account of your result.

3. For the function $f(x_1, x_2) = x_1^3 x_2^4$, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation.

4. For the function $f(x_1, x_2) = x_1^2 x_2^4 x_3^5$, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation.

5. For the function $f(x_1, x_2) = x_1^2 + x_2^2$, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation.

6. For the function $f(x_1, x_2) = a_1x_1^2 + a_2x_2^2 + a_3x_3^2$, determine the first- and second-order partial derivatives, and arrange in vector/matrix notation.

7. Consider the following specific Cobb-Douglas production function:

$$y = 50x_1^{1/2} x_2^{2/3}$$

Find the first- and second-order partial derivatives, and determine the signs. What is the economic interpretation of the signs of these derivatives?

8. Consider the following 3-input version of a Cobb-Douglas production function

$$y = Ax_1^{\alpha} x_2^{\beta} x_3^{\gamma}, \qquad A > 0, \quad 0 < \alpha, \beta, \gamma < 1$$

Find the first- and second-order partial derivatives, and determine the signs. What is the economic interpretation of the signs of these derivatives?

9. Complete the exercise begun in example 11.11. That is, for the function

$$f(x_1, x_2, x_3) = x_1^2 e^{3x_2 + x_1 x_3} + 2x_2^3/x_1$$

show that $f_{12} = f_{21}$, and $f_{13} = f_{31}$, which are implications of Young's theorem.

11.3 The First-Order Total Differential

In section 5.2, we derived the expression for the first-order total differential of the function $y = f(x)$, which is written as

$$dy = f'(x)\, dx$$

For a specific value of x, say $x = a$, we can use this expression as a means of approximating the change in y, Δy, generated by a change in the value of x, Δx, within a neighborhood of the point $x = a$. That is,

$$\Delta y = f(a + \Delta x) - f(a)$$

can be approximated by

$$dy = f'(a)\, dx$$

This is illustrated in figure 11.10, where the change in x is $\Delta x (\equiv dx)$, the actual change in y is Δy, the approximation of the change in y is dy and so $\epsilon = \Delta y - dy$ is the approximation error. In section 5.2 we illustrated how $\epsilon/\Delta y \to 0$ as $\Delta x \to 0$. In other words, the approximation can be made as accurate as one wishes by choosing the change in x to be *small*.

An analogous result applies to functions of more than one variable, although a geometric interpretation is possible only for the case with $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$. This is easy to see if we consider changing the variables one at a time, which mirrors the operation of partial differentiation. Thus, for the function

$$y = f(x_1, x_2)$$

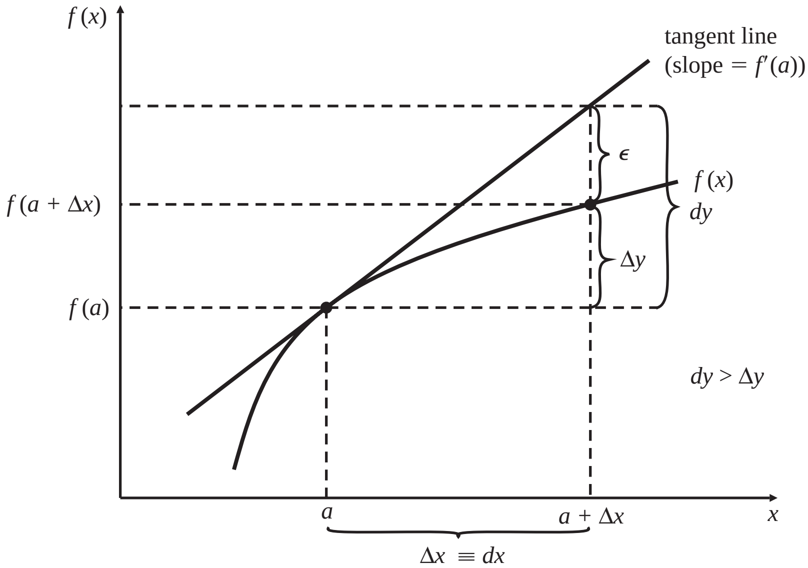


Figure 11.10 Total differential as an approximation to a change in the function value for a change in x

defined in some neighborhood of the point $(a_1, a_2) \in \mathbb{R}^2$, we could change only x_1 ($dx_1 \equiv \Delta x_1$) and get the result that

$$\Delta y = f(a_1 + \Delta x_1, a_2) - f(a_1, a_2)$$

which can be approximated by

$$dy = f_1(a_1, a_2)\, dx_1 \qquad \text{(i.e., } x_2 \text{ fixed, implying that } dx_2 = 0)$$

Similarly we could change x_2 ($dx_2 \equiv \Delta x_2$) only and get the result that

$$\Delta y = f(a_1, a_2 + \Delta x_2) - f(a_1, a_2)$$

which can be approximated by

$$dy = f_2(a_1, a_2)\, dx_2 \qquad \text{(i.e., } x_1 \text{ fixed, implying that } dx_1 = 0)$$

Allowing for both x_1 and x_2 to change leads us to the following result:

Definition 11.3 The **first-order total differential** for the function $y = f(x_1, x_2)$ is

$$dy = f_1(x_1, x_2)\, dx_1 + f_2(x_1, x_2)\, dx_2$$

Using the total differential to approximate actual changes in the function value for given changes in x_1 and x_2 corresponds geometrically to using the tangent plane as an approximation of the function in the same way as we used the tangent line for functions of one variable. This is illustrated in figure 11.11.

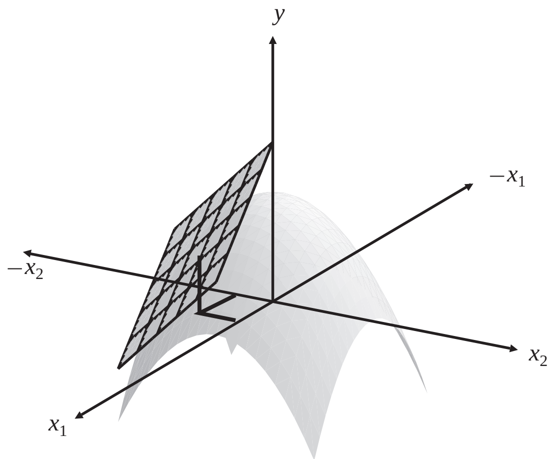


Figure 11.11 A segment of a tangent plane to the function $y = 1 - x_1^2 - x_2^2$

Recall from chapter 5 that the total differential provides an exact approximation for linear functions, $y = ax + b$. This result extends to functions of more than one variable. Consider, for example, the function $y = f(x_1, x_2)$, where

$$f(x_1, x_2) = 2x_1 + 3x_2$$

This function implies that for every 1-unit increase in x_1, the function value f increases by 2 units while, for every 1-unit increase in x_2, the function value f increases by 3 units. Now consider a change in x_1 of amount +4 units and a change in x_2 of amount +5 units. The change in $x_1(dx_1 = 4)$ leads to an increase in y of $2(4) = 8$ while the change in $x_2(dx_2 = 5)$ leads to an increase in y of $3(5) = 15$. That is, the total effect is

$$dy = f_1\, dx_1 + f_2\, dx_2 = 2(4) + 3(5) = 23 \tag{11.1}$$

For a general (nonlinear) function $f(x_1, x_2)$

$$dy = f_1\, dx_1 + f_2\, dx_2$$

is an *approximation* of the total amount by which f changes for changes in the variables x_1 and x_2 of amounts dx_1 and dx_2, respectively. The reason that dy is generally just an approximation for Δy is that for nonlinear functions the partial derivatives f_1 and f_2 change as x_1 and x_2 change. To see that this is so, consider the following two examples. Suppose that we compute the value of the function $f(x_1, x_2) = 2x_1 + 3x_2$ at the points (1, 1) and (5, 6). In moving from (1, 1) to (5, 6), we have $dx_1 = +4$ and $dx_2 = +5$, and we find that the actual change in the function value is

$$\Delta y = f(5, 6) - f(1, 1) = 28 - 5 = 23 \tag{11.2}$$

which corresponds precisely to the result obtained in (11.1) using the total differential.

However, consider the nonlinear function $f(x_1, x_2) = x_1x_2$ evaluated at the points (1, 1) and (5, 6), so that again $dx_1 = +4$ and $dx_2 = +5$. We find that the actual change in the function value is

$$\Delta y = f(5, 6) - f(1, 1) = 30 - 1 = 29 \tag{11.3}$$

Noting that $f_1 = x_2$ and $f_2 = x_1$, we find that the total differential, evaluated at the initial point (1,1), gives the result

$$dy = f_1\, dx_1 + f_2\, dx_2 = 1(+4) + 1(+5) = 9 \tag{11.4}$$

For these *noninfinitessimal* values of dx_1 and dx_2, we find that the formula for the total differential does not in this case give a very impressive approximation to the actual change in the function value (compare equations 11.3 and 11.4). We leave it to the reader to try values $dx_1 = 0.2$ and $dx_2 = 0.3$ to see how the percentage error in using this formula depends on the size of the changes dx_i.

Implicit Differentiation

In chapter 2 we introduced the possibility of a value y being defined *implicitly* as a function of x, $x \in \mathbb{R}$. A simple example is

$$2y + 4x - 10 = 0$$

where we can see that once we have specified an x value, then there is a specific value of y which satisfies the equation. A little algebra indicates that we could explicitly define y as a function of x, with

$$y = -2x + 5$$

and we can see that the derivative is $dy/dx = -2$.

However, finding an explicit solution for y from an equation involving x and y is not always so simple, and so a procedure is useful for finding dy/dx when y is implicitly defined. For example, in

$$e^{x^2+y} - 5 = 0$$

a given value for x implies a specific y if the equality is to be satisfied. Rather than solve explicitly for y, finding dy/dx can be done through the process of **implicit differentiation.**

First, presume for the moment that the above equation does imply that y can be defined as a function of x; namely $y = f(x)$. This being the case, we can write the equation as

$$e^{x^2+f(x)} - 5 = 0$$

Upon differentiation of each term with respect to the variable x, we get (using the chain rule)

$$\left(\frac{d}{dx}[x^2 + f(x)]\right)e^{x^2+f(x)} = [2x + f'(x)]e^{x^2+f(x)} = 0$$

and upon dividing by $e^{x^2+f(x)}$, which cannot be zero, we have that

$$2x + f'(x) = 0$$
$$\Rightarrow\ f'(x) \text{ or } \frac{dy}{dx} = -2x$$

which gives the desired result. It turns out in this case that we can check the result quite easily by first directly solving for y as a function of x. If we take the natural logarithm of both sides of the original equation, $e^{x^2+y} = 5$, we get

$$[x^2 + y]\ln e = \ln 5$$

which, since $\ln e = 1$, implies that

$$y = \ln 5 - x^2$$

and so $dy/dx = -2x$.

However, following the series of steps above to implicitly differentiate a function is a little tedious. A more convenient method is to first write the relationship between x and y as the implicit function

$$F(x, y) = 0$$

and then totally differentiate this expression to get

$$F_x\, dx + F_y\, dy = 0$$

Rearranging terms gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

This is a useful method for computing the relationship between a change in x (dx) and a change in $y(dy)$ within the neighborhood of some point (x_0, y_0) wherever $F(x, y) = 0$ implies that y is a function of x. The appropriate theorem, which we give without proof, is

Theorem 11.2 **Implicit Function Theorem (for two variables)** Let $F(x, y) = 0$ be an implicit function with continuous first derivatives which is satisfied at some point (x_0, y_0) and is defined in some neighborhood of this point. If $F_y \neq 0$ at this point, then there is a function $y = f(x)$ defined in some neighborhood of $x = x_0$ corresponding to the relationship defined by $F(x, y) = 0$ such that

(i) $y_0 = f(x_0)$, and
(ii) $f'(x_0) = -F_x/F_y$

This theorem gives the conditions under which it is possible to presume that the implicit function $F(x, y) = 0$ does imply an explicit functional relationship $y = f(x)$ and how to compute its derivative. The key point is that $F_y \neq 0$. The following example illustrates how to apply the theorem.

Example 11.13 Illustrate the implicit function theorem using the function

$$F(x, y) = x^2 + y^2 - 25 = 0$$

Solution

This is the equation of a circle in $\mathbb{R}^2$ with center at the origin, (0, 0), and radius 5. Since we can also write this equation as either

$$y^2 = 25 - x^2$$

or

$$x^2 = 25 - y^2$$

it follows that $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$. Otherwise, we would have that some number when squared takes on a negative value, which is not a real number. So we choose $x_0 = 3$, $y_0 = 4$, a specific point that satisfies $F(x, y) = 0$. Since $F_y(x, y) = 2y$, it follows that $F_y \neq 0$ at $y_0 = 4$, and so the key condition is satisfied. Thus, in the neighborhood of (3, 4), we can think of y as a function of x, and we have

$$f'(x) = \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y}$$

Now, at the point (3, 4), we have

$$\frac{dy}{dx} = -\frac{6}{8} = -\frac{3}{4}$$

For this example it is easy to solve explicitly for $y = f(x)$ in the neighborhood of the point (3, 4), with $y = \sqrt{25 - x^2}$, and to check the value of the derivative by the following steps:

$$\frac{dy}{dx} = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) \Rightarrow \frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}}$$

and at $x_0 = 3$ we get $dy/dx = -3/4$.

These results are illustrated in figure 11.12. Notice in this figure that at the points (5, 0) and (−5, 0) it is not possible to see the relationship between x and y as a function $y = f(x)$. We cannot construct a neighborhood (i.e., an open interval) around x for which there is a single value y associated with every value of x within the interval, as required by the definition of a function. The implicit function theorem indicates this difficulty since $F_y = 2y = 0$ at values $x = 5$ or $x = -5$ (i.e., since $y = 0$ in each case). ■

Example 11.14 Use the implicit function theorem to show that

$$x^2y^3 + 3xy^2 + y = 22$$

implies an explicitly defined function $y = f(x)$ at the point (1, 2) and find the value of the derivative dy/dx at this point.

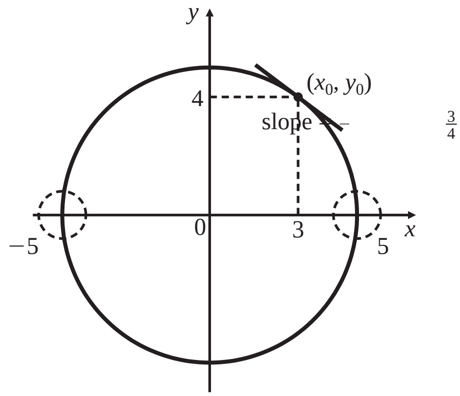


Figure 11.12 Figure to accompany example 11.13

Solution

Let $F(x, y) = x^2y^3 + 3xy^2 + y - 22 = 0$. This function is satisfied at the point $(1, 2)$, $F_y = 3x^2y^2 + 6xy + 1$ so $F_y(1, 2) = 25 \neq 0$. Hence we can perceive this relationship as a function $y = f(x)$ defined in the neighborhood of the point $(1, 2)$, and the derivative is

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{(2xy^3 + 3y^2)}{(3x^2y^2 + 6xy + 1)}$$

Thus, at the point (1, 2), we have $dy/dx = -28/25$. ■

An analogous theorem, which is a generalization of theorem 11.2, holds for functions of more than one variable.

Theorem 11.3 **Implicit Function Theorem** Let $F(x_1, x_2, \ldots, x_n, y) = 0$ be an implicit function, with continuous first derivatives, which is satisfied at some point $(x_1^0, x_2^0, \ldots, x_n^0, y^0)$ and is defined in some neighborhood of this point. If $F_y \neq 0$ at this point, then there is a function $y = f(x_1, x_2, \ldots x_n)$ defined in some neighborhood of $\mathbf{x} = \mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ such that

(i) $y^0 = f(\mathbf{x}^0)$, and
(ii) $f_i(\mathbf{x}^0) = -F_{x_i}/F_y$

Example 11.15 Use implicit differentiation to find the partial derivatives $\partial y/\partial x_1$ and $\partial y/\partial x_2$ implied by the relationship

$$F(x_1, x_2, y) = 3x_1x_2 + x_2y^2 + x_1^2x_2y - 10 = 0$$

Solution

First, we find

$$\begin{aligned} F_{x_1} &= 3x_2 + 2x_1x_2y \\ F_{x_2} &= 3x_1 + y^2 + x_1^2y \\ F_y &= 2x_2y + x_1^2x_2 \end{aligned}$$

According to the implicit function theorem we can perceive this relationship as a function $y = f(x_1, x_2)$ within any neighborhood of a point, provided that $F_y \neq 0$ at that point. So, at any point where $F_y \neq 0$, we have

$$\frac{\partial y}{\partial x_1} = -\frac{F_{x_1}}{F_y} = -\frac{(3x_2 + 2x_1x_2y)}{\left(2x_2y + x_1^2x_2\right)}$$

and

$$\frac{\partial y}{\partial x_2} = -\frac{F_{x_2}}{F_y} = -\frac{\left(3x_1 + y^2 + x_1^2y\right)}{\left(2x_2y + x_1^2x_2\right)}$$ ■

As for the case of functions of one variable, a means of remembering the formula for implicit differentiation is to totally differentiate the function $F(\mathbf{x}, y) = 0$ and take the appropriate ratios. For the case of $n = 2$, we get

$$dF = F_{x_1}\,dx_1 + F_{x_2}\,dx_2 + F_y\,dy = 0$$

Then, to find $\partial y/\partial x_1$, we set $dx_2 = 0$ (since x_2 is held fixed when finding the partial derivative with respect to x_1), and so

$$F_{x_1}\,dx_1 + F_y\,dy = 0 \quad \Rightarrow \quad \frac{\partial y}{\partial x_1} = -\frac{F_{x_1}}{F_y}$$

Similarly, for the partial derivative with respect to x_2, we set $dx_1 = 0$.

Level Curves and Level Sets

For a function $y = f(x_1, x_2)$, the set of (x_1, x_2) pairs that will generate some specific level for y, $\bar{y}$, is called a **level set.** If, in the two-variable case, we can solve the equation explicitly for x_2 in terms of x_1 and the fixed value $\bar{y}$, then we have the equation for a **level curve** in (x_1, x_2)-space for any assumed value of $\bar{y}$. The derivative dx_2/dx_1 is then the slope of the level curve.

An alternative method for computing this slope is to use the total differential. The advantage of doing so is that this technique works for functions which are complicated enough that it is difficult to solve explicitly for x_2 as a function of x_1. A level curve for a function, $y = f(x_1, x_2)$ is written

$$\bar{y} = f(x_1, x_2) \quad \text{or} \quad f(x_1, x_2) - \bar{y} = 0$$

In this form we can implement the implicit function theorem, since it is equivalent to the relationship

$$F(x_1, x_2) \equiv f(x_1, x_2) - \bar{y} = 0$$

Thus we have that

$$dF = f_1\, dx_1 + f_2\, dx_2 = 0 \qquad \text{for } y \text{ fixed.}$$

It is useful to express this relationship in the following manner:

$$\left.\frac{dx_2}{dx_1}\right|_{y=\bar{y}} \quad \text{or} \quad \left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{f_1}{f_2}$$

The condition $y = \bar{y}$ or $dy = 0$ is explicit recognition that we are evaluating the derivative dx_2/dx_1 specifically along a level curve where the value of y is fixed at some level $(y = \bar{y})$ or, in other words, the value of y is not changing $(dy = 0)$.

Example 11.16 Use the total differential to compute the slope of the level curves for the function $f(x_1, x_2) = 2x_1 + 3x_2$.

Solution

Along any level curve of $f(x_1, x_2) = 2x_1 + 3x_2$ it follows that

$$dy = f_1\, dx_1 + f_2\, dx_2 = 2\, dx_1 + 3\, dx_2 = 0$$

which implies that

$$3\, dx_2 = -2\, dx_1$$

which in turn implies that

$$\left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{2}{3}$$

as indicated in figure 11.13. ■

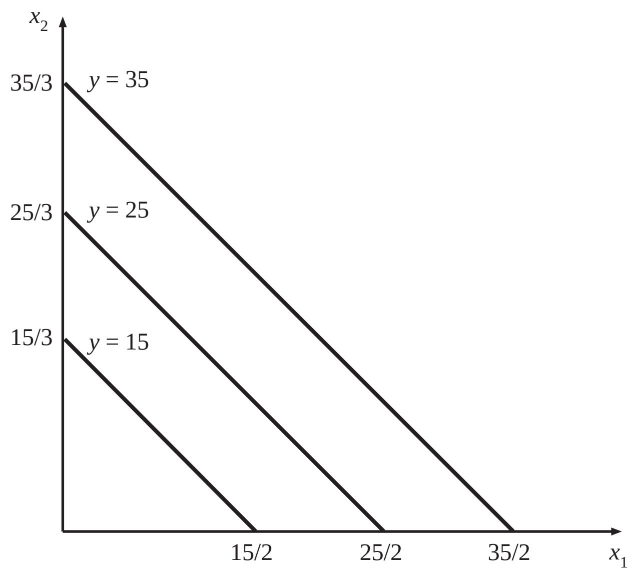


Figure 11.13 Representative level curves for $y = 2x_1 + 3x_2$ (example 11.16)

Example 11.17 Use the total differential to compute the slope of the level curves for the function $f(x_1, x_2) = x_1^2 x_2$.

Solution

The partial derivatives of the function $f(x_1, x_2) = x_1^2 x_2$ are

$$f_1 = 2x_1 x_2$$

and

$$f_2 = x_1^2$$

It follows that the slope of a level curve is

$$\left.\frac{dx_2}{dx_1}\right|_{y=\bar{y}} = -\frac{f_1}{f_2} = -\frac{2x_1 x_2}{x_1^2} = -\frac{2x_2}{x_1}$$

Unlike the example with a linear function, the slope of a level curve for this example depends on the values of x_1 and x_2. ■

Often we will just write dx_2/dx_1 to indicate the slope of level curves, with the qualification that $y = \bar{y}$ or $dy = 0$ to be taken for granted.

The Production Function: Isoquants and the Marginal Rate of Technical Substitution

For a production function written $y = f(x_1, x_2)$, the level curve $x_2 = g(x_1)$ defined by the set

$$\{(x_1, x_2) : \bar{y} = f(x_1, x_2)\} \Rightarrow x_2 = g(x_1)$$

represents the input combinations which will generate the same level of output $y = \bar{y}$. The graph of this curve is called an **isoquant** (where "iso" means equal) because all the input combinations that satisfy the condition above lead to the same quantity of output being produced. Along an isoquant $dy = 0$, and so we can find the slope of an isoquant by setting the total differential to zero:

$$dy = f_1\, dx_1 + f_2\, dx_2 = 0$$

Solving for dx_2/dx_1 along $dy = 0$ gives the slope of the isoquant:

$$\left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{f_1}{f_2}$$

The negative of this slope is called the **marginal rate of technical substitution** (MRTS):

$$\text{MRTS} = -\left.\frac{dx_2}{dx_1}\right|_{dy=0} = \frac{f_1}{f_2}$$

The MRTS is the (marginal) rate at which the firm can substitute one input for the other and continue to produce the same level of output.

Example 11.18 Find the marginal rate of technical substitution for the production function $y = f(x_1, x_2) = x_1^{1/3} x_2^{1/2}$.

Solution

Setting the total differential of this production function to zero gives

$$dy = f_1\, dx_1 + f_2\, dx_2 = \left(\frac{1}{3} x_1^{-2/3} x_2^{1/2}\right) dx_1 + \left(\frac{1}{2} x_1^{1/3} x_2^{-1/2}\right) dx_2 = 0$$

and so we get the slope of an isoquant to be

$$\left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{\frac{1}{3}x_1^{-2/3}x_2^{1/2}}{\frac{1}{2}x_1^{1/3}x_2^{-1/2}} = -\frac{2x_2}{3x_1}$$

and so the marginal rate of technical substitution is

$$\text{MRTS} = -\left.\frac{dx_2}{dx_1}\right|_{dy=0} = \frac{2x_2}{3x_1}$$ ■

A graph of an isoquant for this function is "strictly convex to the origin"; that is (roughly speaking), it can be seen that it bends in toward the origin (see figure 11.14). There is, however, a more formal method of determining whether the shape of an isoquant, or any other level curve, is strictly convex to the origin. Let $x_2 = g(x_1)$ represent the general equation of an isoquant. If dx_2/dx_1 is negative, then the slope of the isoquant is negative, while if in addition we find that $d^2x_2/dx_1^2 > 0$, then the isoquant is strictly convex to the origin. (With a weak inequality, $d^2x_2/dx_1^2 \geq 0$, we just say convex to the origin.) If we draw a graph of an isoquant (see figure 11.14) for the production function in example 11.18, we find that it does indeed have a strictly convex to the origin shape. To see this using

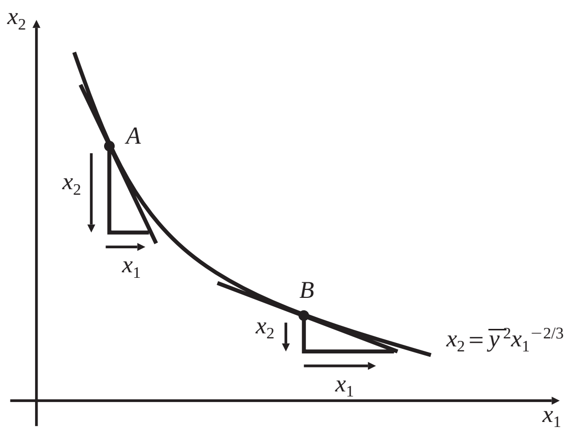


Figure 11.14 Representative isoquant for the production function $y = x_1^{1/3}x_2^{1/2}$ used in example 11.18

the formal method just described, notice that along an isoquant

$$x_1^{1/3} x_2^{1/2} = \bar{y}$$

which implies that

$$x_2 = g(x_1) = \frac{\bar{y}^2}{x_1^{2/3}}$$

is the equation for an isoquant. The first two derivatives of this function are

$$\frac{dx_2}{dx_1} = g'(x_1) = -\frac{2}{3}\bar{y}^2 x_1^{-5/3} < 0 \qquad \text{for } x_1,\ \bar{y} > 0$$

$$\frac{d^2x_2}{dx_1^2} = g''(x_1) = \frac{10}{9}\bar{y}^2 x_1^{-8/3} > 0 \qquad \text{for } x_1,\ \bar{y} > 0$$

which implies that the isoquants are downward sloping and strictly convex to the origin. Notice that the positive second derivative in conjunction with a negative first derivative means that the slope at a point like B in figure 11.14 is smaller in absolute value than at a point like A.

For a general production function with two inputs, we have

$$\text{MRTS } = -\left.\frac{dx_2}{dx_1}\right|_{y=\bar{y}} = \frac{f_1}{f_2}$$

Diagrammatically the condition that isoquants be strictly convex to the origin corresponds to the fact that as we move rightward along an isoquant, increasing x_1 and decreasing x_2, the MRTS is decreasing (i.e., the slope of the curve decreases in absolute value). This implies that

$$\frac{d[-dx_2/dx_1]}{dx_1} = \frac{d[f_1(x_1, x_2)/f_2(x_1, x_2)]}{dx_1} < 0$$

where x_2 is a function of x_1, $x_2 = g(x_1)$, along an isoquant. Therefore, employing the quotient rule and chain rule, and using $x_2 = g(x_1)$, we have that

$$\frac{d}{dx_1}\left[\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}\right] = \frac{(f_{11} + f_{12}g'(x_1))f_2 - (f_{21} + f_{22}g'(x_1))f_1}{f_2^2}$$

$$= \frac{1}{f_2^2}\left[f_{11}f_2 + f_{12}f_2\left(\frac{-f_1}{f_2}\right) - f_{12}f_1 - f_{22}f_1\left(\frac{-f_1}{f_2}\right)\right]$$

$$= \frac{1}{f_2^3}\left[f_{11}f_2^2 - f_1f_2f_{12} - f_1f_2f_{12} + f_{22}f_1^2\right]$$

$$= \frac{1}{f_2^3}\left[f_{11}f_2^2 - 2f_1f_2f_{12} + f_{22}f_1^2\right]$$

where the second equality uses $g'(x_1) = dx_2/dx_1 = -f_1/f_2$ and Young's theorem. Thus an isoquant, or any level curve for a general function with $f_2 > 0$, has the strictly convex to the origin shape provided that

$$f_{11}f_2^2 - 2f_1f_2f_{12} + f_{22}f_1^2 < 0 \tag{11.5}$$

Recall from chapter 2 that such a function is said to be strictly quasiconcave. In fact, economists usually impose the condition of strict concavity on production functions. Any function which is strictly concave is also strictly quasiconcave and so has strictly convex to the origin level curves. We investigate these relationships more fully in section 11.5.

Example 11.19 Consider the Cobb-Douglas production function

$$y = Ax_1^{\alpha}x_2^{\beta}, \qquad A > 0,\ 0 < \alpha,\ \beta < 1$$

Setting the total differential of this production function to zero gives

$$dy = f_1\,dx_1 + f_2\,dx_2 = \left(\alpha A x_1^{\alpha-1}x_2^{\beta}\right)dx_1 + \left(\beta A x_1^{\alpha}x_2^{\beta-1}\right)dx_2 = 0$$

and so we get the slope of an isoquant to be

$$\left.\frac{dx_2}{dx_1}\right|_{dy=0} = -\frac{\alpha A x_1^{\alpha-1}x_2^{\beta}}{\beta A x_1^{\alpha}x_2^{\beta-1}} = -\frac{\alpha x_2}{\beta x_1}$$

Now the marginal rate of technical substitution is

$$\text{MRTS} = -\left.\frac{dx_2}{dx_1}\right|_{dy=0} = \frac{\alpha x_2}{\beta x_1}$$

To show that the shape of an isoquant is convex to the origin, let $x_2 = g(x_1)$ represent the general equation of an isoquant, and compute the first two derivatives:

$$Ax_1^{\alpha}x_2^{\beta} = \bar{y}$$

This implies that the equation of an isoquant is

$$x_2 = g(x_1) = \left(\frac{\bar{y}}{Ax_1^{\alpha}}\right)^{1/\beta} = \left(\frac{\bar{y}}{A}\right)^{1/\beta} x_1^{-\alpha/\beta}$$

The first two derivatives of this function are

$$\frac{dx_2}{dx_1} = g'(x_1) = -\frac{\alpha}{\beta}\left(\frac{\bar{y}}{A}\right)^{1/\beta} x_1^{-1-\alpha/\beta} < 0 \qquad \text{for } x_1, \bar{y} > 0$$

$$\frac{d^2x_2}{dx_1^2} = g''(x_1) = -\frac{\alpha}{\beta}\left(-1 - \frac{\alpha}{\beta}\right)\left(\frac{\bar{y}}{A}\right)^{1/\beta} x_1^{-2-\alpha/\beta} > 0 \qquad \text{for } x_1, \bar{y} > 0$$

The signs of these derivatives imply that the isoquants are downward sloping and strictly convex to the origin. ■

A clear understanding of the relationship between isoquants and the production function is a critical component to the development of the theory of firm behavior, as will become apparent in chapter 13. The description of *consumer behavior* requires as well that we understand the relationship between the utility function, which represents consumers' preferences, and the level sets or contours of this function, which are called **indifference curves.** Although a utility function does not reflect a physical quantity, as does a production function, the mathematical and graphical analysis of the two subjects are remarkably similar.

The Utility Function: Indifference Curves and the Marginal Rate of Substitution

It is convenient to represent an individual's preferences over hypothetical consumption bundles by a utility function, $u(\mathbf{x})$ defined on $\mathbb{R}^n_+$. The level sets of a utility function are $\{\mathbf{x} : u(\mathbf{x}) = \bar{u}, \mathbf{x} \in \mathbb{R}^n_+\}$, and for $\mathbf{x} \in \mathbb{R}^2$ level sets or level curves are called indifference curves. Although these are analogous to the isoquants of the production function, there is one important difference. To say that output has doubled is a meaningful statement, whereas such a remark makes no sense when applied to utility. Therefore we say that the measurement of the level of a physical product has cardinal properties but utility measurements have only ordinal properties. Thus in figure 11.15 it is only the relative ordering of the four bundles (B_1, B_2, B_3, B_4) that has meaning, with comparisons between the size of differences in utilities being irrelevant. Either of the two sets of utility numbers indicated could be used since in both cases $u(B_1) > u(B_2) = u(B_3) > u(B_4)$, indicating that among these bundles B_1 is most preferred, followed by equal preference between B_2 and B_3 (indifference) with B_4 least preferred.

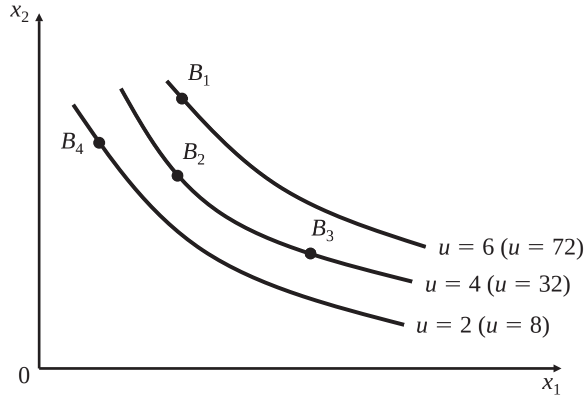


Figure 11.15 Indifference curves

Thus it is the shape of the indifference curves *only* that is relevant in consumer theory. This implies that if $u : \mathbb{R}^n_+ \to \mathbb{R}$ describes a preference ordering, then so would any other function

$$\tilde{u}(\mathbf{x}) = T[u(\mathbf{x})], \qquad T' > 0$$

obtained from u by applying a **positive monotonic transformation** T. For example, suppose that we find that the Cobb-Douglas utility function

$$u = Ax_1^a x_2^b, \qquad A, a, b > 0$$

accurately represents a specific preference ordering for a given a and b, then

$$\tilde{u} = \frac{1}{A^k} u^k = x_1^{ka} x_2^{kb}, \qquad k > 0$$

represents the same preference ordering. In particular, by choosing $k = 1/(a + b)$ the utility function $\tilde{u}$ is seen to have exponents which add up to 1 and so we can write the utility function

$$u = x_1^{\alpha} x_2^{1-\alpha}, \qquad 0 < \alpha < 1$$

without loss of generality to describe any preferences that are Cobb-Douglas. (Note that $\alpha = a/(a + b)$, and so $1 - \alpha = 1 - a/(a + b) = b/(a + b)$.)

Example 11.20 The utility functions $u = x_1^{1/3} x_2^{2/3}$ and $\tilde{u} = [u]^6 = x_1^2 x_2^4$ represent the same preferences. To see this, note that the slope of any level curve at a point in $\mathbb{R}^2_+$ is the same for both functions:

$$\text{For } u, \quad \left.\frac{dx_2}{dx_1}\right|_{du=0} = -\frac{u_1}{u_2} = -\frac{\frac{1}{3}x_1^{-2/3}x_2^{2/3}}{\frac{2}{3}x_1^{1/3}x_2^{-1/3}} = -\frac{1}{2}\frac{x_2}{x_1}$$

$$\text{For } \tilde{u}, \quad \left.\frac{dx_2}{dx_1}\right|_{d\tilde{u}=0} = -\frac{\tilde{u}_1}{\tilde{u}_2} = -\frac{2x_1x_2^4}{4x_1^2x_2^3} = -\frac{1}{2}\frac{x_2}{x_1}$$

■

In the context of the production function, we have discussed conditions that must be imposed to ensure that its level curves or contours are strictly convex to the origin or, equivalently, the condition for the function to be strictly quasiconcave (see equation 11.5). We repeat these below for the utility function $u = u(x_1, x_2)$. This condition for strict quasi-concavity is less general than that given in section 2.4, because it requires differentiability.

A utility function $u = u(x_1, x_2)$ that satisfies $u_2 > 0$ is strictly quasiconcave if and only if

$$u_{11}u_2^2 - 2u_1u_2u_{12} + u_{22}u_1^2 < 0 \tag{11.6}$$

To understand the economic significance of this condition, we need to develop the notion of the **marginal rate of substitution** (MRS), which is equivalent to the MRTS in production theory. The MRS is the (marginal) rate at which the consumer can substitute one good for the other and continue to remain on the same indifference curve. Formally it is the negative of the slope of the indifference curve. Thus

$$\text{MRS} = -\left.\frac{dx_2}{dx_1}\right|_{u=\bar{u}} = \frac{u_1}{u_2}$$

We establish the intuition for this concept through the following two examples.

Example 11.21 Consider the utility function

$$u(x_1, x_2) = x_1^2 x_2$$

with first-order partial derivatives

$$u_1 = 2x_1x_2$$
$$u_2 = x_1^2$$

It follows that the MRS is

$$\text{MRS} = -\left.\frac{dx_2}{dx_1}\right|_{du=0} = \frac{u_1}{u_2} = \frac{2x_2}{x_1}$$

The MRS, which is the absolute value of the slope of the indifference curve, is increasing in x_2 and decreasing in x_1. ■

It is generally presumed that indifference curves are *likely* to have this shape, that is, be strictly convex to the origin. The rationale given for this supposition is that as one moves from a bundle with more of x_2 and less of x_1 (i.e., x_1 becomes relatively scarcer), a reduction in a unit of x_1 requires a greater increase in x_2 to compensate the consumer (i.e., to remain on a given indifference curve).

Example 11.22 Find the shape of the indifference curves for the case of *perfect substitutes* $u(x_1, x_2) = x_1 + x_2$.

Solution

The total differential for $u(x_1, x_2) = x_1 + x_2$ is

$$du = (1)\, dx_1 + (1)\, dx_2$$

Along any indifference curve $du = 0$ and so $dx_2 = -dx_1$; i.e., $dx_2/dx_1 = -1$. This implies that

$$\text{MRS} = -\left.\frac{dx_2}{dx_1}\right|_{du=0} = 1$$

We can see that if from some initial consumption bundle, such as point A in figure 11.16, we substitute k units of one commodity for k units of the other, i.e., $dx_2 = -dx_1$, then the consumer remains on the same indifference curve. Since the consumer is always indifferent in a one-for-one trade-off between these commodities, we say that the commodities represent perfect substitutes for this consumer. ■

The intuition and analysis of the concepts of MRS and MRTS carry over to functions of more than two variables. If $y = f(x_1, x_2, \ldots, x_n)$, then $f_i dx_i$ represents the contribution to the change in the y variable from a change dx_i in the variable x_i, $i = 1, 2, \ldots, n$. The following generalization of definition 11.3 applies.

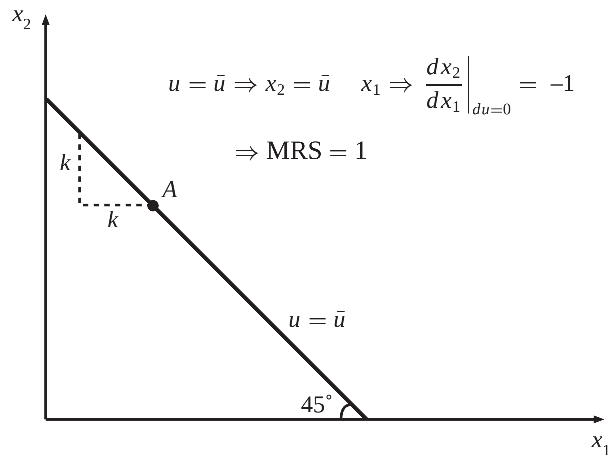


Figure 11.16 Representative indifference curve for goods that are perfect substitutes

Definition 11.4 The **first-order total differential** for the function $y = f(x_1, \ldots, x_n)$ is

$$dy = f_1\,dx_1 + f_2\,dx_2 + \cdots + f_n\,dx_n = \sum_{i=1}^{n} f_i\,dx_i$$

To compute the MRS for a consumer between two commodities or the MRTS for a firm between two inputs set $dx_i = 0$ for all x_i except the two under consideration. Thus, setting $dx_i = 0$ for every $i = 1, 2, \ldots, n$ except $i = k$ and $i = l$ gives us

$$dy = f_k\,dx_k + f_l\,dx_l = 0$$

which implies that

$$\left.\frac{dx_k}{dx_l}\right|_{dy=0, dx_i=0} = -\frac{f_l}{f_k} \qquad \text{for } i \neq k, l$$

The following example illustrates this point.

Example 11.23 Consider the MRTS between pairs of inputs for the production function

$$y = f(x_1, x_2, x_3) = Ax_1^{\alpha}x_2^{\beta}x_3^{\gamma}, \qquad \alpha, \beta, \gamma > 0$$

Total differentiation of this production function gives

$$dy = f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3$$

Along an isoquant $dy = 0$, and so, for example, setting $dx_1 = 0$, gives us

$$\text{MRTS}_{2,3} = -\left.\frac{dx_3}{dx_2}\right|_{dy=0,dx_1=0} = \frac{f_2}{f_3} = \frac{\beta A x_1^{\alpha} x_2^{\beta-1} x_3^{\gamma}}{\gamma A x_1^{\alpha} x_2^{\beta} x_3^{\gamma-1}} = \frac{\beta x_3}{\gamma x_2}$$

A similar procedure will produce the others ($\text{MRTS}_{1,2}$, etc.). ■

EXERCISES

1. For the function $u(x_1, x_2) = 5x_1 + 3x_2$:

  (a) Find the total differential.

  (b) Draw the level curve for $\bar{u} = 120$.

  (c) Use the pair of points (12, 20) and (18, 10) to illustrate that the MRS = 5/3 and derive this result from the total differential in part (a).

2. For the function $u(x_1, x_2) = ax_1 + bx_2$:

  (a) Find the total differential.

  (b) Draw a representative level curve for $u = \bar{u}$.

  (c) Use the expression for the total differential to illustrate that the MRS $= a/b$.

3. Use the total differential to find the MRTS for the production function $y = x_1 x_2$. Show that the isoquants are strictly convex to the origin.

4. Use the total differential to find the $\text{MRTS}_{2,3}$ for the production function $y = x_1 x_2 x_3$. Show that the relevant isoquants in (x_2, x_3)-space (i.e., the plane with x_1 held fixed) are strictly convex to the origin.

5. Use the total differential to find the slopes of the indifference curves for each of the following utility functions:

  (a) $u(x_1, x_2) = x_1 x_2$

  (b) $\hat{u}(x_1, x_2) = x_1^2 x_2^2$

  (c) $\tilde{u}(x_1, x_2) = B x_1^K x_2^K$, where $B, K > 0$ are constants

6. Note that the slopes of the indifference curves for each of the utility functions of question 5 are the same. This is because of the functional relationship between the utility functions. That is, $\hat{u} = F(u) = u^2$ and $\tilde{u} = F(u) = Bu^K$.

In both cases the derivative of the F function, $F'(\cdot)$, is positive on $u > 0$. We call such an F function a positive monotonic transformation of u. Show that for any such F used to generate a function of some original function $u(x_1, x_2)$, the resulting function has indifference curves (or more generally level curves) with the same shape.

7. Use the total differential to find the MRTS for the production function

$$y = \left[0.3x_1^{-3} + 0.7x_2^{-3}\right]^{-1/3}$$

Show that its isoquants are strictly convex to the origin.

8. Use the total differential to find the MRTS for the production function

$$y = A\left[\delta x_1^{-r} + (1-\delta)x_2^{-r}\right]^{-1/r}, \qquad A > 0,\ r > -1$$

Show that its isoquants are strictly convex to the origin.

9. Show that the points $A = (8, 1)$, $B = (4, 4)$, and $C = (2, 16)$ are on the same indifference curve generated by the function $u = x_1^2 x_2$. Compute the ratio $|\Delta x_2|/|\Delta x_1|$ between the pairs B and C, and A and B. How do these ratios relate to the notion of the marginal rate of substitution and the fact that it is falling as x_1 rises (and x_2 falls) moving along the indifference curve from left to right?

11.4 Curvature Properties: Concavity and Convexity

An important aspect of the shape of a function is its curvature. This property is usually described by using the second-order derivatives that explain how the first derivatives change. We discussed this process in section 5.5 for functions of one variable. Figure 11.17 summarizes those results. For $f'' > 0$ the function is convex, which means that for $f' > 0$ the function increases more rapidly as x increases while for $f' < 0$ the function value falls less quickly. For $f'' < 0$, the function is concave, which means that for $f' > 0$ the function value increases less quickly as x increases while for $f' < 0$ the function value falls more quickly.

One might expect that a direct extension for functions on $\mathbb{R}^n$ could be made by simply considering each second-order partial f_{ii} separately and checking its sign. The problem with this approach, however, is that it isolates only a finite number of specific directions in which one can travel from a point, (i.e., by changing each x_i separately while holding all other variables fixed). In fact there are an infinite number of paths that one can take from some point, as illustrated in figure 11.3. As a result we must work a bit harder to determine concavity and convexity for functions defined on $\mathbb{R}^2$ or $\mathbb{R}^n$ than for functions defined on $\mathbb{R}$.

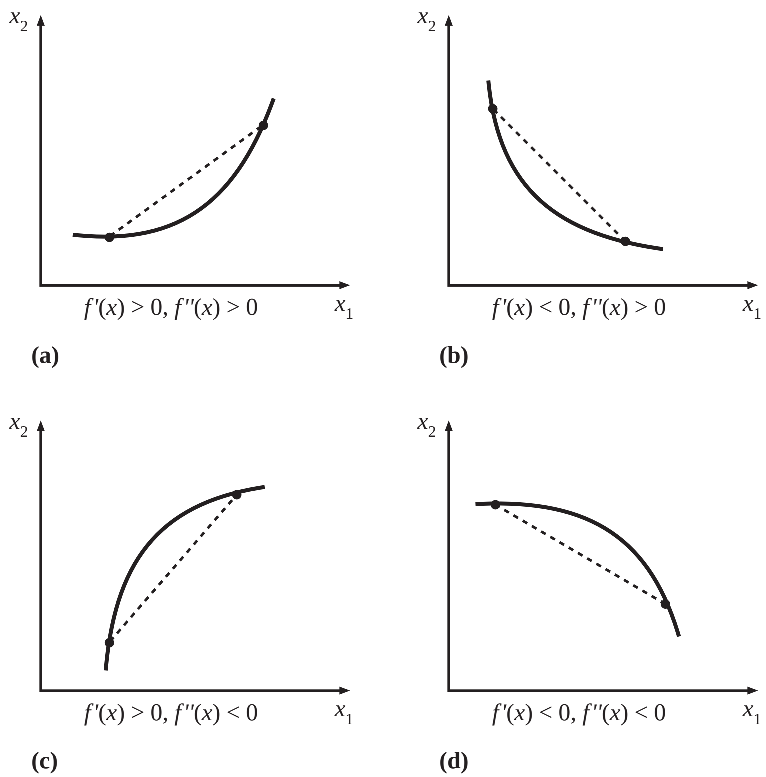


Figure 11.17 Strictly convex, (a) and (b), and strictly concave, (c) and (d), functions

Example 11.24 The second-order derivatives of the function $f(x_1, x_2) = x_1^2 + x_2^2 - 5x_1x_2$ are

$$f_{11} = 2, \quad f_{22} = 2, \quad f_{12} = -5$$

Although both f_{11} and f_{22} are positive, this function is not strictly convex (in all directions), as we see in figure 11.18. The cross-partial derivative ($f_{12} = -5$) also plays a role in determining the curvature of a function, as we will see formally in this section. ■

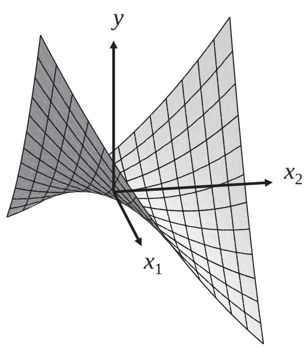


Figure 11.18 Graph of $f(x_1, x_2) = x_1^2 + x_2^2 - 5x_1x_2$ in example 11.24

Consider a function $y = f(x)$ defined on $\mathbb{R}$. Recall that the first-order total differential at the point $x = x^0$ is

$$dy = f'(x^0)\,dx$$

Notice that dy is a function of both x and dx, but we often regard dx as a given constant. Upon taking the total differential of dy, which is written d^2y, we get

$$d^2y = d[dy] = \frac{d[dy]}{dx}\,dx = \frac{d[f'(x)\,dx]}{dx}\,dx$$
$$= f''(x)\,dx\,dx = f''(x)\,dx^2$$

which is called the second-order total differential of $f(x)$. Since the term $dx^2 \equiv (dx)^2$ is strictly positive for any value $dx \neq 0$, it follows that d^2y has the same sign as $f''(x)$. Therefore the determination of convexity and concavity, which rely on the sign of $f''(x)$, can be equally well presented using the sign of d^2y. That is, since a function is convex if $f''(x) \geq 0$ and concave if $f''(x) \leq 0$, then $d^2y = [f''(x)]\,dx^2 \geq 0$ for convex functions and $d^2y = [f''(x)]\,dx^2 \leq 0$ for concave functions.

The same conditions relating the sign of d^2y to concavity/convexity apply to functions defined on $\mathbb{R}^n$. We will develop this result first for the case of $\mathbb{R}^2$; that is, $y = f(x_1, x_2)$. Recall that the first-order total differential for $y = f(x_1, x_2)$ is

$$dy = \frac{\partial f(x_1, x_2)}{\partial x_1}\,dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\,dx_2$$

or

$$dy = f_1\,dx_1 + f_2\,dx_2$$

The second-order total differential is the total differential of dy; that is, $d[dy]$ or d^2y where

$$d[dy] = \frac{\partial[dy]}{\partial x_1}\,dx_1 + \frac{\partial[dy]}{\partial x_2}\,dx_2$$
$$= \frac{\partial[f_1\,dx_1 + f_2\,dx_2]}{\partial x_1}\,dx_1 + \frac{\partial[f_1\,dx_1 + f_2\,dx_2]}{\partial x_2}\,dx_2$$
$$= [f_{11}\,dx_1 + f_{21}\,dx_2]\,dx_1 + [f_{12}\,dx_1 + f_{22}\,dx_2]\,dx_2$$
$$= f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2$$

since $f_{21} = f_{12}$ by Young's theorem.

Definition 11.5 The **second-order total differential** for the function $y = f(x_1, x_2)$ is

$$d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2$$

From this expression we can see that d^2y depends on the cross-partial second derivative f_{12} as well as on f_{11} and f_{22}. We can now state sufficient conditions for a function to be strictly convex or strictly concave.

Theorem 11.4 If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 > 0$ whenever at least one of dx_1 or dx_2 is nonzero, then $y = f(x_1, x_2)$ is a strictly convex function.

Theorem 11.5 If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable and $d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 < 0$ whenever at least one of dx_1 or dx_2 is nonzero, then $y = f(x_1, x_2)$ is a strictly concave function.

Example 11.25 Use the sign of the second-order total differential to show that the function $y = x_1^2 + x_2^2$ is strictly convex.

Solution

The second-order partial derivatives of the function $y = x_1^2 + x_2^2$ are

$$f_{11} = 2, \quad f_{12} = 0, \quad f_{22} = 2$$

It follows that

$$d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 = 2\,dx_1^2 + 2\,dx_2^2$$

Since $dx_1^2 \geq 0$ and $dx_2^2 \geq 0$ and both are zero only if $dx_1 = dx_2 = 0$, then the conditions for strict convexity in theorem 11.4 are satisfied. ■

The conditions for strict convexity and strict concavity for a function $y = f(x_1, x_2)$ given in theorems 11.4 and 11.5 are sufficient but not necessary. The reason for this is illustrated by the following example:

Example 11.26 The function $y = x_1^4 + x_2^4$ is strictly convex, yet $d^2y = 0$ if $x_1 = x_2 = 0$. To see this, notice that

$$f_1 = 4x_1^3, \quad f_2 = 4x_2^3$$
$$f_{11} = 12x_1^2, \quad f_{22} = 12x_2^2, \quad f_{12} = 0$$

and so

$$\begin{aligned} d^2y &= f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 \\ &= 12x_1^2\,dx_1^2 + 12x_2^2\,dx_2^2 \end{aligned}$$

which takes on the value zero in the cases indicated. However, this function can be shown to be strictly convex (see figure 11.19).

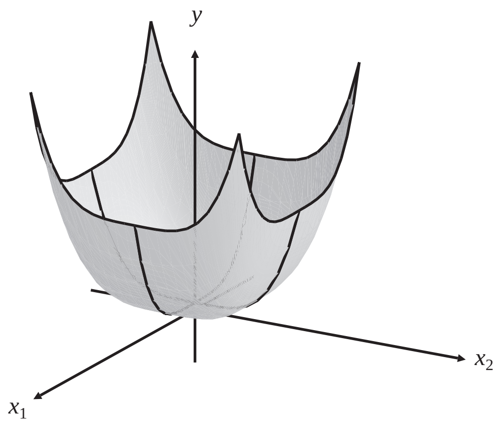


Figure 11.19 Graph of $y = x_1^4 + x_2^4$ (example 11.26) ■

The weaker conditions of concavity and convexity, which allow for linear functions or functions with linear segments, are given by the following.

Theorem 11.6 If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable, then it is convex if and only if

$$d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 \geq 0$$

Theorem 11.7 If the function $y = f(x_1, x_2)$ defined on $\mathbb{R}^2$ is twice continuously differentiable, then it is concave if and only if

$$d^2y = f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 \leq 0$$

Notice that in the case of (weak) convexity or (weak) concavity, the condition on d^2y involves a weak inequality and is both necessary and sufficient.

Example 11.27 Use the sign of the second-order total differential to show that the function $y = 5 - (x_1 + x_2)^2$ is concave.

Solution

First- and second-order partial derivatives of this function are

$$f_1 = -2(x_1 + x_2), \quad f_2 = -2(x_1 + x_2)$$
$$f_{11} = -2, \quad f_{22} = -2, \quad f_{12} = -2$$

It follows that

$$\begin{aligned} d^2y &= f_{11}\,dx_1^2 + 2f_{12}\,dx_1\,dx_2 + f_{22}\,dx_2^2 \\ &= -2\,dx_1^2 - 4\,dx_1\,dx_2 - 2\,dx_2^2 \\ &= -2\,(dx_1 + dx_2)^2 \le 0 \end{aligned}$$

and so the function is concave. To see that this function is not strictly concave, notice that along the set of (x_1, x_2) values satisfying $x_2 = a - x_1$, where a is any constant, we have $y = (x_1 + a - x_1)^2 = a^2$. These (x_1, x_2) values generate horizontal linear segments of the graph of f as illustrated in figure 11.20. ■

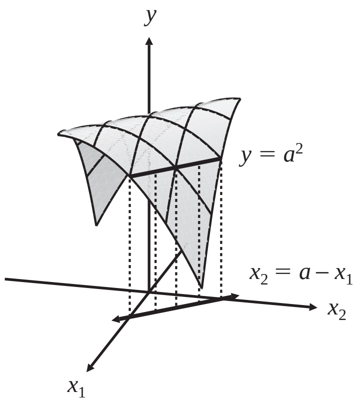


Figure 11.20 Linear segment on the surface of the graph of $y = 5 - (x_1 + x_2)^2$ in example 11.27

This analysis concerning concavity and convexity carries over to functions of more than two variables. Recall that the first-order total differential for $y = f(x_1, x_2, \ldots, x_n)$ is

$$dy = f_1\,dx_1 + f_2\,dx_2 + \cdots + f_n\,dx_n = \sum_{i=1}^{n} f_i\,dx_i$$

Since each of the partial derivatives f_i is a function of $(x_1, x_2, \ldots, x_n)$, dy is also a function of $(x_1, x_2, \ldots, x_n)$, and so we can find its total differential $d[dy]$ or d^2y as follows:

$$\begin{aligned} d^2y &= \frac{\partial[dy]}{\partial x_1}\,dx_1 + \frac{\partial[dy]}{\partial x_2}\,dx_2 + \cdots + \frac{\partial[dy]}{\partial x_n}\,dx_n \\ &= \frac{\partial\left[\sum_{i=1}^{n} f_i\,dx_i\right]}{\partial x_1}\,dx_1 + \frac{\partial\left[\sum_{i=1}^{n} f_i\,dx_i\right]}{\partial x_2}\,dx_2 + \cdots \\ &\quad + \frac{\partial\left[\sum_{i=1}^{n} f_i\,dx_i\right]}{\partial x_n}\,dx_n \end{aligned} \tag{11.7}$$

Each term in equation (11.7) represents n partial derivatives. For example,

$$\begin{aligned}\frac{\partial\left[\sum_{i=1}^{n} f_i\,dx_i\right]}{\partial x_1} &= \frac{\partial\left[f_1\,dx_1 + f_2\,dx_2 + \cdots + f_n\,dx_n\right]}{\partial x_1}\\ &= \frac{\partial f_1}{\partial x_1}\,dx_1 + \frac{\partial f_2}{\partial x_1}\,dx_2 + \cdots + \frac{\partial f_n}{\partial x_1}\,dx_n\\ &= f_{11}\,dx_1 + f_{21}\,dx_2 + \cdots + f_{n1}\,dx_n\end{aligned}$$

It follows that since each of the n terms in equation (11.7) represents n terms, there are n^2 terms altogether. It is tedious to write these out in full. However, by using summation notation, we get

$$d^2y = \sum_{j=1}^{n}\left(\frac{\partial\left[\sum_{i=1}^{n} f_i\,dx_i\right]}{\partial x_j}\,dx_j\right) = \sum_{j=1}^{n}\sum_{i=1}^{n} f_{ij}\,dx_i\,dx_j$$

which in turn indicates how we can write d^2y in matrix notation

$$d^2y = \mathbf{dx}^T \nabla_2 F\,\mathbf{dx} \tag{11.8}$$

where $\mathbf{dx}^T = [dx_1\,dx_2\,\cdots\,dx_n]$ is the vector of changes dx_i and

$$\nabla_2 F = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix}$$

is the Hessian matrix of f.

The conditions regarding the relationship between the sign of d^2y and concavity or convexity for functions of two variables carry over directly to functions of n variables using the above expression for d^2y given by equation (11.8).

Determining the sign of d^2y directly can involve quite a lot of algebraic manipulation even for rather simple functions of only two variables. Fortunately results from chapter 10 on quadratic forms offer a simpler way of determining whether d^2y is positive or negative (or neither). To see this, notice from equation (11.8) that d^2y is in fact a quadratic form with matrix $\nabla_2 F$ and vector $\mathbf{dx}$. Moreover, from Young's theorem (i.e., $f_{ij} = f_{ji}$) it follows that the matrix $\nabla_2 F$ is symmetric. We can therefore use results from the theory of quadratic forms to establish alternative sets of conditions for checking whether a function $y = f(\mathbf{x})$ is concave or convex. For example, from theorem 10.17 we know that the quadratic form d^2y is positive (i.e., the matrix $\nabla_2 F$ is positive definite) if and only if all the leading principal minors of the matrix $\nabla_2 F$ are positive. This provides an alternative method of determining whether a function is strictly convex. Writing the Hessian as H (i.e., $\nabla_2 F \equiv H$), we have the following rules for determining convexity/concavity of a function.

Theorem 11.8 For any function $y = f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$ which is twice continuously differentiable with Hessian H, it follows that:

1. the function f is strictly convex on $\mathbb{R}^n$ if H is positive definite for all $\mathbf{x} \in \mathbb{R}^n$ (i.e., $d^2 y = \mathbf{dx}^T H\, \mathbf{dx} > 0$).
2. the function f is strictly concave on $\mathbb{R}^n$ if H is negative definite for all $\mathbf{x} \in \mathbb{R}^n$ (i.e., $d^2 y = \mathbf{dx}^T H\, \mathbf{dx} < 0$).
3. the function f is convex on $\mathbb{R}^n$ if and only if H is positive semi-definite for all $\mathbf{x} \in \mathbb{R}^n$ (i.e., $d^2 y = \mathbf{dx}^T H\, \mathbf{dx} \geq 0$).
4. the function f is concave on $\mathbb{R}^n$ if and only if H is negative semi-definite for all $\mathbf{x} \in \mathbb{R}^n$ (i.e., $d^2 y = \mathbf{dx}^T H\, \mathbf{dx} \leq 0$).

Notice that the conditions on H are only sufficient in the case of strict convexity/concavity, while the conditions are both necessary and sufficient in the case of (weak) convexity/concavity. This is consistent with our earlier discussion regarding functions of two variables.

Recall from section 10.3 that the conditions for a matrix to be positive or negative definite depend on the signs of the leading principal minors of the matrix. We collect these results in the context of the Hessian (i.e., the quadratic form $d^2 y = \mathbf{dx}^T H \mathbf{dx}$) in the following theorem. Recall that $|H_k|$ refers to the leading principal minor of order k while we indicate by $|H_k^*|$ any principal minor of order k, so $|H_k^*|$ refers to more than one minor. Thus the leading principal minors are

$$|H_1| = f_{11} \quad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \quad \ldots$$

$$|H_n| = |H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

It is tedious to write out all of the $|H_k^*|$ determinants, and so we do this only for $n = 3$ here:

$$|H_1^*| = f_{11}, f_{22}, f_{33}$$

$$|H_2^*| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix}, \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix}$$

$$|H_3^*| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

Since $|H_n^*|$ represent the determinants of matrices formed by interchanging row-column pairs, it follows that we only need to check the sign of $|H_n| = |H|$ to know the sign of all the $|H_n^*|$ values.

Theorem 11.9 Let H be the Hessian matrix associated with a twice continuously differentiable function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. It follows that:

1. H is positive definite on $\mathbb{R}^n$ if and only if its leading principal minors are positive; $|H_1| > 0, |H_2| > 0, |H_3| > 0, \ldots, |H_n| = |H| > 0$ for $\mathbf{x} \in \mathbb{R}^n$. In this case $d^2y > 0$ and so f is strictly convex.
2. H is negative definite on $\mathbb{R}^n$ if and only if its leading principal minors alternate in sign beginning with a negative value for $k = 1$;

$$|H_1| < 0, |H_2| > 0, \ldots, |H_n| = |H| \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases}$$

   for $\mathbf{x} \in \mathbb{R}^n$. In this case $d^2y < 0$ and so f is strictly concave.
3. H is positive semidefinite on $\mathbb{R}^n$ if and only if all of its principal minors are positive or zero; $|H_1^*| \geq 0, |H_2^*| \geq 0, |H_3^*| \geq 0, \ldots, |H_n^*| = |H| \geq 0$ for $\mathbf{x} \in \mathbb{R}^n$. In this case $d^2y \geq 0$ and so f is convex. Moreover, if f is convex, this set of conditions must hold.
4. H is negative semidefinite on $\mathbb{R}^n$ if and only if all of its principal minors alternate in sign beginning with a negative or zero value for $k = 1$;

$$|H_1^*| \leq 0, |H_2^*| \geq 0, \ldots, |H_n^*| = |H| \begin{cases} \geq 0 & \text{if } n \text{ is even} \\ \leq 0 & \text{if } n \text{ is odd} \end{cases}$$

   for $\mathbf{x} \in \mathbb{R}^n$. In this case $d^2y \leq 0$ and so f is concave. Moreover, if f is concave this set of conditions must hold.

The following examples illustrate how to use the results in theorem 11.9 to determine the concavity/convexity properties of a function.

Example 11.28 Use theorem 11.9 to determine the convexity/concavity property of the function $y = f(x_1, x_2) = (x_1 + x_2)^{1/2}$ defined on $\mathbf{x} \in \mathbb{R}^2_{++}$.

Solution

The second-order partial derivatives are

$$f_{11} = -\frac{1}{4}(x_1 + x_2)^{-3/2}$$
$$f_{12} = f_{21} = -\frac{1}{4}(x_1 + x_2)^{-3/2}$$
$$f_{22} = -\frac{1}{4}(x_1 + x_2)^{-3/2}$$

Since all of these are negative, we check first for strict concavity of f:

$$|H_1| = f_{11} < 0, \quad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11}f_{22} - f_{12}^2 = 0$$

Note that $|H_2| = 0$, and so we cannot apply result 2 of theorem 11.9. Therefore we check for (weak) concavity. $|H_1^*| = f_{11}, f_{22}$, both of which are negative. $|H_2^*| = |H_2| \geq 0$ (because $|H_2| = 0$). Therefore f is concave. ■

The following example illustrates that the conditions for weak concavity or convexity are not determined simply by relaxing the strict inequalities to weak inequalities only on the *leading* principal minors. All minor determinants must be evaluated when checking for (weak) convexity or (weak) concavity.

Example 11.29 Use theorem 11.9 to determine the convexity/concavity property of the function $y = f(x_1, x_2) = 3x_1 + x_2^2$ defined on $\mathbb{R}^2$.

Solution

The first- and second-order partial derivatives of this function are

$$f_1 = 3, \quad f_2 = 2x_2$$
$$f_{11} = 0, \quad f_{12} = f_{21} = 0, \quad f_{22} = 2$$

To check for strict concavity/convexity we determine the signs for the leading principal minors of H

$$|H_1| = f_{11} = 0, \quad |H_2| = f_{11}f_{22} - f_{12}^2 = 0$$

and so neither the condition for strict concavity nor strict convexity is satisfied.

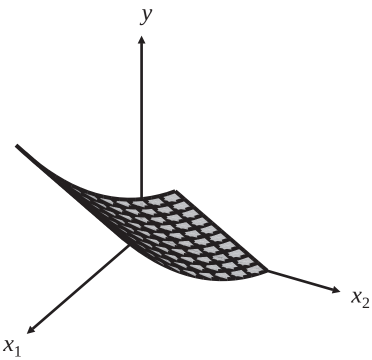


Figure 11.21 Graph of $y = 3x_1 + x_2^2$ in example 11.29

It is an error to conclude that since the series $|H_k|$ in this example satisfies trivially both $|H_k| \leq 0$ and $|H_k| \geq 0$, $k = 1, 2$, that f is both weakly convex and weakly concave. The only function for which this is true is the linear function, and the graph of f in figure 11.21 clearly indicates that this is not the case. The appropriate condition to check for weak convexity or weak concavity involves checking the sign of both $|H_1^*|$ values, f_{11} and f_{22}:

$$|H_1^*| = \begin{cases} f_{11} = 0 \\ f_{22} > 0 \end{cases}$$

and so $|H_1^*| \geq 0$. $|H_2^*| = f_{11}f_{22} - f_{12}^2 = 0$ and so $|H_2^*| \geq 0$. Therefore, the function is convex. (Note that since one of the $|H_1^*|$ values is positive, this function is not also concave.) ■

Example 11.30 Use theorem 11.9 to determine the convexity/concavity property of the function

$$y = f(x_1, x_2, x_3) = x_1^{\alpha} + x_2^{\beta} + x_3^{\gamma}, \quad \mathbf{x} \in \mathbb{R}^3_{++}, 0 < \alpha, \beta, \gamma < 1$$

Solution

The first-order partials of this function are

$$f_1 = \alpha x_1^{\alpha-1}, \quad f_2 = \beta x_2^{\beta-1}, \quad f_3 = \gamma x_3^{\gamma-1}$$

Since all second-order cross-partials are zero, the second-order partial derivatives are

$$f_{11} = (\alpha - 1)\alpha x_1^{\alpha-2} < 0 \qquad \text{since } \alpha - 1 < 0$$

$$f_{22} = (\beta - 1)\beta x_2^{\beta-2} < 0 \qquad \text{since } \beta - 1 < 0$$

$$f_{33} = (\gamma - 1)\gamma x_3^{\gamma-2} < 0 \qquad \text{since } \gamma - 1 < 0$$

$$f_{ij} = 0, \quad i \neq j$$

As a result we get the following value of $|H|$:

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$$= \begin{vmatrix} (\alpha - 1)\alpha x_1^{\alpha-2} & 0 & 0 \\ 0 & (\beta - 1)\beta x_2^{\beta-2} & 0 \\ 0 & 0 & (\gamma - 1)\gamma x_3^{\gamma-2} \end{vmatrix}$$

And so we get

$$|H_1| = (\alpha - 1)\alpha x_1^{\alpha-2} < 0$$

$$|H_2| = \left[(\alpha - 1)\alpha x_1^{\alpha-2}\right]\left[(\beta - 1)\beta x_2^{\beta-2}\right] > 0$$

$$|H_3| = \left[(\alpha - 1)\alpha x_1^{\alpha-2}\right]\left[(\beta - 1)\beta x_2^{\beta-2}\right]\left[(\gamma - 1)\gamma x_3^{\gamma-2}\right] < 0$$

which implies that the function is strictly concave. ■

This example illustrates an interesting general point regarding the relationship between conditions for a function of one variable to be convex (concave) and the conditions for a function of more than one variable to be convex (concave). For instance, if for $y = f(x), x \in \mathbb{R}$, we have that $f''(x) \leq 0$ (or $d^2y \leq 0$), then the function is concave. If we have a function of many variables $y = f(x_1, x_2, \ldots, x_n)$ which has the property that all cross-partials are zero, then such a function is concave if and only if every second-order partial f_{ii} is negative or zero. In particular, additively separable functions which can be written as

$$f(x_1, x_2, \ldots, x_n) = g^1(x_1) + g^2(x_2) + \cdots + g^n(x_n)$$

have this property (see definition 11.2). We prove this in the following theorem:

Theorem 11.10 An additively separable function $y = f(\mathbf{x})$ is concave *if and only if* $f_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$.

Proof

We need to show that $d^2y = \mathbf{dx}^T H\,\mathbf{dx} \leq 0$ for all vectors $\mathbf{dx}$ if and only if $f_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$. First, note that since $f_{ij} = 0$ for $i \neq j$, we can write d^2y as

$$d^2y = \mathbf{dx}^T H\,\mathbf{dx} = \sum_{j=1}^{n}\sum_{i=1}^{n} f_{ij}\,dx_i\,dx_j$$
$$= \sum_{i=1}^{n} f_{ii}\,dx_i^2$$

We first show that if $f_{ii} \leq 0$ for all i, then $d^2y \leq 0$. Since $dx_i^2 \geq 0$ for any dx_i and $f_{ii} \leq 0$ for all i, then every term in d^2y is negative or zero, which proves the sufficiency part of the theorem (i.e., if all $f_{ii} \leq 0$ then $d^2y \leq 0$).

We now need to show that the claim that $d^2y \leq 0$ for any vector $\mathbf{dx}$ requires that $f_{ii} \leq 0$ for all i. First, suppose the contrary; that is, suppose that one of the $f_{ii} > 0$. Without loss of generality, suppose that $f_{11} > 0$. Then choose a vector $\mathbf{dx}$ with $dx_1 \neq 0$ and $dx_j = 0$ for $j = 2, 3, \ldots, n$. The result is that

$$d^2y = f_{11}\,dx_1^2 > 0$$

which is a contradiction. This proves the necessity part of the theorem (i.e., that $d^2y \leq 0$ only if all $f_{ii} \leq 0$). ■

Clearly, we can establish similar theorems for the cases of (weak) concavity and (strict and weak) convexity. Some of these results are left as problems in the exercises. The intuitive point brought out by theorem 11.10 is that if a function of many variables has zero cross-partial derivatives, then it will be concave if and only if it is concave in each direction x_i (i.e., with respect to each variable x_i). For example, the function $f(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$, $x_1 > 0$ and $x_2 > 0$, is strictly concave on $\mathbb{R}^2_{++}$ with respect to each variable x_i $(f_{11}, f_{22} < 0)$, as illustrated in figure 11.22, while the function $f(x_1, x_2) = x_1^{1/2} + x_2^2$ is strictly concave on $\mathbb{R}^2_{++}$ with respect to x_1 (i.e., $f_{11} < 0$) but strictly convex with respect to x_2(i.e., $f_{22} > 0$) and so is neither concave nor convex, as illustrated in figure 11.23.

If the cross-partial derivatives are nonzero, the necessity part of theorem 11.10 continues to hold:

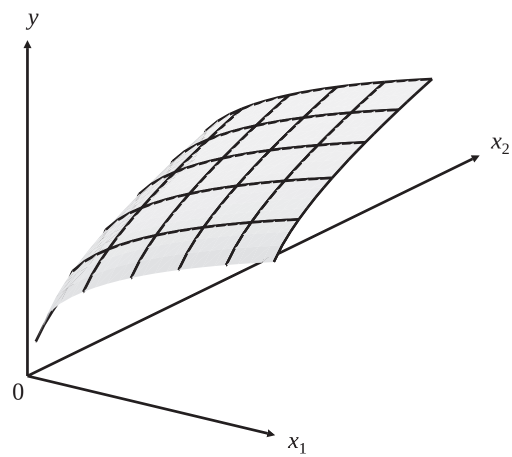


Figure 11.22 Graph of $y = x_1^{1/2} + x_2^{1/2}$

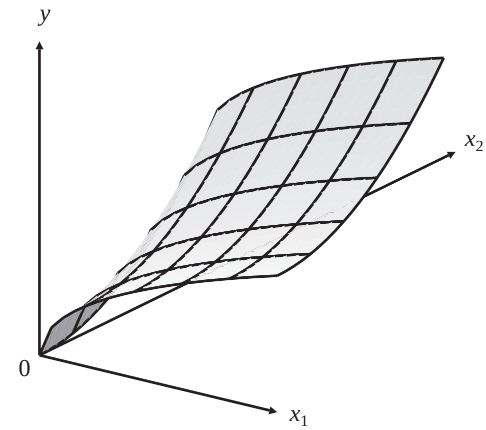


Figure 11.23 Graph of $y = x_1^{1/2} + x_2^2$

Theorem 11.11 If a function $y = f(\mathbf{x})$ is concave, then it must be the case that $f_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$.

Proof

We need to prove that if

$$d^2y = \mathbf{dx}^T H\, \mathbf{dx} = \sum_{j=1}^{n} \sum_{i=1}^{n} f_{ij}\, dx_i\, dx_j \leq 0$$

for all vectors **dx** (except for the zero vector), then it must be the case that each second-order partial f_{ij} with $i = j$ (i.e., each f_{ii}) must be ≤ 0. Without loss of generality, consider in particular the case $i = 1$. Choose the vector **dx** with $dx_1 \neq 0$ but $dx_j = 0$ for all $j \neq 1$. The expression above for d^2y then becomes

$$d^2y = f_{11}\, dx_1^2$$

and $d^2y \leq 0$ clearly requires that $f_{11} \leq 0$. This proves the theorem. ■

We close this section by illustrating the important result that if the exponents on the inputs of a Cobb-Douglas production function sum to a number less than one, then the function is strictly concave. Although we show this for the case of two inputs, it holds for any number of inputs.

Example 11.31 Show that the Cobb-Douglas production function

$$y = f(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$

with $0 < \alpha, \beta < 1$, and $\alpha + \beta < 1$ is strictly concave on $x_1, x_2 > 0$.

Solution

The second-order partial derivatives are

$$f_{11} = \alpha(\alpha - 1)Ax_1^{\alpha-2}x_2^{\beta}$$

$$f_{12} = \alpha\beta Ax_1^{\alpha-1}x_2^{\beta-1}$$

$$f_{22} = \beta(\beta - 1)Ax_1^{\alpha}x_2^{\beta-2}$$

Since $0 < \alpha < 1$, we have $\alpha - 1 < 0$ and so $|H_1| = f_{11} < 0$. Now we need to show that $|H_2| = f_{11}f_{22} - f_{12}^2 > 0$. That is, we need to show that

$$f_{11}f_{22} > f_{12}^2$$

Thus, making the substitutions for the second-order partial derivatives, we need to show that

$$\left[\alpha(\alpha - 1)Ax_1^{\alpha-2}x_2^{\beta}\right]\left[\beta(\beta - 1)Ax_1^{\alpha}x_2^{\beta-2}\right] > \left[\alpha\beta Ax_1^{\alpha-1}x_2^{\beta-1}\right]^2$$

The following steps demonstrate that for $\alpha + \beta < 1$ the inequality above does indeed hold:

$$\begin{aligned}
\alpha\beta(\alpha - 1)(\beta - 1)A^2x_1^{2\alpha-2}x_2^{2\beta-2} &> \alpha^2\beta^2A^2x_1^{2\alpha-2}x_2^{2\beta-2} \\
(\alpha - 1)(\beta - 1) &> \alpha\beta \\
\alpha\beta - \alpha - \beta + 1 &> \alpha\beta \\
-\alpha - \beta &> -1 \\
\alpha + \beta &< 1
\end{aligned}$$

■

We have shown how the total differential can be used as a test to determine whether a function is either concave or convex (or neither). We have demonstrated two methods of doing so: (i) direct determination of the sign of the second order total differential, d^2y, as in example 11.27; (ii) using the results which relate quadratic forms to positive or negative (semi-) definite matrices to determine the sign of the second-order total differential, d^2y, as in example 11.30.

EXERCISES

1. Use theorem 11.6 to show that the function $f(x_1, x_2) = (x_1 + x_2)^2$ is convex (see example 11.27).

2. Use theorem 11.5 to show that the function $f(x_1, x_2) = 10 - x_1^2 - x_2^2$ is strictly concave (see example 11.27).

3. Use theorem 11.9 to show that the function $f(x_1, x_2) = x_1^{1/2} x_2^{1/3}$, defined on $\mathbb{R}^2_{++}$, is strictly concave.

4. Use theorem 11.9 to show that the function $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$, defined on $\mathbb{R}^2_{++}$, is concave.

5. Use theorem 11.7 to show that the function $y = (x_1 + x_2)^{1/2}$, defined on $\mathbb{R}^2_{++}$, is a concave function. Show that there are linear segments on the surface of this function and so the function is not strictly concave (see example 11.27).

6. *Show that an additively separable function $y = f(x_1, x_2, \ldots, x_n)$ is convex if and only if $f_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$ (see theorem 11.10).

7. *Show that an additively separable function $y = f(x_1, x_2, \ldots, x_n)$ is strictly concave if $f_{ii} < 0$ for all $i = 1, 2, \ldots, n$ (see theorem 11.10).

8. *Show that if a function $y = f(x_1, x_2, \ldots, x_n)$ is convex, then it must be the case that $f_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$ (see theorem 11.11).

11.5 More Properties of Functions with Economic Applications

We have developed some important properties of functions, such as concavity and convexity, in earlier sections, and we have applied them to a number of economic relationships. In this section we add to these results as preparation for some of the arguments used in chapters 12 and 13.

Concavity/Convexity and Quasiconcavity/Quasiconvexity

In section 11.3 we developed the important proposition that a function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, which is increasing in each of its arguments, has strictly convex to the origin level curves if the following condition is satisfied:

$$f_2^{-3}\left\{f_{11} f_2^2 - 2 f_1 f_2 f_{12} + f_{22} f_1^2\right\} < 0 \tag{11.9}$$

A function satisfying this condition is quasiconcave, a term which was defined in chapter 2. (The material in section 2.4 on concavity, convexity, quasiconcavity, and quasiconvexity should be reviewed before proceeding with the material in this section.) All functions that are concave are also quasiconcave, but the reverse is not true. The first part of this relationship is illustrated in the following example.

Example 11.32 Show that if a function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, with $f_1 > 0$ and $f_2 > 0$, satisfies the conditions of theorem 11.9 for a strictly concave function, then it is also quasiconcave.

Solution

From theorem 11.9 the following conditions imply that a function is strictly concave

$$d^2y = \mathbf{dx}^T H\, \mathbf{dx} < 0 \qquad \text{for any vector } \mathbf{dx}$$

We need to show that this condition, in conjunction with $f_1 > 0$ and $f_2 > 0$, implies that f also satisfies equation (11.9).

Since f is strictly concave, then $d^2y < 0$ for any vector $\mathbf{dx}$ (i.e., $\mathbf{dx}^T$) and it follows that $d^2y < 0$ for the particular choice $\mathbf{dx}^T = [f_2 \quad -f_1]$. Substitution of this vector into the expression above for d^2y gives

$$[f_2 \quad -f_1] \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix} < 0$$

which implies that

$$[(f_{11}f_2 - f_1 f_{12}) \quad (f_2 f_{12} - f_{22} f_1)] \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix} < 0$$

Therefore

$$f_2(f_{11}f_2 - f_1 f_{12}) - f_1(f_2 f_{12} - f_1 f_{22}) < 0$$

implying that

$$f_{11}f_2^2 - 2f_1 f_2 f_{12} + f_{22} f_1^2 < 0$$

Since $f_2 > 0$, this shows that the function is also quasiconcave. (See equation 11.9.)

■

Figures 11.24 and 11.25, which are graphs of the functions $y = x_1^{1/4} x_2^{1/2}$ and $y = x_1 x_2^2$ on $\mathbf{x} \in \mathbb{R}_+^2$, along with representative contours, illustrate the difference between concavity and quasiconcavity. The shape of the contours indicates that the level sets of these functions are strictly convex to the origin and so both functions are quasiconcave. It is clear in figure 11.25, however, that the function $y = x_1 x_2^2$ is not concave. It is especially illuminating to cut through this function for a given value of x_2, as was done in figure 11.9, to see how the function actually has a *convex* shape in the x_1 direction.

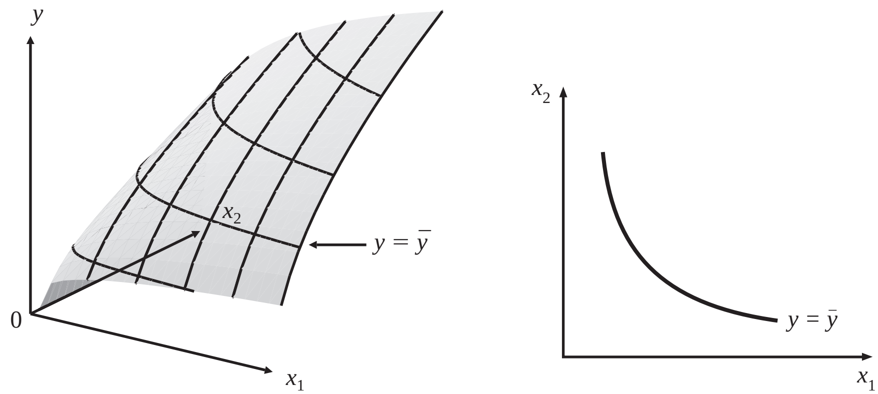


Figure 11.24 Graph of $y = x_1^{1/4} x_2^{1/2}$ and a representative level curve

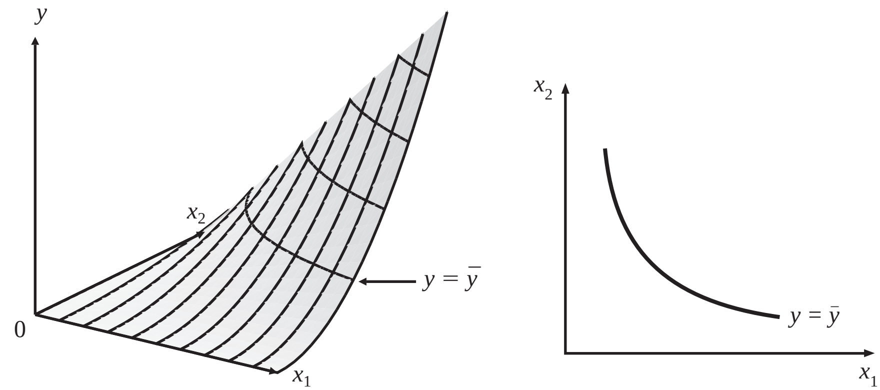


Figure 11.25 Graph of $y = x_1 x_2^2$ and a representative level curve

Sufficient conditions for quasiconcavity and quasiconvexity are given in theorem 11.12 and the various relations between concave (convex) and quasiconcave (quasiconvex) functions are summarized in theorem 11.13. First, we need to define the bordered Hessian, $\bar{H}$, for the function f.

Definition 11.6 Suppose that the function f defined on $\mathbb{R}^n$ has continuous first- and second-order partial derivatives. The **bordered Hessian** of the function f is

$$\bar{H} = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}$$

Notice that the bordered Hessian is formed by taking the Hessian matrix and adding $[0\ \ f_1\ \ f_2\ \ \cdots\ \ f_n]$ as a first column and a first row (i.e., the first derivatives of the function preceded by a zero).

We refer to the matrix composed of the first $k+1$ row and column elements (leading principal minors) as

$$\bar{H}_k = \begin{bmatrix} 0 & f_1 & f_2 & \cdots & f_k \\ f_1 & f_{11} & f_{12} & \cdots & f_{1k} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k & f_{k1} & f_{k2} & \cdots & f_{kk} \end{bmatrix}$$

implying that

$$|\bar{H}_1| = \begin{vmatrix} 0 & f_1 \\ f_1 & f_{11} \end{vmatrix}, \quad |\bar{H}_2| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix}$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & f_{11} & f_{12} & f_{13} \\ f_2 & f_{21} & f_{22} & f_{23} \\ f_3 & f_{31} & f_{32} & f_{33} \end{vmatrix}$$

and so on. Notice that $|\bar{H}_1| = -f_1^2$ and so must be nonpositive. For this reason no mention is made concerning the sign of $|\bar{H}_1|$ in the following theorem, which provides sufficient conditions for a function to be quasiconcave or quasiconvex:

Theorem 11.12 Suppose that f is a function defined on $\mathbb{R}^n$ and that f has continuous first- and second-order partial derivatives. Let $\bar{H}$ represent the bordered Hessian of f.

(i) If $|\bar{H}_2| > 0$, $|\bar{H}_3| < 0, \ldots, |\bar{H}_n| = |\bar{H}| > 0$ (n even), < 0 (n odd) for all $\mathbf{x} \in \mathbb{R}^n_+$, then f is quasiconcave.

(ii) If $|\bar{H}_2| < 0$, $|\bar{H}_3| < 0, \ldots, |\bar{H}_n| = |\bar{H}| < 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$, then f is quasiconvex.

Theorem 11.13 (i) Any convex function is quasiconvex, although the reverse does not necessarily hold.

(ii) Any concave function is quasiconcave, although the reverse does not necessarily hold.

Example 11.33 Use theorem 11.12 to show that the function $f(x_1, x_2) = x_1 x_2^2$ defined on $\mathbb{R}^2_+$ is quasiconcave.

Solution

Since the first leading principal minor in this case is $\bar{H}_2 = \bar{H}$, we only have this determinant to evaluate

$$
\begin{aligned}
|\bar{H}_2| = |\bar{H}| &= \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 0 & x_2^2 & 2x_1x_2 \\ x_2^2 & 0 & 2x_2 \\ 2x_1x_2 & 2x_2 & 2x_1 \end{vmatrix} \\
&= 0 \begin{vmatrix} 0 & 2x_2 \\ 2x_2 & 2x_1 \end{vmatrix} - x_2^2 \begin{vmatrix} x_2^2 & 2x_2 \\ 2x_1x_2 & 2x_1 \end{vmatrix} \\
&\quad + 2x_1x_2 \begin{vmatrix} x_2^2 & 0 \\ 2x_1x_2 & 2x_2 \end{vmatrix} \\
&= -x_2^2\left(2x_1x_2^2 - 4x_1x_2^2\right) + 2x_1x_2\left(x_2^2 2x_2\right) \\
&= 6x_1x_2^4 > 0 \qquad \text{for } x_1, x_2 > 0
\end{aligned}
$$

Thus f satisfies the conditions for quasiconcavity. ■

Example 11.34 Show that for an increasing function f defined on $\mathbb{R}^2_+$, with continuous first- and second-order partial derivatives, the condition for quasiconcavity given by equation (11.9) is consistent with that given in theorem 11.12.

Solution

A function f defined on $\mathbb{R}^2_+$ is quasiconcave according to theorem 11.12 if

$$|\bar{H}_2| = |\bar{H}| = \begin{vmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{vmatrix} > 0$$

which implies that

$$\begin{aligned} 0\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} - f_1\begin{vmatrix} f_1 & f_{12} \\ f_2 & f_{22} \end{vmatrix} + f_2\begin{vmatrix} f_1 & f_{11} \\ f_2 & f_{21} \end{vmatrix} \\ = -f_1(f_1 f_{22} - f_2 f_{12}) + f_2(f_1 f_{21} - f_2 f_{11}) \\ = -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11} > 0 \end{aligned}$$

which, upon multiplying throughout by -1 and rearranging, implies that

$$f_{11} f_2^2 - 2f_1 f_2 f_{12} + f_{22} f_1^2 < 0$$

If $f_2 > 0$, then this condition is equivalent to the condition given by equation (11.9). ■

The Cobb-Douglas function, defined on $\mathbb{R}^n_+$,

$$f(x_1, x_2, \ldots, x_n) = Ax_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad A > 0,\ \alpha_i > 0,\ \forall i$$

is quasiconcave for the stated conditions but is concave only if $\sum \alpha_i \leq 1$ and strictly concave only if $\sum \alpha_i < 1$. We effectively showed this to be the case for $n = 2$ through example 11.31.

Homogeneous Functions

When we study the theory of the firm in a competitive setting, we impose the assumption of concavity (rather than quasiconcavity) on the firm's production function. Concavity rules out the possibility of increasing returns to scale (IRS)—a phenomenon inconsistent with a market composed of *many* small firms, because under increasing returns to scale it is more efficient for firms to merge or increase in size than to remain "small." If we restrict ourselves to production functions which are homogeneous then the analysis of returns to scale is substantially simplified.

Definition 11.7 A function f defined on $\mathbb{R}^n$ is **homogeneous of degree** k if

$$f(sx_1, sx_2, \ldots, sx_n) = s^k f(x_1, x_2, \ldots, x_n)$$

If f is a production function which is homogeneous of degree k, then multiplying the level of all inputs by the same factor s will increase output by the factor s^k. For example, if $f(x_1, x_2) = x_1x_2^2$ is the production function, we can see that it is homogeneous of degree 3, since

$$\begin{aligned} f(sx_1, sx_2) &= (sx_1)(sx_2)^2 = sx_1s^2x_2^2 = s^3x_1x_2^2 \\ &= s^3 f(x_1, x_2) \end{aligned}$$

In particular, if we were to double all input levels ($s = 2$), we would increase output by a factor of 8 ($2^3 = 8$). This is an example of increasing returns to scale. Performing the same analysis for the function $f(x_1, x_2) = x_1^{1/4}x_2^{1/2}$, we see that this function is homogeneous of degree 3/4. In this case, if we were to double all input levels we would increase output by a factor of $2^{3/4} \doteq 1.68$. This is a case of decreasing returns to scale. Upon doubling inputs for the production function $y = x_1^{1/2}x_2^{1/2}$, we see that output doubles (see figure 11.26). This is a case of constant returns to scale.

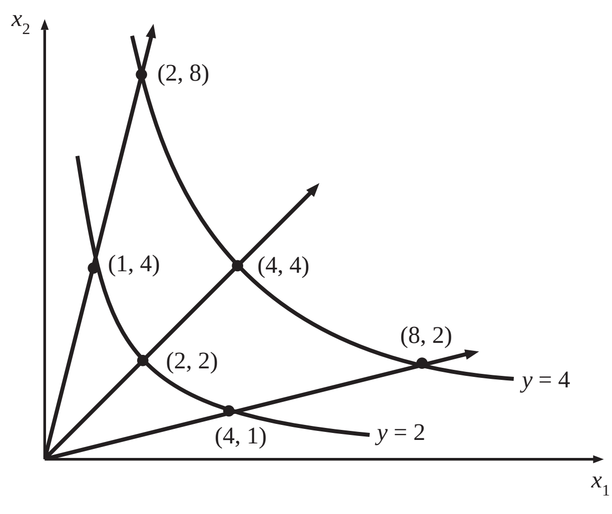


Figure 11.26 Isoquants for the production function $y = x_1^{1/2}x_2^{1/2}$

To see how homogeneity is a restriction that all functions do not satisfy, consider the following example of a nonhomogeneous function and the accompanying graph of a few of its representative isoquants.

Example 11.35 Show that the production function $y = f(x_1, x_2) = x_1^{1/2}x_2^{1/3} + x_2^{3/2}$ is not homogeneous.

Solution

We have

$$
\begin{aligned}
f(sx_1, sx_2) &= (sx_1)^{1/2}(sx_2)^{1/3} + (sx_2)^{3/2} \\
&= s^{5/6}x_1^{1/2}x_2^{1/3} + s^{3/2}x_2^{3/2}
\end{aligned}
$$

This function cannot be written in the form

$$
f(sx_1, sx_2) = s^k f(x_1, x_2) = s^k\left(x_1^{1/2}x_2^{1/3} + x_2^{3/2}\right)
$$

and so it is not homogeneous, and its returns-to-scale properties depend on the values of x_1 and x_2. See figure 11.27 where it is shown that beginning with input bundle (16, 1) and doubling each input leads to an output level which is *less* than double while beginning with input bundle (1, 2.38) and doubling each input leads to an output which is *more* than double.

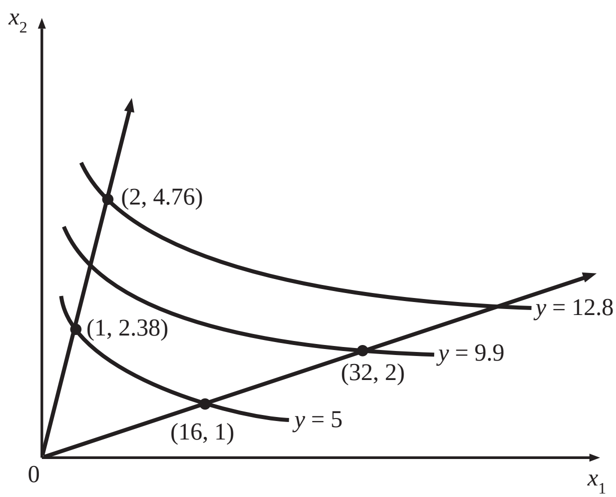


Figure 11.27 Isoquants for the production function $y = x_1^{1/2}x_2^{1/3} + x_2^{3/2}$ ■

From the preceding examples, the following theorem should be clear:

Theorem 11.14 Suppose the production function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n_+$, is homogeneous of degree k. That is,

$$f(s\mathbf{x}) = s^k f(\mathbf{x})$$

This production function displays:

(i) Increasing returns to scale if $k > 1$
(ii) Constant returns to scale if $k = 1$
(iii) Decreasing returns to scale if $k < 1$

Euler's Theorem

The following theorem, referred to as **Euler's theorem,** leads to an interesting result for the special case of a production function which is homogeneous of degree $k = 1$. First, we present the general case for any value of k.

Theorem 11.15 **(Euler's theorem)** If $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n_+$, is homogeneous of degree k, then the following condition holds:

$$f_1 x_1 + f_2 x_2 + \cdots + f_n x_n = kf(x_1, x_2, \ldots, x_n)$$

Proof

Since f is homogeneous of degree k, we have

$$f(sx_1, sx_2, \ldots, sx_n) = s^k f(x_1, x_2, \ldots, x_n)$$

Writing each sx_i as z_i and differentiating both sides with respect to s gives

$$\frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial s} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial s} + \cdots + \frac{\partial f}{\partial z_n}\frac{\partial z_n}{\partial s} = ks^{k-1} f(x_1, x_2, \ldots, x_n)$$

which implies that

$$f_1 x_1 + f_2 x_2 + \cdots + f_n x_n = ks^{k-1} f(x_1, x_2, \ldots, x_n)$$

Since this condition holds for any $s > 0$, it holds for $s = 1$ which implies that

$$f_1x_1 + f_2x_2 + \cdots + f_nx_n = kf(x_1, x_2, \ldots, x_n) \tag{11.10}$$

■

Euler's theorem states that if f is homogeneous of degree k, then multiplying the marginal product of each input i by the level of that input and summing ($\sum_{i=1}^{n} f_i x_i$) gives a value equal to k times the value of output. For the specific case of $k = 1$, we have that $\sum_{i=1}^{n} f_i x_i$ equals the value of output.

Example 11.36 Show that for $y = f(x_1, x_2) = x_1^{1/4}x_2^{3/4}$ it follows that $f_1x_1 + f_2x_2 = f(x_1, x_2)$.

Solution

$$f_1 = \frac{1}{4}x_1^{-3/4}x_2^{3/4}, \quad f_2 = \frac{3}{4}x_1^{1/4}x_2^{-1/4}$$

and so

$$\begin{aligned} f_1x_1 + f_2x_2 &= \left(\frac{1}{4}x_1^{-3/4}x_2^{3/4}\right)x_1 + \left(\frac{3}{4}x_1^{1/4}x_2^{-1/4}\right)x_2 \\ &= \frac{1}{4}x_1^{1/4}x_2^{3/4} + \frac{3}{4}x_1^{1/4}x_2^{3/4} = x_1^{1/4}x_2^{3/4} = f(x_1, x_2) \end{aligned}$$

■

Elasticity of Substitution

The elasticity of substitution between inputs, σ, is defined as

$$\sigma = \frac{\text{proportionate rate of change of the input ratio}}{\text{proportionate rate of change of the MRTS}}$$

when MRTS, the marginal rate of technical substitution, is the slope of the isoquant. In figure 11.28 we have drawn two level curves (isoquants) for two different production functions f and g. The proportionate rate of change in the input ratio (i.e., the numerator of σ) in moving from point x^0 to $\bar{x}$ is the same for either isoquant f or g. However, the proportionate rate of change of the slopes of the isoquants—the MRTS (i.e., the denominator of σ)—is less for g than f. Thus g displays greater elasticity of substitution. To make such comparisons mathematically, we use the following definition:

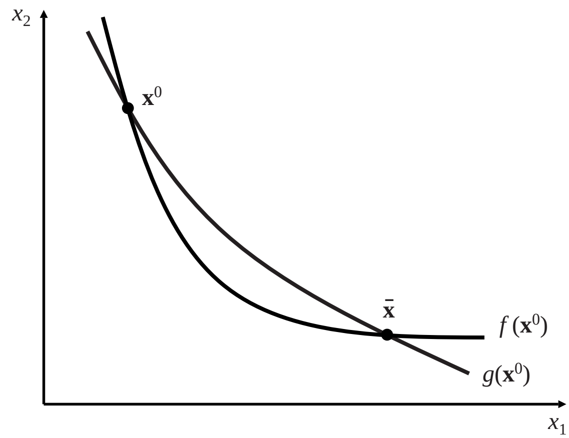


Figure 11.28 Production function associated with level curve g showing a greater elasticity of substitution than that associated with level curve f

Definition 11.8 The **elasticity of substitution** between inputs for a production function $y = f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2_+$ which has continuous marginal product functions is defined as

$$\sigma = \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)}$$

The constant elasticity of substitution (CES) production function has the property, as its name suggests, that the value of σ is the same at any point on any isoquant. The Cobb-Douglas production function has this property, with $\sigma = 1$.

Example 11.37 Show that the Cobb-Douglas production function $y = Ax_1^{\alpha}x_2^{\beta}$, $A, \alpha, \beta > 0$ defined on $\mathbb{R}^2_{++}$ has $\sigma = 1$ everywhere.

Solution

First we find the partial derivatives f_1 and f_2:

$$f_1 = \alpha A x_1^{\alpha-1}x_2^{\beta}, \quad f_2 = \beta A x_1^{\alpha}x_2^{\beta-1}$$

and so

$$\frac{f_1}{f_2} = \frac{\alpha x_2}{\beta x_1} \quad \text{or} \quad \frac{x_2}{x_1} = \frac{\beta}{\alpha}\frac{f_1}{f_2}$$

This implies that

$$\ln\left(\frac{x_2}{x_1}\right) = \ln\left(\frac{f_1}{f_2}\right) + \ln\left(\frac{\beta}{\alpha}\right)$$

Since $\ln(\beta/\alpha)$ is a constant, we see that

$$\sigma = \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)} = 1$$

■

Example 11.38 Find the elasticity of substitution for the CES production function $y = f(L, K) = [\delta L^{-r} + (1-\delta)K^{-r}]^{-1/r}$, $0 < \delta < 1$, $r > -1$ where inputs $L, K > 0$ refer to labor and capital respectively.

Solution

We have

$$f_L = -\frac{1}{r}[\delta L^{-r} + (1-\delta)K^{-r}]^{-1/r-1}(-r\delta L^{-r-1})$$

$$f_K = -\frac{1}{r}[\delta L^{-r} + (1-\delta)K^{-r}]^{-1/r-1}(-r(1-\delta)K^{-r-1})$$

which implies that

$$\text{MRTS} = \frac{f_L}{f_K} = \frac{\delta}{1-\delta}\left(\frac{K}{L}\right)^{r+1}$$

Taking logs gives

$$\ln\left(\frac{f_L}{f_K}\right) = \ln\left(\frac{\delta}{1-\delta}\right) + (r+1)\ln\left(\frac{K}{L}\right)$$

or

$$\ln\left(\frac{K}{L}\right) = \frac{1}{1+r}\ln\left(\frac{f_L}{f_K}\right) - \frac{1}{1+r}\ln\left(\frac{\delta}{1-\delta}\right)$$

Since the last term is just a constant, we have

$$\sigma = \frac{d[\ln(K/L)]}{d[\ln(f_L/f_K)]} = \frac{1}{1+r}$$

■

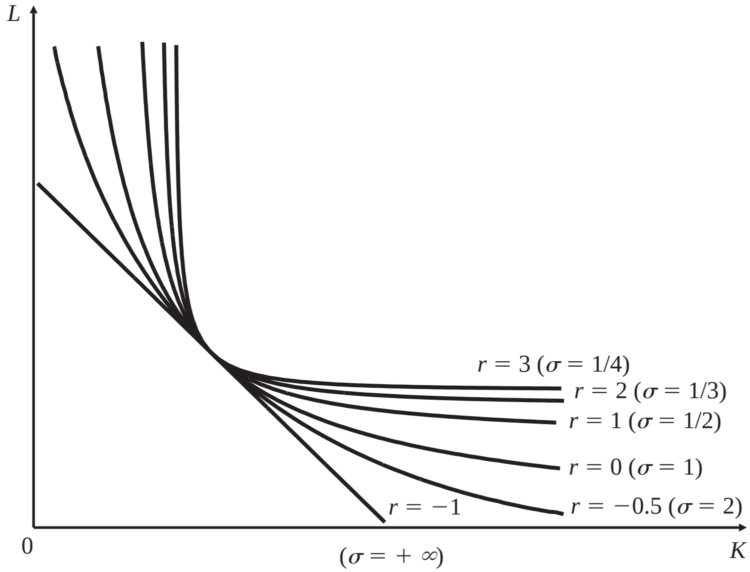


Figure 11.29 Isoquants for the CES production function $y = [0.5L^{-r} + 0.5K^{-r}]^{-1/r}$

The smaller r is in example 11.38, the greater is σ. This is illustrated in figure 11.29. The following special cases for the value of r are of interest:

Case 1 $r \to +\infty$, which implies that $\sigma \to 0$. Here the MRTS approaches zero if $L > K$ and $+\infty$ if $K > L$. This means that isoquants become a right angle and it is not possible to substitute between inputs.

Case 2 $r = 0$, which implies that $\sigma = 1$. We cannot use the formula for the CES in this case because $-1/r$ is undefined at $r = 0$. However, we know that $\sigma = 1$ corresponds to the Cobb-Douglas case and it can be shown that the Cobb-Douglas is a limiting case of the CES as $r \to 0$.

Case 3 $r \to -1$ (from the right since $r > -1$), which implies that $\sigma \to +\infty$. This is the case of perfect substitutability and the isoquants are linear.

Some of these issues are taken up in section 13.1 on cost minimization with the CES production function.

In these examples it is easy to write $\ln(x_2/x_1)$ as a function of $\ln(f_1/f_2)$, and so we could apply the formula given in definition 11.8. Since this is not always the case, an alternative and equivalent formula may be used

$$\sigma = \frac{f_1 f_2(f_1 x_1 + f_2 x_2)}{x_1 x_2 |\bar{H}|} \tag{11.11}$$

where $|\bar{H}| = -f_1^2 f_{22} + 2 f_1 f_2 f_{12} - f_2^2 f_{11}$ is the determinant of the bordered Hessian for f.

EXERCISES

1. Show that the following functions defined on $\mathbb{R}^2_{++}$ are quasiconcave. Which are also concave?

(a) $f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$

(b) $f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$

(c) $f(x_1, x_2) = x_1^2 x_2^3$

2. Show that the following functions defined on $\mathbb{R}^2_{++}$ are quasiconvex. Which are also convex?

(a) $f(x_1, x_2) = x_1^2 + x_2^2$

(b) $f(x_1, x_2) = 3x_1^4 + 5x_2^2$

(c) $f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 x_2^3$

3. Show that:

(a) $y = x_1^{1/4} x_2^{1/3} x_3^{1/4}$ satisfies the conditions of quasiconcavity and strict concavity on $\mathbb{R}^3_{++}$.

(b) $y = x_1^{1/2} x_2^{1/3} x_3^{1/4}$ satisfies the conditions of quasiconcavity but not strict concavity on $\mathbb{R}^3_{++}$.

4. Show that the production function $y = x_1^{1/2} x_2^{2/3}$ is homogeneous of degree $7/6$.

5. For the Cobb-Douglas production function $f(x_1, x_2) = A x_1^{\alpha} x_2^{\beta}$, $A > 0$, α, $\beta > 0$ defined on $\mathbb{R}^2_{++}$, show that Euler's theorem applies so that

$$f_1 x_1 + f_2 x_2 = k f(x_1, x_2)$$

where k is the degree of homogeneity of f.

6. Show that using the formula given by equation (11.11) to find the elasticity of substitution, σ, for the Cobb-Douglas production function yields the same result as was found in example 11.39.

11.6 Taylor Series Expansion*

In section 5.6 we presented the Taylor series expansion for functions of one variable. In this section we do the same for functions of n variables. Since it is the **remainder formula** that is most useful in economic analysis, we focus on this particular form of the Taylor series expansion. We first review it for the one variable case. Then we illustrate the formula for two variables and finally present the general case for n variables.

The **Taylor series** expansion for functions of one variable, $y = f(x)$, $x \in \mathbb{R}$, is given in definition 5.11. We present below the same expressions as in section 5.6 except with some notational changes. These changes are introduced because in this chapter we have used n to represent the number of variables of a function, while in section 5.6 we used n to represent the number of terms in the formula for the Taylor series expansion.

The following expression is the remainder formula of the Taylor series expansion for functions of one variable taken to the pth term for the remainder:

$$f(\hat{x}) = f(x^0) + \sum_{k=1}^{p-1} \left[\frac{f^{(k)}(x^0)(\hat{x} - x^0)^k}{k!} \right] + R_p$$

with

$$R_p = \frac{f^{(p)}(\xi)(\hat{x} - x^0)^p}{p!}$$

The point $x = \xi$ lies between x^0 and $\hat{x}$ and the function is assumed to possess derivatives to the pth order.

The particular case of $p = 2$ proves to be the most useful, and so we focus on it. It is also helpful to note that since $\hat{x} - x^0$ is just some change in x, we can set $dx = \hat{x} - x^0$. Therefore we get the following expression for the case $p = 2$:

$$f(\hat{x}) = f(x^0) + f'(x^0)\, dx + \frac{f''(\xi)\, dx^2}{2} \tag{11.12}$$

Notice in equation (11.12) that $f'(x^0)\, dx$ is just dy evaluated at $x = x^0$, while $f''(\xi)\, dx^2/2$ is just $d^2y/2$ evaluated at the point ξ. Therefore, remembering that total differentials are functions of x, and noting that $\Delta y = f(\hat{x}) - f(x^0)$, we can rewrite equation (11.12) in the following convenient manner:

$$\Delta y = dy\,(x^0) + \frac{d^2 y(\xi)}{2} \tag{11.13}$$

Equation (11.13) emphasizes the fact that the first-order total differential is an estimate of the amount by which a function changes given some change in x (see section 11.3). This estimate is made exact in the Taylor series formula by applying the remainder term, which if we choose $p = 2$, is simply one-half times the second-order total differential. The problem with applying this formula to obtain a precise value of Δy, however, is that the second-order total differential must be evaluated at some point ξ between x^0 and $\hat{x}$, and there is no straightforward procedure for finding this value precisely.

Nonetheless, the formula is still very useful. For example, if a function is strictly concave (everywhere), then we know that the second-order total differential is negative (everywhere) and so $d^2y(\xi) < 0$ for any ξ. This allows us to conclude that dy is always an overestimate of Δy for any strictly concave function. This conclusion follows directly from simple analysis of equation (11.13):

$$dy(x^0) - \Delta y = -\frac{1}{2}d^2y(\xi) > 0 \quad \text{if} \quad d^2y(\xi) < 0$$

The opposite result, that dy is always an underestimate of Δy for any strictly convex function follows in an analogous manner.

The great advantage of writing the Taylor series expansion in terms of total differentials is that it extends immediately to functions of n variables. Moreover the conclusion about dy being an overestimate (underestimate) of Δy for strictly concave (convex) functions also continues to hold in general. We expand each element of equation (11.13) for the case of $n = 2$ in order to illustrate what the Taylor series expansion looks like for functions of more than one variable. This exercise also points out the convenience of the total differential notation.

$$\Delta y = f(\hat{x}_1, \hat{x}_2) - f\left(x_1^0, x_2^0\right),$$

$$\mathbf{dx} = \begin{bmatrix} \left(\hat{x}_1 - x_1^0\right) \\ \left(\hat{x}_2 - x_2^0\right) \end{bmatrix}$$

and so for the first-order total differential, we get

$$dy(\mathbf{x}^0) = \begin{bmatrix} f_1\left(x_1^0, x_2^0\right) & f_2\left(x_1^0, x_2^0\right) \end{bmatrix} \begin{bmatrix} \left(\hat{x}_1 - x_1^0\right) \\ \left(\hat{x}_2 - x_2^0\right) \end{bmatrix}$$

For the second-order total differential, we get (using notation $H \equiv \nabla_2 F$)

$$\begin{aligned} d^2y(\xi_1, \xi_2) &= \mathbf{dx}^T H \, \mathbf{dx} \\ &= \begin{bmatrix} \left(\hat{x}_1 - x_1^0\right) & \left(\hat{x}_2 - x_2^0\right) \end{bmatrix} \begin{bmatrix} f_{11}(\xi_1, \xi_2) & f_{12}(\xi_1, \xi_2) \\ f_{21}(\xi_1, \xi_2) & f_{22}(\xi_1, \xi_2) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \left(\hat{x}_1 - x_1^0\right) \\ \left(\hat{x}_2 - x_2^0\right) \end{bmatrix} \end{aligned}$$

We summarize the results of this section with definition 11.9, two theorems, and an example. In chapter 12 we will see how to use these results to interpret second-order conditions of optimization problems.

Definition 11.9 The **remainder formula** for the Taylor series expansion for a function defined on $\mathbb{R}^n$, $y = f(\mathbf{x})$, expanded around the point $\mathbf{x}^0$ and taken to two terms is

$$f(\hat{\mathbf{x}}) = f(\mathbf{x}^0) + dy(\mathbf{x}^0) + \frac{1}{2}d^2y(\boldsymbol{\xi})$$

where $\boldsymbol{\xi}$ lies between $\mathbf{x}^0$ and $\hat{\mathbf{x}}$.

Theorem 11.16 If $y = f(\mathbf{x})$ is a strictly concave function, then using the first-order total differential to estimate the change in the function value caused by moving away from $\mathbf{x}^0$ to any other point $\hat{\mathbf{x}}$ always leads to an overestimate. That is, for $\Delta y \equiv f(\hat{\mathbf{x}}) - f(\mathbf{x}^0)$ we have that

$$\Delta y = dy(\mathbf{x}^0) + \frac{1}{2}d^2y(\xi) \Rightarrow \Delta y < dy(\mathbf{x}^0)$$

since $d^2y < 0$ for $y = f(\mathbf{x})$ strictly concave.

Theorem 11.17 If $y = f(\mathbf{x})$ is a strictly convex function, then using the first-order total differential to estimate the change in the function value caused by moving away from $\mathbf{x}^0$to any other point $\hat{\mathbf{x}}$ always leads to an underestimate. That is, for $\Delta y \equiv f(\hat{\mathbf{x}}) - f(\mathbf{x}^0)$ we have that

$$\Delta y = dy(\mathbf{x}^0) + \frac{1}{2}d^2y(\xi) \Rightarrow \Delta y > dy(\mathbf{x}^0)$$

since $d^2y > 0$ for $y = f(\mathbf{x})$ strictly convex.

Note that there are similar results to the above two propositions for (weakly) concave and (weakly) convex functions. The inequalities just need to be changed to be weak (i.e., $\geq$ and $\leq$ rather than $>$ and $<$).

Example 11.39 Here we illustrate the result that the total differential overestimates the actual change in the function value for the concave function $y = f(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$ on $x_1, x_2 > 0$.

It is easy to show that this function is strictly concave, and so theorem 11.16 applies. To illustrate, consider the initial point $\mathbf{x}^0 = (4, 9)$ and changes to x_1 and x_2 of amounts $dx_1 = 12$ and $dx_2 = 27$. This implies that the *new* point is $\hat{\mathbf{x}} = (16, 36)$. The resulting change in the value of the function is

$$\Delta y = f(16, 36) - f(4, 9) = [4 + 6] - [2 + 3] = 5$$

If we use $dy(\mathbf{x}^0)$ to estimate Δy, we get

$$\begin{aligned} dy(\mathbf{x}^0) &= \left[f_1\left(x_1^0, x_2^0\right) \quad f_2\left(x_1^0, x_2^0\right) \right] \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1/2}{\left(x_1^0\right)^{1/2}} & \dfrac{1/2}{\left(x_2^0\right)^{1/2}} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1/2}{(4)^{1/2}} & \dfrac{1/2}{(9)^{1/2}} \end{bmatrix} \begin{bmatrix} 12 \\ 27 \end{bmatrix} \\ &= \left(\frac{1}{4}\right) 12 + \left(\frac{1}{6}\right) 27 = 7\frac{1}{2} \end{aligned}$$

and so we see that $dy = 7\frac{1}{2}$ is indeed an overestimate of $\Delta y = 5$. (See figure 11.30)

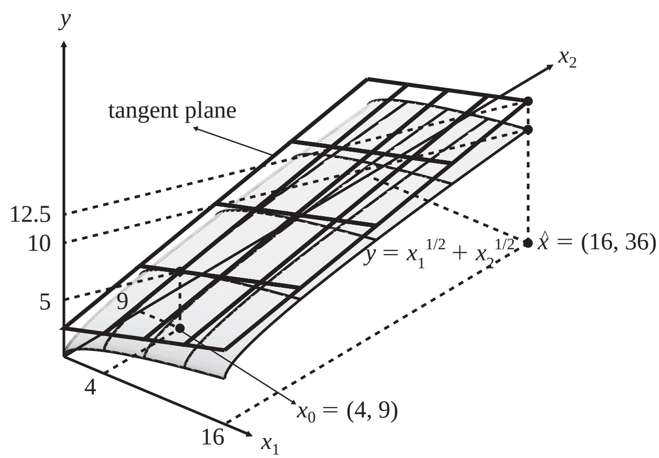


Figure 11.30 Tangent plane that overestimates the change in the function value for the strictly concave function $y = x_1^{1/2} + x_2^{1/2}$ (example 11.39) ■

The equation for the tangent plane to $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, at the point $\mathbf{x} = \mathbf{x}^0$ is

$$T(x_1, x_2, \ldots, x_n) = f(\mathbf{x}^0) + [\, f_1(\mathbf{x}^0) \quad f_2(\mathbf{x}^0) \quad \ldots \quad f_n(\mathbf{x}^0)\,] \begin{bmatrix} (x_1 - x_1^0) \\ (x_2 - x_2^0) \\ \vdots \\ (x_n - x_n^0) \end{bmatrix}$$

or, in more compact notation

$$T(\mathbf{x}) = f(\mathbf{x}^0) + \nabla \mathbf{f}^T (\mathbf{x} - \mathbf{x}^0)$$

We can see from this expression that the idea of estimating changes in the value of a function using the total differential (i.e., ignoring all terms of the Taylor series expansion beyond the first derivative terms) is the same as using the tangent plane at a point to *estimate* the function itself. (See also figure 11.30 and example 11.39 for a specific function.) The *estimate* can become as accurate as one wishes by choosing dx_i values to be *small*. That is, as $\mathbf{dx} \to 0$, the function $T(\mathbf{x})$ approximates $f(\mathbf{x})$ with precision approaching 100%. Note that for functions defined on $\mathbb{R}^n$, $n > 2$, there is no geometric interpretation of the function $T(\mathbf{x})$. In this case we refer to $T(\mathbf{x})$ as the tangent *hyperplane* to $f(\mathbf{x})$ at the point $\mathbf{x}^0$.

EXERCISES

1. Illustrate the result that using the first-order total differential leads to overestimates of the change in a function value for the function

$$y = 10 - \left(x_1^2 + x_2^2\right).$$

Use the initial point $\mathbf{x} = (1, 1)$ and changes in the x_i values of $dx_1 = 2$ and $dx_2 = 3$.

2. Illustrate the result that using the first-order total differential leads to underestimates of the change in a function value for the function

$$y = x_1^2 + x_2^2.$$

Use the initial point $\mathbf{x} = (1, 2)$ and changes in the x_i values of $dx_1 = 3$ and $dx_2 = 1$.

CHAPTER REVIEW

Key Concepts

additively separable function
bordered Hessian
cross-partial derivatives
elasticity of substitution
Euler's theorem
first-order total differential
gradient vector
Hessian matrix
homogeneous function
implicit differentiation
implicit function theorem
indifference curves
isoquant
level curve
level set
marginal rate of substitution (MRS)
marginal rate of technical substitution (MRTS)
partial derivative
positive monotonic transformation
remainder formula
second-order total differential
Taylor series
Young's theorem

Review Questions

1. What is a gradient vector?
2. Why is it convenient to express the first-order and second-order partial derivatives in vector/matrix notation?
3. What is meant by "implicit differentiation," and why is it useful?
4. What properties does a function require in order for the implicit function theorem to apply?
5. What is involved in finding the second-order total differential of a function that is additively separable?
6. How does the sign of the second-order total differential relate to concavity/convexity of a function?
7. Distinguish between concave and quasiconcave functions, and between convex and quasiconvex functions.
8. How would you determine if a production function were homogeneous?
9. What in general can be said about the returns-to-scale properties of production functions that are homogeneous?
10. What in general can be said about the returns-to-scale properties of production functions that are not homogeneous?
11. What is meant by the "remainder formula" for the Taylor series expansion?

Review Exercises

1. Find the marginal-product functions for the production function

$$f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$$

for $A > 0$, $0 < \alpha, \beta < 1$ and $x_1, x_2 > 0$.

2. Compute all the second-order partial derivatives for the function given in question 1. Determine the signs of these and provide an economic interpretation.

3. Compute all the first- and second-order derivatives of the function

$$f(x_1, x_2, x_3) = ax_1 + x_2^{\beta}x_3^{\gamma}$$

and show that Young's theorem applies.

4. For each of the following functions find the total differential, and use this to draw a representative level curve.

   (a) $y = x_1 + x_2$ for $x_1, x_2 \geq 0$

   (b) $y = x_1^{1/2}x_2^{1/3}$ for $x_1, x_2 > 0$

   (c) $y = x_1^3x_2^2$ for $x_1, x_2 > 0$

   In which cases are the level curves strictly convex to the origin? Explain by finding the second derivatives of the functions of these level curves.

5. Use theorem 11.9 to show that the function

$$y = x_1^{1/4}x_2^{1/2}$$

defined on $\mathbb{R}^2_{++}$ is strictly concave.

6. Show that the function

$$y = Ax_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3},$$

$A > 0$, $\alpha_i > 0\ \forall i$ defined on $\mathbb{R}^3_{++}$ is quasiconcave. Show that it is concave only if we add the restriction that $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$.

7. Use a Venn diagram to show the following relationships between:

   (a) The set of functions that is quasiconcave and the set that is concave.

   (b) The set of functions that is quasiconvex and the set that is convex.

8. Illustrate Euler's theorem for the Cobb-Douglas production function

$$f(x_1, x_2, x_3) = Ax_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \qquad A > 0, \alpha_i > 0 \; \forall \, i$$

defined on $\mathbb{R}^3_{++}$, and $\sum \alpha_i = 1$.

9. Consider the following specific CES production function defined on $x_1 > 0$, $x_2 > 0$:

$$y = f(x_1, x_2) = \left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-1/2}$$

(a) Find an expression for the MRTS, and show that isoquants are strictly convex to the origin.

(b) Use the determinant condition in theorem 11.12 to show that f is quasiconcave.

(c) Show that f is concave.

(d) Show that f is homogeneous, and find its degree of homogeneity.

(e) Show that the following result (from Euler's theorem) applies to f

$$f_1 x_1 + f_2 x_2 = kf(x_1, x_2)$$

where k is the degree of homogeneity of f.

(f) Use the formula given in definition 11.9 to find the elasticity of substitution between the inputs for this function.

10. Do parts (a), (d), and (e) of question 10 for the general CES production function

$$y = A\left[\delta x_1^{-r} + (1 - \delta)x_2^{-r}\right]^{-1/r}, \qquad A > 0, 0 < \delta < 1, r > -1$$

defined on $x_1 > 0$ and $x_2 > 0$.

Chapter 12 Optimization of Functions of *n*-Variables

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Price-Discriminating Monopoly with Linear Demands and Costs
- Cournot Equilibrium with n Identical Firms: Example
- Two-Plant Monopoly: Example
- Optimal Input Quantities for a Competitive Firm: Example
- Multiproduct Monopoly Revisited: Example
- Multiplant Monopoly with Linear Costs: Example
- Two-Plant Monopoly with Capacity Constraints: Example

The idea of optimization is fundamental in economics, and the mathematical methods of optimization underlie most economic models. For example, the theory of demand is based on the model of a consumer who chooses the best ("most preferred") bundle of goods from the set of affordable bundles. The theory of supply is based on the model of a firm choosing inputs in such a way as to minimize the cost of producing any given level of output, and then choosing output to maximize profit. Rationality and optimization are virtually synonymous in economics.

In a formal sense, by optimization we mean the maximization or minimization of a function over some given set. The significance of the concept of optimization is therefore that it gives us a well-defined mathematical procedure for obtaining the solutions to economic models: the "predictions" of the model are based upon the solution to the optimization problem it contains.

We already considered optimization methods for functions of one variable in chapter 6. In this chapter we extend these to functions of any number of variables though, as in chapter 11, we continue to focus on functions that are twice continuously differentiable, namely $f \in C^2$. We begin by considering the *unconstrained problem*, in which any point in $\mathbb{R}^n$ is allowed to be a possible solution. In the last section of this chapter we modify this to consider the case, often arising in economics, in which permissible values of at least one of the variables are restricted to a subset of the real line. In chapter 13 we consider the important problem of *functional constraints*.

Few difficulties will be presented by the methods in this chapter and the next if the reader has a good grasp of the material in chapters 6 and 11, and for the discussion of second-order conditions in section 12.2, it will also be useful to have studied the material on quadratic forms in section 10.3 and their relationship to concavity discussed in chapter 11.

12.1 First-Order Conditions

In chapter 6 we defined **extreme values** and **stationary values** of a function of one variable. Similar definitions apply here. An extreme value is a maximum or a minimum of the function, while a stationary value in the present case has the following definition:

Definition 12.1 A stationary value of a function f over the domain $\mathbb{R}^n$ occurs at a point $(x_1^*, \dots, x_n^*)$ at which the n-equalities

$$f_1(x_1^*, \dots, x_n^*) = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$f_n(x_1^*, \dots, x_n^*) = 0$$

hold simultaneously.

As in the case of functions of one variable, not all stationary points need give extreme values, because of the possibility of points of inflection. In the case of functions of n variables, there is the further possibility that a stationary value may be a **saddle point,** at which the function takes on a maximum with respect to changes in some of the x-values and a minimum with respect to others. For example, consider the case of a function of two variables. Holding one variable constant, say x_2, it becomes a function of x_1 only. If at some $x_1 = x_1^*$ we have $f_1(x_1^*, \bar{x}_2) = 0$, where $\bar{x}_2$ is the fixed value of x_2, then we know from the discussion in chapter 6 that this could correspond to a minimum, a maximum, or a point of inflection of the function, thought of only as a function of x_1. Likewise, if we fix the value of x_1 and consider the function only as a function of a single variable, x_2, the point at which $f_2(\bar{x}_1, x_2^*) = 0$ could correspond to a minimum, a maximum, or a point of inflection of this function. A further difficulty occurs when considering the possibility of allowing both x_1 and x_2 to change in moving away from the point (x_1^*, x_2^*). The complications that can arise are similar to those associated with the function $f(x_1, x_2) = x_1^2 + x_2^2 - 5x_1x_2$ discussed in example 11.24 and figure 11.18. This function appears to possess a minimum point at $(x_1^*, x_2^*) = (0, 0)$ as one changes x_1 only or x_2 only, with the function being convex in each of these two directions. However, as we see in figure 11.24, this function does not display a minimum when we change x_1 and x_2 simultaneously. This means that if we have a point (x_1^*, x_2^*) at which both partial derivatives are zero

$$f_1(x_1^*, x_2^*) = 0$$
$$f_2(x_1^*, x_2^*) = 0$$

it does not necessarily imply we have either a minimum or a maximum. Figure 12.1 illustrates four possibilities. Cases (c) and (d) represent saddle points. Only in cases (a) and (b) do we have extreme values of the function, in the first case a minimum and in the second a maximum.

The purpose of this discussion is to show that it is not *sufficient* for $(x_1^*, \ldots, x_n^*)$ to yield an extreme value of the function that the conditions in definition 12.1 are satisfied. Just as in the case of functions of one variable, we are going to have to

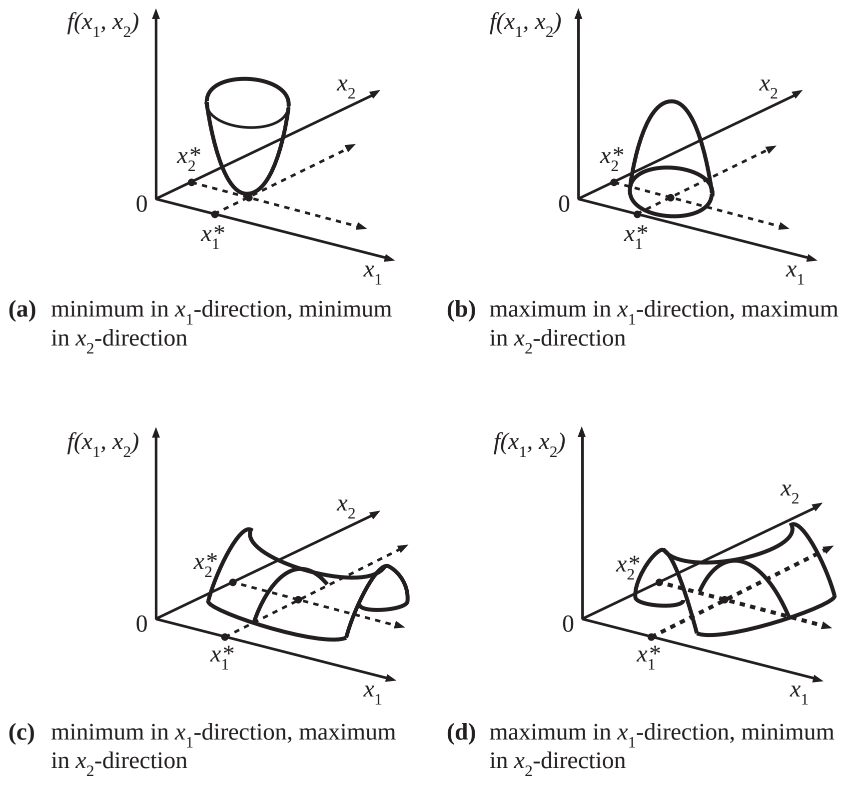


Figure 12.1 Possible cases of stationary values of $f(x_1, x_2)$

develop second-order conditions to help us distinguish among maximum points, minimum points, and other points such as saddle points.

However, we can show that for $(x_1^*, \ldots, x_n^*)$ to yield a maximum or a minimum it is *necessary* that the conditions in definition 12.1 be satisfied. Consider figure 12.1 (b). If we are at the peak of the hill, it must be impossible to move away from the point (x_1^*, x_2^*) *in any direction* in such a way as to increase the value of the function. That implies that it must be impossible to move either in the x_1-direction or the x_2-direction and increase the value of the function. It is a necessary condition, therefore, that the partial derivatives at this point are zero. A similar argument applies to the case of the minimum in figure 12.1(a). We can put these intuitive arguments a little more generally as follows:

Theorem 12.1 If at a point $(x_1^*, \ldots, x_n^*)$ we have a local maximum of the function f, so that

$$f(x_1^*, \ldots, x_n^*) \geq f(x_1, \ldots, x_n)$$

for all points $(x_1, \ldots, x_n)$ in a (possibly small) neighborhood of $(x_1^*, \ldots, x_n^*)$, then the conditions

$$\begin{gathered} f_1(x_1^*, \ldots, x_n^*) = 0 \\ \cdots\cdots\cdots\cdots \\ f_n(x_1^*, \ldots, x_n^*) = 0 \end{gathered}$$

hold simultaneously.

To see this, consider the total differential

$$dy = df = f_1(x_1^*, \ldots, x_n^*)\, dx_1 + \cdots + f_n(x_1^*, \ldots, x_n^*)\, dx_n$$

Suppose that at $(x_1^*, \ldots, x_n^*)$ any one of the partial derivatives is nonzero, say the first. Then, if $f_1 > 0$, choose $dx_1 > 0$, and all other $dx_i = 0$, $i \neq 1$, and we can increase the value of the function; if $f_1 < 0$, choose $dx_1 < 0$ and all other $dx_i = 0$, $i \neq 1$, and we can again increase the value of the function. Only if $f_1 = 0$, is it not possible to find a $dx_1 \neq 0$ that will increase the value of the function.

Figure 12.2 illustrates for the case $n = 2$. In (a) we have a nice well-behaved hill-shaped function. At its peak, the tangent plane is horizontal. The slope of the tangent plane in the x_1-direction is f_1, and the slope in the x_2-direction is f_2, and since the plane is horizontal each of these slopes is zero. In (b) we show the level curves for the function, and the point $(x_1^*, \ldots, x_n^*)$ again gives the maximum of the function.

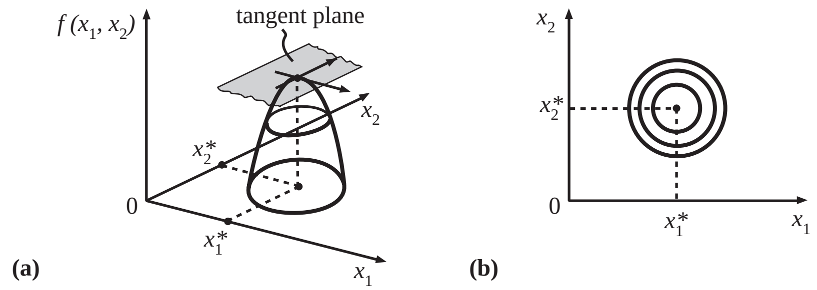


Figure 12.2 (x_1^*, x_2^*) yields a maximum of the function

Since they follow exactly the same lines, we leave it to the reader to supply the details of the argument establishing theorem 12.2.

Theorem 12.2 If at a point $(x_1^*, \ldots, x_n^*)$ we have a minimum of the function f, so that

$$f(x_1^*, \ldots, x_n^*) \leq f(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n)$ in a (possibly small) neighborhood of $(x_1^*, \ldots, x_n^*)$, then the conditions

$$\begin{gathered} f_1(x_1^*, \ldots, x_n^*) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ f_n(x_1^*, \ldots, x_n^*) = 0 \end{gathered}$$

hold simultaneously.

We now consider some mathematical examples before going on to some economic applications of these important theorems.

Example 12.1 Find stationary values of the following functions:

(i) $y = 2x_1^2 + x_2^2$
(ii) $y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1x_2$
(iii) $y = 4x_1^2 - x_1x_2 + x_2^2 - x_1^3$
(iv) $y = 2x_1^2 + x_2^2 + 4x_3^2 - x_1 + 2x_3$
(v) $y = x_1^2 - x_2^2$

Solution

(i) The first-order conditions are

$$4x_1 = 0, \quad 2x_2 = 0$$

These can only be satisfied at $x_1^* = x_2^* = 0$. Therefore (0, 0) is a stationary point.

(ii) The first-order conditions are

$$\begin{aligned} 4 - 2x_1 + x_2 &= 0 \\ 2 - 2x_2 + x_1 &= 0 \end{aligned}$$

which can be written as

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Solving by Cramer's rule gives

$$\begin{aligned} x_1^* &= \frac{10}{3} = 3.33 \\ x_2^* &= \frac{8}{3} = 2.67 \end{aligned}$$

Thus (3.33, 2.67) gives a stationary value of the function.

(iii) The first-order conditions are

$$\begin{aligned} 8x_1 - x_2 - 3x_1^2 &= 0 \\ -x_1 + 2x_2 &= 0 \end{aligned}$$

From the second equation we have

$$x_2 = \frac{x_1}{2}$$

So substituting into the first gives

$$8x_1 - 0.5x_1 - 3x_1^2 = 0$$

implying that

$$x_1^* = \frac{7.5}{3} = 2.5 \quad \text{and} \quad x_2^* = 1.25$$

Thus (2.5, 1.25) gives a stationary value of the function. Note, however, that $x_1 = x_2 = 0$ also gives a stationary value. Therefore the function has *two* stationary points, (0, 0) and (2.5, 1.5).

(iv) The first-order conditions are

$$\begin{aligned} 4x_1 - 1 &= 0 \\ 2x_2 &= 0 \\ 8x_3 + 2 &= 0 \end{aligned}$$

These yield, respectively,

$$x_1^* = 0.25, \quad x_2^* = 0, \quad x_3^* = -0.25$$

Thus (0.25, 0, −0.25) gives a stationary value of the function.

(v) The first-order conditions are

$$2x_1 = 0, \quad -2x_2 = 0$$

Thus $x_1^* = x_2^* = 0$, and (0, 0) gives a stationary value of the function. ■

Multiproduct Monopoly

A monopoly produces two outputs, x_1 and x_2, with the linear demand functions

$$\begin{aligned} x_1 &= 100 - 2p_1 + p_2 \\ x_2 &= 120 + 3p_1 - 5p_2 \end{aligned}$$

As we found in chapter 6, it is useful to have demand functions in inverse form, with price a function of quantity, when solving profit-maximization problems. To obtain the inverse-demand functions in this case, we treat p_1 and p_2 as unknowns and solve the equations simultaneously to express them as functions of the quantities. Thus we can write the demand functions as the system

$$\begin{bmatrix} 2 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 100 - x_1 \\ 120 - x_2 \end{bmatrix}$$

and using Cramer's rule to solve gives

$$p_1 = \frac{5(100 - x_1) + (120 - x_2)}{7}$$

$$p_2 = \frac{2(120 - x_2) + 3(100 - x_1)}{7}$$

which in turn give the inverse demand functions

$$p_1 = 88.57 - 0.71x_1 - 0.14x_2$$
$$p_2 = 77.14 - 0.29x_1 - 0.43x_2$$

Note that in the inverse-demand functions, when goods are substitutes, each output enters with a negative sign in the demand function of the other. An increase, say, in x_1 causes a fall in its price, which then causes a fall in demand for good 2, and therefore a fall in p_2 at any given output x_2.

Next, assume that the firm's cost function takes the form

$$C = 50 + 10x_1 + 20x_2$$

We then have the firm's profit function

$$\begin{aligned}\pi(x_1, x_2) &= p_1x_1 + p_2x_2 - C \\ &= 88.57x_1 - 0.71x_1^2 - 0.14x_1x_2 + 77.14x_2 \\ &\quad - 0.29x_1x_2 - 0.43x_2^2 - 50 - 10x_1 - 20x_2 \\ &= 78.57x_1 + 57.14x_2 - 0.71x_1^2 - 0.43x_2^2 - 0.43x_1x_2 - 50\end{aligned}$$

To find the first-order conditions, we apply theorem 12.1 to obtain

$$\pi_1(x_1^*, x_2^*) = 78.57 - 1.42x_1^* - 0.43x_2^* = 0$$
$$\pi_2(x_1^*, x_2^*) = 57.14 - 0.86x_2^* - 0.43x_1^* = 0$$

and using Cramer's rule gives the profit-maximizing outputs

$$x_1^* = \frac{\begin{vmatrix} 78.57 & 0.43 \\ 57.14 & 0.86 \end{vmatrix}}{1.04} = 41.35$$
$$x_2^* = \frac{\begin{vmatrix} 1.42 & 78.57 \\ 0.43 & 57.14 \end{vmatrix}}{1.04} = 45.53$$

To find the corresponding prices, we insert these outputs into the demand functions to obtain

$$p_1^* = \$52.84, \quad p_2^* = \$45.57$$

This result gives the firm's maximum profit as $\pi^* = \$2885.64$. All this effort, of course, assumes that we have found a true maximum of the function. One way of

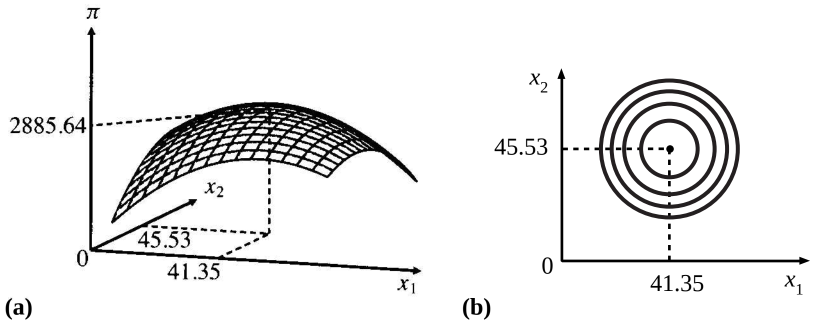


Figure 12.3 Two-output monopoly profit function

checking is to consider the graph of the function, drawn in figure 12.3. In (a) we show the graph in three dimensions: we have a nice strictly concave shape, and the point (41.35, 45.53) is clearly a true maximum. In (b) we show the level curves of the function, again with the indicated maximum. In the next section we consider second-order conditions which allow us to carry out this check quite generally.

Cournot Duopoly

Two firms produce identical outputs and sell into a market with the linear demand function

$$p = 100 - (q_1 + q_2)$$

where q_i is the output of firm $i = 1, 2$. We assume that each firm's production cost is zero. Each firm wants to maximize its profits, given by

$$\pi_i = pq_i = 100q_i - (q_1 + q_2)q_i, \qquad i = 1, 2$$

The basic idea here is that market price depends on the total output of the two firms, and so each firm's profit depends on how much output the other firm produces, as well as on its own output. This form of close interdependence is characteristic of oligopolistic markets, of which the two-firm duopoly is a special case. The difficulty the firm faces in solving its profit-maximizing problem is to figure out what output the other firm will produce, since until it does that it cannot compute the profit that will result from any choice of its own output. The assumption about this, made by the French economist Augustin Cournot, is that each firm takes the output of the other as a given parameter when choosing its own output, and the market equilibrium is then given as the solution of the pair of simultaneous equations that results. We now see how this works out.

If we maximize firm i's profit, treating the other firm's output as a given parameter, we obtain for the two firms the two equations

$$\frac{\partial \pi_1}{\partial q_1} = 100 - 2q_1 - q_2 = 0$$
$$\frac{\partial \pi_2}{\partial q_2} = 100 - 2q_2 - q_1 = 0$$

Solving these for the outputs gives

$$q_1 = q_2 = 33.33 \quad \text{and} \quad p = 100 - 66.67 = \$33.33$$

Thus, as we might expect from the symmetry of the example, the firms end up sharing the market equally.

We illustrate this solution in figure 12.4. From the first-order conditions above we have the two equations

$$q_1 = \frac{100 - q_2}{2}$$
$$q_2 = \frac{100 - q_1}{2}$$

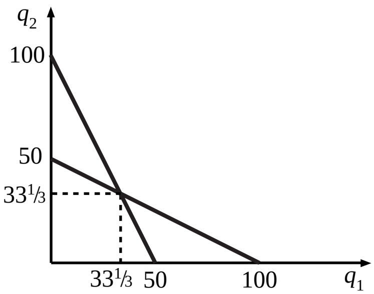


Figure 12.4 Cournot equilibrium

These equations give the output of one firm which is best (profit maximizing) for it at each possible level of output of the other firm, and so they are often known as "best response" or "best reply" or simply "reaction" functions. In figure 12.4 we graph these functions, and note that the point at which they intersect gives the Cournot equilibrium outputs we just derived.

As question 2 of the exercises asks you to confirm, the total output in this market lies between the monopoly and perfectly competitive levels, which is a nice feature of a model with more than one, but still relatively few, firms. We explore this feature further in the next example (see the website).

EXERCISES

1. Find the stationary values of the following functions:

(a) $y = 3x_1^2 + 2x_2^2 + 5$

(b) $y = 2x_1^2 - 4x_2^2 + 1$

(c) $y = 2x_1 + x_2 - 3x_1^2 - 4x_2^2 + x_1x_2$

(d) $y = 0.5x_1 - 2x_1^2 + 4x_2 - 3x_2^2 + 2x_1x_2$

(e) $y = 2x_1^3 - 3x_1x_2 + x_1^2 - 2x_2^2$

(f) $y = x_1x_2^{0.5}(10 - x_1 - x_2)^{0.4}$

(g) $y = (x_1^2 + x_2^4 + x_3^6)^2$

(h) $y = 4x_1^2 + 2x_2^2 + x_3^2 - x_1 + 5x_3$

(i) $y = x_1^3 + x_2^3 - 3x_1x_2$

(j) $y = 2(x_1 - x_2)^2 - x_1^4 - x_2^4$

2. In the example of Cournot duopoly, calculate the following:

(a) The output that would be sold if prices were set equal to marginal cost.

(b) The output that would be sold if the two firms acted jointly as a profit-maximizing, monopolist.

Show that the Cournot equilibrium output lies between these extremes.

3. A firm sells some output in a perfectly competitive market, where the price is \$60 per unit, and some on a market in which it has a monopoly, with a demand function $p_2 = 100 - q_2$, where q_2 is output in the monopoly market. Its total-cost function is $C = (q_1 + q_2)^2$, where q_1 is output in the competitive market. Find the profit-maximizing outputs in the two markets and discuss the nature of the equilibrium. Suppose now that the price in the competitive market falls to \$10. Find the new profit-maximizing solution, and discuss how it compares with the original one.

4. Take the discriminating monopoly example of this section, but assume now that it has the cost function $C = (q_1 + q_2)^2$. Find the profit-maximizing solutions in the cases where it does and does not practice price discrimination. Discuss your results.

5. Two firms produce identical outputs in a market with the demand function

$$p = 10 - 0.1(q_1 + q_2)$$

They have cost functions

$$C_1 = 0.25q_1 \quad \text{and} \quad C_2 = 0.5q_2$$

First find and discuss the Cournot duopoly equilibrium. Then assume that the firms adopt the joint profit-maximizing solution; in other words, they maximize the sum of their profits. Compare the result to the Cournot equilibrium.

6. A monopoly has the demand function $p = D(q, a)$, where a is its advertising expenditure, and p and q are price and output. What would you expect to be true of the signs of the partials D_a and D_{qa}? Advertising is measured in dollars, and so the cost function is $C = C(q) + a$. Find and discuss the firm's profit-maximizing equilibrium.

7. A monopolist faces the demand function

$$p = 100 - (q_1 + q_2)$$

and produces identical outputs from two plants with cost functions

$$C_1 = 2q_1^2, \quad C_2 = 3q_2^2$$

Find the profit-maximizing price and total output and the corresponding output from each plant. Explain, both in this numerical problem and generally, why the marginal costs of the plants are equalized at the optimum. Illustrate your answer in a diagram.

12.2 Second-Order Conditions

We know that the first-order conditions cannot in themselves distinguish between maximum values, minimum values, points of inflection and saddle points, because they hold at each of these. We now develop second-order conditions which tell us when we can be sure that a point satisfying the first-order conditions is certainly a true maximum or minimum. We begin with an intuitive discussion and at the end of this section give a somewhat more rigorous account in terms of the Taylor series expansion.

Suppose that $(x_1^*, \ldots, x_n^*)$ is a stationary point of the function f. If we make small changes in the x-vector *in any direction* from this point, and the result is to *reduce* the value of f, then this point must yield a local maximum of the function. Similarly, if we move away from this point a small distance in *any* direction and this *increases* the value of the function, this point must yield a local minimum. Finally, if moving in some directions increases the value of the function, while moving in other directions reduces the function value, then the stationary point must be a saddle point.

Note the emphasis on "in *any* direction." If simply moving in the x_i-direction (parallel to the x_i-axis), for each x_i reduces the value of the function, this need not imply we have a maximum, since these are only a small subset of all the possible directions in which we could go. Figure 12.5 illustrates for a function of two variables. The bulge in the function means that (x_1^*, x_2^*) gives a stationary value which is not a local maximum, since moving in the direction indicated—not parallel to either axis—increases the value of the function.

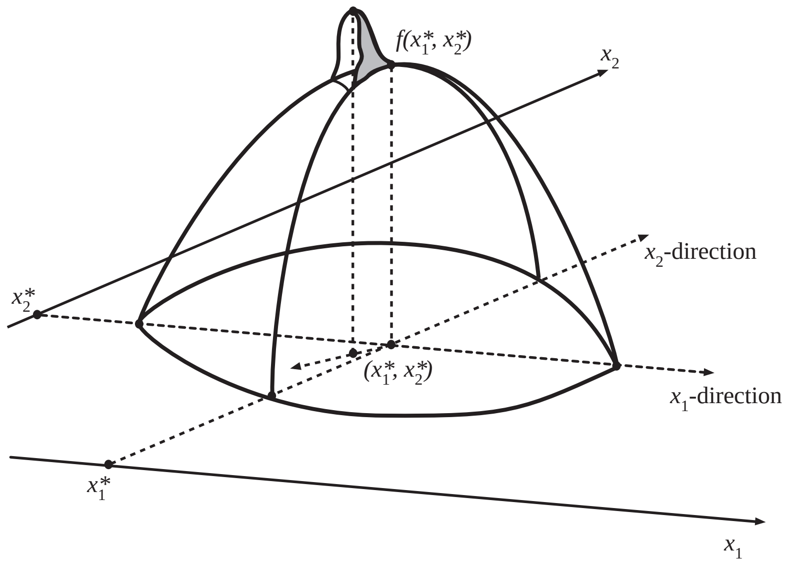


Figure 12.5 A movement away from (x_1^*, x_2^*) *not* in the x_1- or x_2-direction increases the value of the function

The requirement to take account of *all possible* directions of change in the x-vector is what complicates the algebra of the second-order conditions just as it complicates the determination of concavity and convexity (see section 11.4). The second-order partial f_{ii} tells us about the curvature of the function in the x_i-direction. If the function value decreases with a move from $(x_1^*, \ldots, x_n^*)$ in the x_i-direction, as is *necessary* if this point is to be a maximum, then we have $f_{ii}(x_1^*, \ldots, x_n^*) < 0$. However, it is not *sufficient* to state this as a condition for all i, because it takes no account of movements away from the x-vector that are not in any x_i-direction. For this, we require the total differential.

Given some function with continuous partial derivatives, its total differential for arbitrary changes dx_i gives the change in the function in an arbitrary direction. Suppose this function is the total differential dy. At the stationary point $(x_1^*, \ldots, x_n^*)$, this total differential $dy = df(x_1^*, \ldots, x_n^*) = 0$. If, for *any* (small) movement away from this point dy becomes negative, that means the function is decreasing and $(x_1^*, \ldots, x_n^*)$ yields a maximum. If, for any (small) movement away, dy becomes positive, this means the function value is increasing and $(x_1^*, \ldots, x_n^*)$ yields a minimum. Thus sufficient conditions for a local maximum or minimum can be expressed in terms of what happens to dy as we move away from $(x_1^*, \ldots, x_n^*)$ in any direction, namely in terms of the second-order differential d^2y. We can put this more formally as follows:

We have a function $y = f(x_1, \ldots, x_n)$ or $y = f(\mathbf{x})$, with $\mathbf{x} = (x_1, \ldots, x_n)$. At a point $\mathbf{x}^*$ consider the total differential

$$dy = f_1(\mathbf{x}^*)\,dx_1 + f_2(\mathbf{x}^*)\,dx_2 + \cdots + f_n(\mathbf{x}^*)\,dx_n$$

where the differentials dx_i are to be interpreted as small given numbers. This is of course just a function of the vector $\mathbf{x}$ and as in chapter 11, we can take the differential of this function to obtain the second-order differential

$$\begin{aligned}
d(dy) \equiv d^2y &= [d(f_1(\mathbf{x}^*)\,dx_1)] + [d(f_2(\mathbf{x}^*)\,dx_2)] + \cdots + [d(f_n(\mathbf{x}^*)\,dx_n)] \\
&= [f_{11}(\mathbf{x}^*)\,dx_1\,dx_1 + f_{12}(\mathbf{x}^*)\,dx_1\,dx_2 + \cdots + f_{1n}(\mathbf{x}^*)\,dx_1\,dx_n] \\
&\quad + [f_{21}(\mathbf{x}^*)\,dx_2\,dx_1 + f_{22}(\mathbf{x}^*)\,dx_2\,dx_2 + \cdots + f_{2n}(\mathbf{x}^*)\,dx_2\,dx_n] \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad + [f_{n1}(\mathbf{x}^*)\,dx_n\,dx_1 + f_{n2}(\mathbf{x}^*)\,dx_n\,dx_2 + \cdots + f_{nn}(\mathbf{x}^*)\,dx_n\,dx_n] \\
&= \sum_i \sum_j f_{ij}(\mathbf{x}^*)\,dx_i\,dx_j
\end{aligned}$$

This expression is a *quadratic form* (see chapter 10). Now suppose that $d^2y < 0$ for arbitrary vectors of values (dx_i) (which may have to be very small). That means that in all directions the value of dy is decreasing around the point $\mathbf{x}^*$. But, if dy is zero at $\mathbf{x}^*$, as it must be if $\mathbf{x}^*$ is an extreme value, then dy is negative in a neighborhood of $\mathbf{x}^*$. That means that the function is decreasing in the neighborhood of $\mathbf{x}^*$, and so if the first-order conditions are satisfied at $\mathbf{x}^*$, then this is a *local* maximum of the function. We refer to this as a local maximum because we only use information about the behavior of dy in a small neighborhood of $\mathbf{x}^*$, not over all possible vectors $\mathbf{x}$. On the other hand, a *global* maximum would be the maximum of the function taken over all possible vectors $\mathbf{x}$.

We have

Theorem 12.3 It is sufficient for $\mathbf{x}^*$ to yield a local maximum of the C^2 function $y = f(\mathbf{x})$ that

$$f_i(\mathbf{x}^*) = 0, \qquad i = 1, \ldots, n$$

and the quadratic form

$$d^2y(\mathbf{x}^*) = \sum_i \sum_j f_{ij}(\mathbf{x}^*)\,dx_i\,dx_j < 0, \qquad i, j = 1, \ldots, n$$

That is, d^2y is negative definite, or since $d^2y = dx^T H\,dx$, the Hessian matrix H is negative-definite.

Proof

By using the Taylor series expansion (definition 11.1 in section 11.6), we can expand $f(\mathbf{x})$ around the point $\mathbf{x}^*$ to get, for any $\hat{\mathbf{x}}$ in the neighborhood of $\mathbf{x}^*$ (i.e., close to $\mathbf{x}^*$),

$$f(\hat{\mathbf{x}}) = f(\mathbf{x}^*) + dy(\mathbf{x}^*) + d^2y(\xi)$$

where ξ lies between $\mathbf{x}^*$ and $\hat{\mathbf{x}}$. Since, $f_i(\mathbf{x}^*) = 0$ for all $i = 1, 2, \ldots, n$, we have

$$dy(\mathbf{x}^*) = [f_1(\mathbf{x}^*) \quad f_2(\mathbf{x}^*) \quad \ldots \quad f_n(\mathbf{x}^*)] \begin{bmatrix} (\hat{x}_1 - x_1^*) \\ (\hat{x}_2 - x_2^*) \\ \vdots \\ (\hat{x}_n - x_1^*) \end{bmatrix} = 0$$

Moreover, since $d^2y(\mathbf{x}) < 0$, it must also be the case that for $\hat{\mathbf{x}}$ sufficiently close to $\mathbf{x}^*$, and hence ξ close to $\mathbf{x}^*$, $d^2y(\hat{\mathbf{x}}) < 0$ and $d^2y(\xi) < 0$. These results together give us

$$f(\hat{\mathbf{x}}) - f(\mathbf{x}^*) = d^2(\xi) < 0$$

In other words, $f(\hat{\mathbf{x}})$ is less than $f(\mathbf{x}^*)$ for any $\hat{\mathbf{x}}$ *near* $\mathbf{x}^*$ if $f_i(\mathbf{x}^*) = 0$ and $d^2y(\mathbf{x}^*) < 0$. Thus $\mathbf{x}^*$ yields a local maximum. ■

From this proof we see that if $f(\mathbf{x})$ is a strictly concave function on $\mathbf{x} \in \mathbb{R}^n$, then, if $f_i(\mathbf{x}^*) = 0$, $i = 1, 2, \ldots, n$, it must be the case that $\mathbf{x}^*$ is a unique global maximum. This follows clearly if $d^2f(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^n$, a sufficient condition for strict concavity of f.

Theorem 12.4 Suppose that $y = f(\mathbf{x})$ is a strictly concave function defined on $\mathbf{x} \in \mathbb{R}^n$. If at $\mathbf{x} = \mathbf{x}^*$ all first derivatives vanish,

$$f_i(\mathbf{x}^*) = 0, \qquad i = 1, 2, \ldots, n$$

then $\mathbf{x}^*$ yields a unique global maximum.

By a similar argument, simply noting that $\mathbf{x}^*$ yields a minimum of the function if d^2y is positive in a neighborhood of $\mathbf{x}^*$, we can also obtain the following two theorems:

Theorem 12.5 It is sufficient for $\mathbf{x}^*$ to yield a local minimum of the function $f(\mathbf{x})$ that

$$f_i(\mathbf{x}^*) = 0, \qquad i = 1, \ldots, n$$

and the quadratic form

$$d^2y = \sum_i \sum_j f_{ij}(\mathbf{x}^*)\, dx_i\, dx_j > 0, \qquad i, j = 1, \ldots, n$$

That is, d^2y is positive definite, or since $d^2y = \mathbf{dx}^T H\, \mathbf{dx}$, the Hessian matrix H is positive definite (see definition 10.13).

Theorem 12.6 Suppose that $y = f(\mathbf{x})$ is a strictly convex function defined on $\mathbf{x} \in \mathbb{R}^n$. If at $\mathbf{x} = \mathbf{x}^*$ all first derivatives vanish,

$$f_i(\mathbf{x}^*) = 0, \qquad i = 1, 2, \ldots, n$$

then $\mathbf{x}^*$ yields a unique global minimum.

We will now apply these conditions to a number of examples. Before doing so, we should note that these conditions are sufficient but not necessary—there are cases in which they are not satisfied at a maximum point. For example, the principal minors may all be zero at the maximum (or minimum) point. At this point you will find it useful to review quadratic forms in section 10.3.

Example 12.2 For each function given in example 12.1, determine whether the stationary point is a local or global maximum, minimum, or saddle point.

Solution

(i)

$$y = 2x_1^2 + x_2^2$$

$$f_1 = 4x_1, \quad f_2 = 2x_2$$

$$f_{11} = 4, \quad f_{12} = 0, \quad f_{21} = 0, \quad f_{22} = 2 \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n$$

$$F = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0 \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n$$

Applying theorem 12.5 shows that the stationary point (0, 0) yields a unique global minimum of the function.

(ii)

$$y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1x_2$$
$$f_1 = 4 - 2x_1 + x_2, \quad f_2 = 2 - 2x_2 + x_1$$
$$f_{11} = -2, \quad f_{12} = 1, \quad f_{21} = 1, \quad f_{22} = -2 \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n$$

$$F = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0 \qquad \text{for any } \mathbf{x} \in \mathbb{R}^n$$

Since $f_{11} = -2 < 0$, applying theorem 12.4 shows that the stationary point (3.33, 2.67) is a maximum.

(iii)

$$y = 4x_1^2 - x_1x_2 + x_2^2 - x_1^3$$
$$f_1 = 8 - 6x_1 - x_2 - 3x_1^2, \quad f_2 = -x_1 + 2x_2$$
$$f_{11} = 8 - 6x_1, \quad f_{12} = -1, \quad f_{21} = -1, \quad f_{22} = 2$$

This function has stationary points (0, 0) and (2.5, 1.5). At the first of these

$$f_{11} = 8 > 0, \quad F = \begin{vmatrix} 8 & -1 \\ -1 & 2 \end{vmatrix} = 15 > 0$$

Therefore applying theorem 12.5 tells us this point yields a minimum of the function.

At (2.5, 1.5),

$$f_{11} = 8 - 15 = -7 < 0, \quad F = \begin{vmatrix} -7 & -1 \\ -1 & 2 \end{vmatrix} = -14 - 1 = -15 < 0$$

Neither theorem 12.3 nor theorem 12.5 can be applied in this case. In fact the function has a saddle point at (2.5, 1.5). This can be seen by noting that $f_{11} < 0$ means that the function reaches a maximum in the x_1-direction, while $f_{22} > 0$ means that it reaches a minimum in the x_2-direction.

(iv)

$$y = 2x_1^2 + x_2^2 + 4x_3^2 - x_1 + 2x_3$$
$$f_1 = 4x_1 - 1, \quad f_2 = 2x_2, \quad f_3 = 8x_3 + 2$$
$$f_{11} = 4, \quad f_{22} = 2, \quad f_{33} = 8, \quad f_{ij} = 0, \qquad i \neq j \text{ for any } \mathbf{x} \in \mathbb{R}^n$$

$$F = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{vmatrix}$$

Thus we have

$$f_{11} = 4 > 0, \quad \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0, \quad \begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{vmatrix} = 64 > 0 \quad \text{for any } \mathbf{x} \in \mathbb{R}^n$$

Applying theorem 12.5 therefore tells us the function has a unique global minimum at the stationary point (0.25, 0, −0.25). ■

Profit-Maximizing Input Choice by a Competitive Firm

Suppose that a competitive firm produces output y using two inputs, labor L, and capital K. The firm faces a product price p and input prices w and r per unit, and has a Cobb-Douglas production function $y = AL^{\alpha}K^{\beta}$. In this classic problem, the firm chooses L and K so as to maximize profit given by

$$\pi(L, K) = pAL^{\alpha}K^{\beta} - wL - rK, \qquad \alpha, \beta > 0$$

The first-order conditions are, using theorem 12.1,

$$\frac{\partial \pi}{\partial L} = \alpha p A L^{\alpha-1}K^{\beta} - w = 0$$

$$\frac{\partial \pi}{\partial K} = \beta p A L^{\alpha}K^{\beta-1} - r = 0$$

By theorem 12.3, this solution gives a local maximum if the associated quadratic form is negative-definite. This amounts to the Hessian matrix being negative-definite, which from part 2 of theorem 11.9 requires the leading principal minors to alternate in sign starting with a negative. (At this stage you should write out the 2×2 Hessian matrix of second-order partial derivatives.) The first leading principal minor of the Hessian in this case is the derivative $\partial^2\pi/\partial L^2$, or

$$|H_1| = \alpha(\alpha - 1)pAL^{\alpha-2}K^{\beta}$$

which is negative if and only if $\alpha < 1$. The second leading principal minor is simply the determinant of H, given by

$$|H| = \alpha\beta p^2 A^2 L^{2\alpha-2}K^{2\beta-2}(1 - \alpha - \beta)$$

which is positive, as required for a maximum if and only if $1 > \alpha + \beta$. (Recall that $\alpha + \beta < 1$ implies decreasing returns to scale, while $\alpha + \beta = 1$ implies constant returns to scale and $\alpha + \beta > 1$ implies increasing returns to scale. The latter two possibilities are not consistent with a competitive market.)

EXERCISES

1. Take each function given in question 1 of the exercises in section 12.1, and determine whether the stationary points give a maximum, a minimum, or a saddle point in each case.

2. Confirm that the solution to the profit-maximization price-discrimination problem in section 12.1 is a true maximum.

3. Confirm that in the Cournot duopoly example the solution to each firm's profit-maximization problem is a true maximum.

4. Confirm that your solution to question 7 of the exercises in section 12.1 yields a true maximum.

5. Solve the competitive firm's profit-maximizing use of labor and capital for the case where $y = L^{0.25}K^{0.5}$, $p = 64$, $w = 2$, and $r = 4$. Show that the solution is a true maximum.

6. Suppose that we repeat question 5 with the production function

$$y = L^{1/2}K^{3/4}$$

   What problem arises? Explain.

7. Consider the problem of maximizing the function

$$y = -\left(x_1^4 + x_2^4\right)$$

   What is the solution? What problem arises in applying theorem 12.3 in this case?

12.3 Direct Restrictions on Variables

In section 6.3 we considered the case in which the variable on which a function was defined was restricted to lie in an interval. For example, if a firm is subject to an output quota, then its output is restricted to lie between zero and some upper limit equal to the quota. In fact in many economic problems it makes sense to restrict the values of the variables to be nonnegative, but it is often implicitly or explicitly assumed that this constraint does not bind at the optimum. Nevertheless, we can often learn interesting things, or resolve puzzles in which the first-order conditions appear to give strange results, by taking such restrictions explicitly into account.

The results we developed in section 6.3, for functions of one variable extend readily to the case of functions of n variables. Thus suppose that each variable

x_i is restricted to an interval $a_i \leq x_i \leq b_i$, $i = 1, \ldots, n$. It can be the case that for some i, a_i is $-\infty$, and for some (not necessarily the same) i, b_i is $+\infty$, but we assume that for at least some i, a_i, and/or b_i are finite. In what follows our remarks are aimed at these variables.

Suppose that the point $\mathbf{x}^*$ gives a maximum of the function, subject to the constraint that each x_i-value lies in its given interval. For each x_i in turn we must then have one of three possible cases, which are illustrated in figure 12.6. [Note that $\mathbf{x}^*_{-i}$ is the vector of fixed values $(x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, x_n^*)$.]

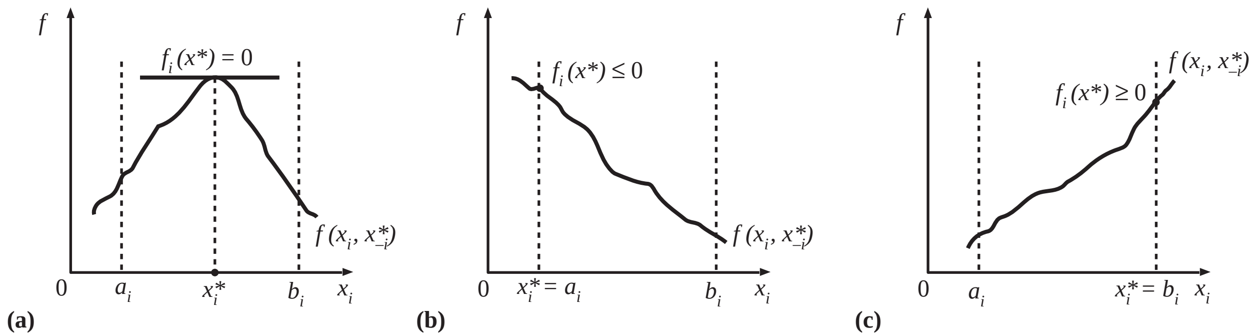


Figure 12.6 Possible solutions when x_i must lie in an interval

Case 1 $a_i < x_i^* < b_i$. In case 1 we must have $f_i(\mathbf{x}^*) = 0$. To see this, consider the component of the total differential df corresponding to x_i, $f_i(\mathbf{x}^*)\,dx_i$. If $f_i(\mathbf{x}^*) \neq 0$, then it is possible to find a suitably small dx_i with the appropriate sign, such that $f_i(\mathbf{x}^*)\,dx_i > 0$. This way the function value can be increased, contradicting the fact that it is at a maximum. Thus we must have $f_i(\mathbf{x}^*) = 0$. This is, of course, the argument we used for the case in which no constraints were imposed.

Case 2 $a_i = x_i^*$. In case 2 we must have $f_i(\mathbf{x}^*) \leq 0$. To see this, suppose that $f_i(\mathbf{x}^*) > 0$. We are free to choose some $dx_i > 0$, since that keeps x_i within the feasible interval, and we then have $f_i(\mathbf{x}^*)\,dx_i > 0$, contradicting the fact that the function is at a maximum. Thus, we can rule out $f_i(\mathbf{x}^*) > 0$ as a possibility. However, if $f_i(\mathbf{x}^*) < 0$, we could only increase the function value by taking some $dx_i < 0$, which is not permissible because it will violate the constraint. Therefore, we cannot rule out the possibility that $f_i(\mathbf{x}^*) < 0$, nor indeed that $f_i(\mathbf{x}^*) = 0$, since in either case we could not increase the function value by permissible variations in x_i.

Case 3 $x_i^* = b_i$. In case 3 we must have $f_i(\mathbf{x}^*) \geq 0$. To see this, suppose that $f_i(\mathbf{x}^*) < 0$. We are free to choose a $dx_i < 0$ such that $f_i(\mathbf{x}^*)\,dx_i > 0$, and so the function value can be increased without violating the constraint. Therefore we can rule this out. On the other hand, if $f_i(\mathbf{x}^*) > 0$, only a $dx_i > 0$ could increase the function value, but that violates the constraint, and so the function value cannot be increased. The function value also cannot be increased by small variations in x_i if $f_i(\mathbf{x}^*) = 0$.

We can express these cases more succinctly in

Theorem 12.7 If $\mathbf{x}^*$ is a solution to the problem

$$\max f(\mathbf{x}) \quad \text{s.t.} \quad a_i \le x_i \le b_i, \qquad i = 1, \ldots, n$$

then one or both of the following conditions must hold:

(i) $f_i(\mathbf{x}^*) \le 0$ and $(x_i^* - a_i) f_i(\mathbf{x}^*) = 0$
(ii) $f_i(\mathbf{x}^*) \ge 0$ and $(b_i - x_i^*) f_i(\mathbf{x}^*) = 0$

for all $i = 1, \ldots, n$.

The reader should confirm that these conditions aptly summarize the cases just considered. If $a_i < x_i^* < b_i$ then *both* conditions hold. Note also that different conditions may hold for different x_i.

By exactly the same type of argument, we can also establish

Theorem 12.8 If $\mathbf{x}^*$ is a solution to the problem

$$\min f(\mathbf{x}) \quad \text{s.t.} \quad a_i \le x_i \le b_i, \qquad i = 1, \ldots, n$$

then one or both of the following conditions must hold:

(i) $f_i(\mathbf{x}^*) \ge 0$ and $(x_i^* - a_i) f_i(\mathbf{x}^*) = 0$
(ii) $f_i(\mathbf{x}^*) \le 0$ and $(b_i - x_i^*) f_i(\mathbf{x}^*) = 0$

Again, if $a_i < x_i^* < b_i$, then *both* conditions hold.

Example 12.3 Solve the following problems:

(i) max $y = 10x_1 - 5x_2$ subject to $0 \le x_1 \le 20$, $0 \le x_2 \le 20$
(ii) max $y = x_1^{1/2} x_2^{1/2}$ subject to $0 \le x_1 \le 10$, $0 \le x_2 \le 10$
(iii) max $y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1 x_2$ subject to $0 \le x_1 \le 10$, $0 \le x_2 \le 10$
(iv) max $y = 4x_1 + 2x_2 - x_1^2 - x_2^2 + x_1 x_2$ subject to $0 \le x_1 \le 1$, $0 \le x_2 \le 2.67$

Solution

(i) This function is linear and increasing in x_1 and linear and decreasing in x_2. In the absence of the interval constraints there would be no solution. We can guess immediately that the solution is at the upper bound of x_1 and the lower bound of x_2:

$$x_1^* = 20, \quad x_2^* = 0$$

This point satisfies the necessary conditions in theorem 12.7, since

$$\begin{aligned} f_1 &= 10 \geq 0, \quad (20 - x_1^*)10 = 0 \\ f_2 &= -5 \leq 0, \quad (x_2^* - 0)(-5) = 0 \end{aligned}$$

at $(20, 0)$.

(ii) This function is monotonically increasing over the given intervals, and so we can guess that the solution is at the upper bounds of the intervals

$$x_1^* = 10, \quad x_2^* = 10$$

This point satisfies the necessary condition in theorem 12.7, since

$$f_1(10, 10) = \frac{1}{2}10^{1/2}10^{1/2} \geq 0, \quad (10 - 10) f_1 = 0$$

$$f_2(10, 10) = \frac{1}{2}10^{1/2}10^{1/2} \geq 0, \quad (10 - 10) f_2 = 0$$

(iii) We solved this problem in example 12.1, where we found the maximum was at $(3.33, 2.67)$. Since this point is interior to both intervals, the constraints are *nonbinding* and this continues to be the solution. In this case we have

$$\begin{aligned} f_1 &= 0, \quad (3.33 - 0) f_1 = (10 - 3.33) f_1 = 0 \\ f_2 &= 0, \quad (2.67 - 0) f_2 = (10 - 2.67) f_2 = 0 \end{aligned}$$

which satisfies the conditions of theorem 12.7.

(iv) Here, we have the same function as previously but the intervals differ. For x_1, the given interval excludes the previously optimal solution. For x_2, the previously optimal value is the upper bound of the interval and is still available. But beware! Even though still available, this value of x_2 *need not be optimal for this new, constrained problem*, and we shall in fact see that this is the case.

We may be tempted to try the upper bounds of the two intervals, the point (1, 2.67), as a candidate for the solution. The partial derivatives of the function are

$$f_1 = 4 - 2x_1 + x_2, \quad f_2 = 2 - 2x_2 + x_1$$

Thus at (1, 2.67) we have

$$f_1 = 4.67 > 0, \quad f_2 = 3 - 5.34 = -2.34 < 0$$

We conclude that this point *cannot* be optimal, because the necessary conditions in theorem 12.7 are violated: we require $f_2 \geq 0$ when x_2 is at the upper bound of its interval.

We can guess at the likely solution by noting first that for all x_1 in the interval [0, 1] and all x_2 in [0, 2.67], we have $f_1 > 0$; this means that the function is *increasing* in x_1. Therefore it makes sense to set x_1 at its upper bound $x_1 = 1$. As we just saw, at (1, 2.67) the partial derivative $f_2 < 0$, suggesting that we can increase the value of the function by reducing x_2. But how far? We can find the answer if we set $x_1 = 1$ in the function y and maximize with respect to x_2 over the interval [0, 2.67]: we solve

$$\max y = 3 + 3x_2 - x_2^2 \quad \text{s.t.} \quad 0 \leq x_2 \leq 2.67$$

We can guess that we will have an interior solution with the first-order condition

$$3 - 2x_2 = 0$$

implying that $x_2^* = 1.5$. To check that this satisfies the necessary conditions, we have

$$f_1 = 4 - 2(1) + 1.5 = 3.5 > 0, \quad f_1(1 - x_1^*) = 0$$
$$f_2 = 2 - 2(1.5) + 1 = 0, \quad f_2(1.5 - 0) = f_2(2.67 - 1.5) = 0$$

and so the conditions hold.

Figure 12.7 illustrates what is going on in this problem. The level curves reflect the concave shape of this function. The peak of the function is at (3.33, 2.67), but in the constrained problem we are restricted to the interval [0, 1] for x_1. Then the point [1, 2.67] is not on the highest attainable level curve. We reach the highest possible level curve by moving to [1, 1.5]. Note that this is a *point of tangency* between the vertical constraint line and the highest possible level curve.

You may be worried about the amount of guesswork in this answer and want a more systematic approach. This takes the following form. Note that, for any value

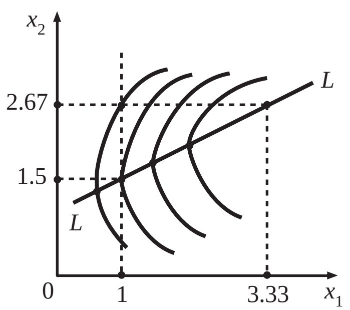


Figure 12.7 Interval constraint that changes optimal values of both variables

of x_1 that may be set, a necessary condition for an optimal solution is that x_2 must maximize the function for that given level of x_1. Thus we can solve the problem

$$\max y = 4x_1 + 2x_2 - x_1^2 - x_1^2 + x_1x_2 \quad \text{s.t.} \quad 0 \leq x_2 \leq 2.67$$

with respect to x_2 taking x_1 as given. From the first-order condition

$$2 - 2x_2 + x_1 = 0$$

this gives the solution value for x_2 as a function of x_1

$$x_2 = 1 + 0.5x_1$$

Substituting for x_2 in the function, we then solve

$$\begin{aligned} \max y &= 4x_1 + 2[1 + 0.5x_1] - x_1^2 - [1 + 0.5x_1]^2 + x_1[1 + 0.5x_1] \\ &= 1 + 5x_1 - 0.75x_1^2 \quad \text{s.t.} \quad 0 \leq x_1 \leq 1 \end{aligned}$$

This maximum is clearly achieved at $x_1^* = 1$, giving the corresponding value $x_2^* = 1 + 0.5x_1^* = 1.5$.

Diagramatically the first step in this procedure amounts to finding the locus of points of tangency of the contours of the function with the vertical lines corresponding to each value of x_1. This locus is denoted LL in figure 12.7. Then the intersection of this locus with the line drawn at $x_1 = 1$ gives the overall solution. ■

Discriminating Monopoly with an Output Quota

Suppose that a monopoly supplies two countries, its own and a foreign country. The inverse-demand functions are

$$\begin{aligned} p_1 &= 100 - q_1 \\ p_2 &= 80 - 2q_2 \end{aligned}$$

and its total-cost function is

$$C = (q_1 + q_2)^2$$

Thus we assume that there are no cost differences in supplying the two countries. It is also possible to prevent arbitrage between the markets without cost. Then, if there are no further constraints, just as in the earlier case of discriminating monopoly we can solve for the profit-maximizing outputs and prices in the two countries by solving

$$\max \pi(q_1, q_2) = 100q_1 + 80q_2 - q_1^2 - 2q_2^2 - (q_1 + q_2)^2$$

We proceed directly to the solution, which gives

$$q_1^* = 22, \quad q_2^* = 6, \quad p_1^* = \$78, \quad p_2^* = \$68, \quad \pi^* = \$1{,}340$$

We can conclude that country 2 has the higher demand elasticity at the optimum.

Now suppose that the government of country 2, the foreign country, accuses the firm of "dumping," because it is selling in the foreign market at a lower price than in its home market. In retaliation, it imposes a quota of 4 on imports of the good. What is the impact on the firm's outputs in both markets, as well as on its profits? (The latter will determine how much it can afford to spend on lobbying to get the decision reversed.) We therefore have to solve the problem

$$\max \pi(q_1, q_2) \quad \text{s.t.} \quad q_2 \leq 4$$

where the profit function is as given above. Note that strictly speaking, we should also impose nonnegativity conditions, but having solved the unconstrained problem, we can be pretty sure that these will not be binding at the solution. Applying theorem 12.7 gives the conditions

$$100 - 4q_1 - 2q_2 = 0$$
$$80 - 2q_1 - 6q_2 \geq 0 \quad \text{and} \quad (4 - q_2)(80 - 2q_1 - 6q_2) = 0$$

First, we show that we must have $q_2 = 4$. If $q_2 < 4$, then the second condition implies that

$$80 - 2q_1 - 6q_2 = 0$$

But we already know that this condition, together with the first, gives a value of $q_2 = 6$, and so we can rule out this possibility. Thus $q_2 = 4$, its quota, and inserting this into the first condition and solving for q_1 gives the overall solution

$$q_1^* = 23, \quad q_2^* = 4, \quad p_1^* = \$77, \quad p_2^* = \$72, \quad \pi^* = \$1{,}330$$

So in country 1 output increases and price falls, while in country 2 price rises and output falls, which certainly makes that country's consumers worse off. The firm also loses a little profit. These effects are illustrated in figure 12.8.

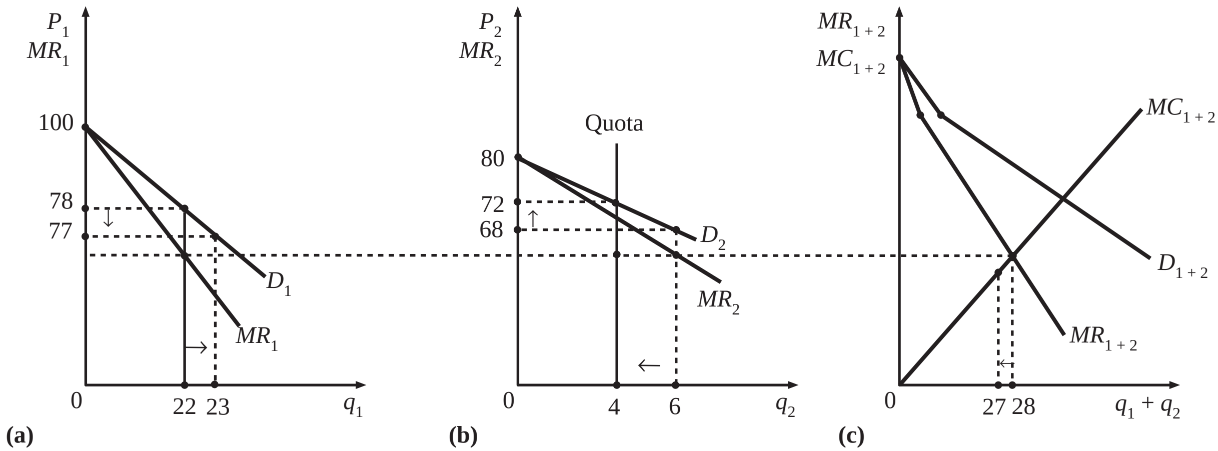


Figure 12.8 Effect of a quota on the discriminating monopoly solution

The least obvious of these results is the effect in country 1. It is an exercise to show that if marginal costs were constant, there would have been no effect on country 1. However, because marginal costs are increasing, the reduced output in country 2 reduces marginal cost below the level of marginal revenue in country 1, and so it is profitable to expand output there.

What happens to the condition on equality of marginal revenues that we saw earlier was a key aspect of the (unconstrained) price-discriminating solution? The marginal revenues in the two markets are given by

$$R_1'(q_1^*) = 100 - 2q_1^* = \$54$$
$$R'(q_2^*) = 80 - 4q_2^* = \$64$$

Thus, as we expected from the fact that the quota is a binding constraint, the firm would like to switch output from country 1 to country 2.

Finally, we note that the marginal profitability of output in market 2 is positive, since we have

$$\pi_2(q_1^*, q_2^*) = 80 - 2q_1^* - 6q_2^* = 10 > 0$$

We call this the **shadow price** of the output quota to the firm, since it represents the rate at which profit increases with a small relaxation of the output quota. Note that the fact that this shadow price is positive explains why the quota is a binding constraint, as the conditions given above show.

EXERCISES

1. Solve the following problems:

   (a) min $y = 3x_1^2 + 2x_2^2 + 5$ subject to $0 \leq x_1 \leq 10, \quad 2 \leq x_2 \leq 10$

   (b) max $y = 2x_1 + x_2 - 3x_1^2 - 4x_2^2 + x_1x_2$ subject to $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$

   (c) max $y = 2x_1 + x_2 - 3x_1^2 - 4x_2^2 + x_1x_2$ subject to $1 \leq x_1 \leq 2$, $1 \leq x_2 \leq 2$

   (d) max $y = 2x_1 + x_2 - 3x_1^2 - 4x_2 + x_1x_2$ subject to $0 \leq x_1 \leq 1$, $1 \leq x_2 \leq 2$

   (e) max $y = 2x_1^3 - 3x_1x_2 + x_1^2 - 2x_2^2$ subject to $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$

   (f) max $y = 2x_1^3 - 3x_1x_2 + x_1^2 - 2x_2^2$ subject to $0 \leq x_1 \leq 1$, $1 \leq x_2 \leq 2$

2. In the model of two-plant monopoly with capacity constraints, assume the demand function changes to $p = 200 - (q_1 + q_2)$. Find and discuss the new profit-maximizing solution. What is the shadow price of a small increase in capacity of plant 2? of plant 1?

3. In the model of discriminating monopoly with an output quota, show that if marginal cost of production is constant there is no effect of the imposition of the quota in country 2 on price and output in country 1. Explain this result.

4. Explain why, if a local maximum occurs at $x = a$ with $f'(a) = 0$, or at $x = b$ with $f'(b) = 0$, then *both* the conditions in theorem 12.7 hold. Show a similar result for theorem 12.8.

CHAPTER REVIEW

Key Concepts

Cournot duopoly
extreme values
saddle point
shadow price
stationary values

Review Questions

1. State and explain necessary conditions for a point $(x_1^*, \ldots, x_n^*)$ to yield a stationary value of a function f over its domain $\mathbb{R}^n$.

2. What types of points make up the set of stationary points?

3. State and explain sufficient conditions for a point $(x_1^*, \ldots, x_n^*)$ to yield a local maximum of a function f over its domain $\mathbb{R}^n$.

4. State and explain sufficient conditions for a point $(x_1^*, \ldots, x_n^*)$ to yield a local minimum of a function f over its domain $\mathbb{R}^n$.

5. Explain why these second-order conditions cannot be framed in terms of the second-order partials f_{ii} only.

6. Explain what is meant by the *saddle point* of a function.

7. State and explain necessary conditions for a point $(x_1^*, \ldots, x_n^*)$ to yield a local maximum of the function f subject to the interval constraints $a_i \leq x_i \leq b_i$, $i = 1, \ldots, n$.

8. State and explain necessary conditions for a point $(x_1^*, \ldots, x_n^*)$ to yield a local minimum of the function f subject to the interval constraints $a_i \leq x_i \leq b_i$, $i = 1, \ldots, n$.

Review Exercises

1. Find the stationary values of the following functions and use the second-order conditions to determine which give maxima, minima, or saddle points:

(a) $y = 0.5x_1^2 + 2x_2^2$

(b) $y = x_1 + x_2 - x_1^2 - x_2^2 + x_1x_2$

(c) $y = 10x_1 + 2x_2 - 0.5x_1^2 - 2x_2^2 + 5x_1x_2$

(d) $y = 2x_1x_2 - x_1^3 - x_2^2$

(e) $y = x_1^3 + x_2^3 - 4x_1x_2$

(f) $y = x_1x_2 + 2/x_1 + 4/x_2$

(g) $y = 2x_1^2 - 4x_2^2$

2. Find the maxima or minima of each of the functions in question 1 subject to the constraints

$$0 \leq x_1 \leq 10 \qquad 1 \leq x_2 \leq 20$$

3. A firm is considering bidding for the franchise to sell cola and hot dogs at a baseball stadium. It estimates the demand functions for cola and hot dogs respectively as

$$D_C = 20 - 4p_C - p_H$$
$$D_H = 15 - p_C - 5p_H$$

where D_C is demand for cola in thousands (of cans), D_H is demand for hot dogs in thousands, p_C is the price of a can of cola in dollars, and p_H is the price of a hot dog. The unit cost of supplying a hot dog is constant at \$0.1, and the unit cost of a can of cola is likewise constant at \$0.5.

(a) Find the upper limit to the amount the firm would bid for the franchise.

(b) Interpret the demand functions, in particular the cross-demand effects.

(c) Suppose that cola and hot dogs must be supplied by two *separate* firms. What modeling issues arise in analyzing the franchise bids of the two firms, and how would you deal with them?

4. A company owns an inventory of 100 units of a good. It must sell the entire inventory over the next three periods. The profit function for sales within any one period is

$$\pi(x_t) = 50x_t - 0.5x_t^2, \qquad t = 1, 2, 3$$

It wishes to maximize the *present value of profit*

$$V = \pi(x_1) + \beta\pi(x_2) + \beta^2\pi(x_3)$$

where $\beta = 0.8$ is its discount factor. Find the optimal values for x_1, x_2, and x_3; illustrate and discuss this solution. [Hint: Use the condition $x_1 + x_2 + x_3 = 100$ to substitute for x_3 in the V function and then maximize with respect to x_1 and x_2.]

5. A monopolist supplies two markets, one at home, the other abroad. The demand functions are

$$q_1 = 10 - p_1, \quad q_2 = 5 - 0.5p_2$$

where q_1 denotes home sales and q_2 foreign sales. The firm's total-cost function is

$$C = 0.5(q_1 + q_2)^2$$

(a) Find its profit-maximizing output and prices (no arbitrage between the markets is possible).

(b) Suppose now that price regulation is imposed in the home market, in the form of a maximum price of \$$b$. What is the effect of this on prices, outputs, and profit? Illustrate and explain your results.

Chapter 13 Constrained Optimization

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Farmer's Land Allocation
- Numerical Version of the Land-Allocation Problem: Example
- Consumer Demand Functions with Cobb-Douglas Utility
- Long-Run Cost Function for a Firm with a Cobb-Douglas Production Function
- Cost-Minimization with a CES Production Function
- Numerical Version of the CES Cost Function Problem: Example
- Optimization with More Than One Constraint
- Points Rationing: Example
- Constraints in Points Rationing
- Second-Order Conditions for the Land-Allocation Problem: Example
- Second-Order Conditions for the Cost-Minimization Problem: Example
- Second-Order Conditions with a Cobb-Douglas Production Function: Example

If, when maximizing or minimizing a function, we are free to consider any value of an x-variable on the real line as a possible solution, then the problem is said to be unconstrained. Most of the techniques developed in chapters 6 and 12 related to this case. In many, probably most, economic problems, however, there exist one or more *constraints* which restrict the set of x-values we are allowed to consider as possible solutions. We already examined one type of constraint in chapters 6 and 12, namely that where the x-values are restricted to lie in some interval. The examples we examined in those chapters showed that such restrictions arise naturally in economic problems and have important effects on the solution. In this chapter we develop techniques for dealing with another type of constraint, that in which the x-variables are restricted to a set of values that satisfy one or more functional equations. The main topic is the derivation and application of the method of **Lagrange multipliers.**

13.1 Constrained Problems and Approaches to Solutions

Suppose that we wish to maximize a function $f(x_1, x_2)$, where $f \in C^2$ is a strictly concave function of the type drawn in figure 12.1 (a) in the previous chapter. In the absence of constraints, we know from theorem 12.1 that the solution (x_1^*, x_2^*) satisfies the conditions $f_i(x_1^*, x_2^*) = 0,\ i = 1, 2$. Now suppose that we impose the constraint

$$g(x_1, x_2) = 0$$

where $g \in C^2$ also. This means that we are only allowed to consider as possible solutions to the problem those pairs of (x_1, x_2)-values which satisfy that equation.

In figure 13.1 the function f, is illustrated (a) in three dimensions and (b) by the associated level curves. The curve G in each part of the figure represents the set of (x_1, x_2)-pairs that satisfy a constraint such as the one above. The problem of maximizing the function subject to the constraint is, stated diagrammatically, the problem of finding a point on the curve G which gives the highest possible value of the function. The answer can be seen in figure 13.1 (a), but it emerges more clearly in figure 13.1 (b). The level curve labeled f^* is clearly the highest level curve of f that can be reached while remaining on the curve G, and so the point (x_1^*, x_2^*) is the solution to the problem: it gives the highest value of the function consistent with satisfying the constraint.

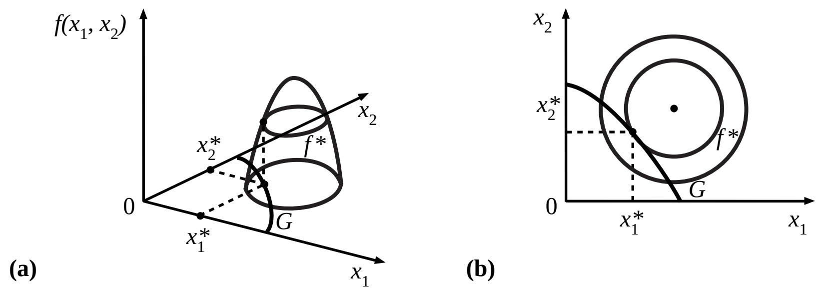


Figure 13.1 Finding the point on the constraint curve that gives the highest function value

The most obvious feature about the point (x_1^*, x_2^*) is that it is a point of tangency between the curve G and the level curve f^*. Given the shapes of the level curves of f and the curve G, it should be clear that any solution to the problem must be at a point of tangency. This could be tested by shifting the curve G slightly while keeping its general shape.

Suppose now that we want to express this diagrammatic solution algebraically—that would certainly simplify the problem of finding the numerical values of the solution. We know (recall section 11.3) that we can write the slope of a level curve of $f(x_1, x_2)$ as

$$\frac{dx_2}{dx_1} = -\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

and the slope of the curve G as

$$\frac{dx_2}{dx_1} = -\frac{g_1(x_1, x_2)}{g_2(x_1, x_2)}$$

Since at a point of tangency these slopes must be equal, we have that at (x_1^*, x_2^*),

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$

Note that this is only one equation in two unknowns and so in itself cannot give us a solution for the values x_1^* and x_2^*. The second equation we need is given by the constraint

$$g(x_1^*, x_2^*) = 0$$

since if x_1^* and x_2^* are optimal, then they must be feasible and so satisfy the constraint. These two equations taken together allow us to solve for the values x_1^* and x_2^*.

Theorem 13.1 If x_1^* and x_2^* is a tangency solution to the constrained maximization problem

$$\max f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = 0$$

then we have that x_1^* and x_2^* satisfy

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$
$$g(x_1^*, x_2^*) = 0$$

Although the diagrammatic approach is suggestive, we now study the problem algebraically. Consider again the constraint $g(x_1, x_2) = 0$, and assume that it can be solved to give, say, x_2 as a function of x_1:

$$x_2 = \gamma(x_1)$$

This is, in fact, simply the function which gives the curve G in figure 13.1. We know from the presentation of implicit differentiation (theorem 11.2) that the derivative of this function is

$$\frac{dx_2}{dx_1} = \gamma'(x_1) = -\frac{g_1(x_1, \gamma(x_1))}{g_2(x_1, \gamma(x_1))}$$

Now, if we substitute $\gamma(x_1)$ for x_2 in the function f, we are left with the unconstrained problem in one variable:

$$\max f(x_1, x_2) = \max f(x_1, \gamma(x_1)) = \max \phi(x_1)$$

We maximize the function ϕ by differentiating and setting the derivative equal to zero:

$$\phi' = f_1(x_1^*, \gamma(x_1^*)) + f_2(x_1^*, \gamma(x_1^*))\gamma'(x_1^*) = 0$$

Then, rearranging this, substituting for $\gamma'(x_1^*)$, and noting that feasibility requires that $x_2^* = \gamma(x_1^*)$ gives

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$

which is the tangency condition of theorem 13.1.

We now set out the *Lagrange multiplier technique* for solving constrained optimization problems, and we shall justify it by showing that it is equivalent to the two approaches we have just examined. We proceed by introducing a new variable, λ, the **Lagrange multiplier,** and by forming the **Lagrange function** or **Lagrangean**

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

We then find a stationary point of $\mathcal{L}$ with respect to x_1, x_2, and λ. (Later we show that this must be a **saddlepoint** of the Lagrangean) This gives the conditions

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1(x_1^*, x_2^*) + \lambda^* g_1(x_1^*, x_2^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = f_2(x_1^*, x_2^*) + \lambda^* g_2(x_1^*, x_2^*) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0$$

We have the three equations to solve for the three unknowns, x_1^*, x_2^*, and λ^*. The justification for the use of the Lagrange method is that these x-values are precisely the solutions to the original constrained maximization problem we are interested in. To see this, rewrite the first two conditions as

$$f_1(x_1^*, x_2^*) = -\lambda^* g_1(x_1^*, x_2^*)$$
$$f_2(x_1^*, x_2^*) = -\lambda^* g_2(x_1^*, x_2^*)$$

Then taking ratios of the left-hand and right-hand sides respectively gives

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$

which is precisely the tangency condition we obtained earlier. Thus eliminating the Lagrange multiplier from the first two conditions in this way gives us the tangency condition, and then the third condition is simply the constraint so that we can solve to obtain the same solution as before. One way of looking at the Lagrange multiplier procedure is as a way of delivering the tangency conditions for an optimal solution.

We can summarize this by

Definition 13.1

The **Lagrange method** of finding a solution (x_1^*, x_2^*) to the problem

$$\max f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = 0$$

consists of deriving the following first-order conditions to find the stationary point(s) of the Lagrange function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

which are

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1(x_1^*, x_2^*) + \lambda^* g_1(x_1^*, x_2^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = f_2(x_1^*, x_2^*) + \lambda^* g_2(x_1^*, x_2^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0$$

How can we be sure that for some arbitrarily given problem there is always a unique solution point at a tangency of the constraint curve and level curve of f, and that the Lagrange procedure always works in that it delivers this point? Exactly what role is played by the assumptions on the shapes of the level curves of the functions f and g? These questions will be considered later, in section 13.3. Here we will simply show the usefulness of the procedure by applying it to a number of examples.

Example 13.1 Solve the constrained maximization problem

$$\max y = x_1^{0.25} x_2^{0.75} \quad \text{s.t.} \quad 100 - 2x_1 - 4x_2 = 0$$

Solution

The Lagrange function is

$$\mathcal{L} = x_1^{0.25} x_2^{0.75} + \lambda(100 - 2x_1 - 4x_2)$$

and the first-order conditions are

$$0.25 x_1^{-0.75} x_2^{0.75} - 2\lambda = 0 \tag{13.1}$$

$$0.75 x_1^{0.25} x_2^{-0.25} - 4\lambda = 0 \tag{13.2}$$

$$100 - 2x_1 - 4x_2 = 0 \tag{13.3}$$

Solving equations (13.1) and (13.2) to eliminate λ gives

$$x_2 = \frac{3}{2} x_1$$

and substituting this into equations (13.3) gives the solution

$$x_1^* = \frac{600}{48}, \quad x_2^* = \frac{300}{16}$$

■

The Student's Time-Allocation Problem

In question 9 of the review exercises at the end of chapter 6, we presented the following problem. A student wishes to allocate her available study time of 60 hours per week between two subjects in such a way as to maximize her grade average.

We can formulate the problem as

$$\max \frac{g_1(t_1) + g_2(t_2)}{2} \quad \text{s.t.} \quad 60 - t_1 - t_2 = 0$$

where t_i is time spent studying subject $i = 1, 2$ and the g_i functions give the expected grade as a function of study time:

$$g_1 = 20 + 20\sqrt{t_1}, \quad g_2 = -80 + 3t_2$$

This is a constrained optimization problem. In chapter 6 we suggested solving it by using the constraint to eliminate t_2. We can write

$$t_2 = 60 - t_1$$

and then solve the unconstrained problem in one variable.

In terms of our discussion in this chapter, $60 - t_1$ is the equivalent of the function γ.

We will now solve the problem using the Lagrange multiplier method. The Lagrange function is

$$\mathcal{L} = \frac{20 + 20\sqrt{t_1} - 80 + 3t_2}{2} + \lambda(60 - t_1 - t_2)$$

The first-order conditions are

$$\frac{\partial L}{\partial t_1} = \frac{10t_1^{-1/2}}{2} - \lambda = 0$$
$$\frac{\partial L}{\partial t_2} = \frac{3}{2} - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 60 - t_1 - t_2 = 0$$

Substituting for λ from the second condition into the first gives

$$5t_1^{-1/2} = 3/2$$

implying that

$$t_1^* = \left(\frac{10}{3}\right)^2 = 11.11$$

Then substituting for t_1 in the constraint gives

$$t_2^* = 60 - 11.11 = 48.89$$

The conditions we obtained under the Lagrange approach suggest some interesting interpretations of the results. We can interpret λ as the value of the *marginal contribution to the grade average* from studying each subject *at the optimal solution*. (Later in this section we show how λ is to be interpreted generally in constrained optimization problems; see theorem 13.2.) Thus note that $3/2$ is the contribution to the grade average made by allocating an extra bit of time to studying subject 2, and this is constant because the function g_2 is linear. Likewise the derivative $g_1' = 5t_1^{-1/2}$ is the marginal contribution to the grade average from studying subject 1, and since g_1 is a concave function, this marginal contribution decreases as t_1 increases. Then, at the optimum time allocation, these two marginal time allocations are equal to each other and to λ:

$$\lambda^* = 5(11.1)^{-1/2} = \frac{3}{2}$$

Figure 13.2 illustrates. The distance OT on the horizontal axis represents total time available (60 hours) so t_1 is measured rightward from 0, and t_2 is measured leftward from T. The optimal allocation is characterized by the condition that the *last little bit of time spent studying each subject makes the same contribution to the grade average*. Diverting a small amount of time from one subject to the other would leave the grade average just about unchanged.

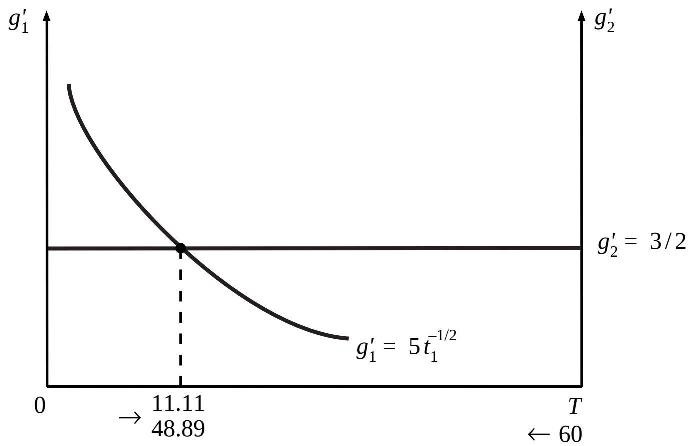


Figure 13.2 Optimal allocation of study time

Note that the individual subject grades at the optimal allocation turn out to be quite different

$$g_1 = 20 + 20\sqrt{11.11} = 86.67$$
$$g_2 = -80 + 3(48.89) = 66.67$$

The student ends up with a worse grade in subject 2, even though she spends more time studying it. What matters, however, is the contribution to grade average *at the margin* made by studying each subject (see question 2 of the exercises at the end of this section).

In the discussion so far, we have considered only maximization problems. The case of minimization problems, however, is treated similarly. In figure 13.3 we show a strictly convex function $f(x_1, x_2)$ and a constraint curve G corresponding to the constraint $g(x_1, x_2) = 0$. The problem is to reach the lowest possible point on the function while remaining on the constraint curve. The solution is clearly at point (x_1^*, x_2^*), a point of tangency. In fact the level curve diagram looks just like that in the maximization problem, the difference being that the value of the function falls as we move inward toward the point T. Thus, exactly as before, we can show that the optimal point must satisfy the tangency conditions

$$\frac{f_1(x_1^*, x_2^*)}{f_2(x_1^*, x_2^*)} = \frac{g_1(x_1^*, x_2^*)}{g_2(x_1^*, x_2^*)}$$
$$g(x_1^*, x_2^*) = 0$$

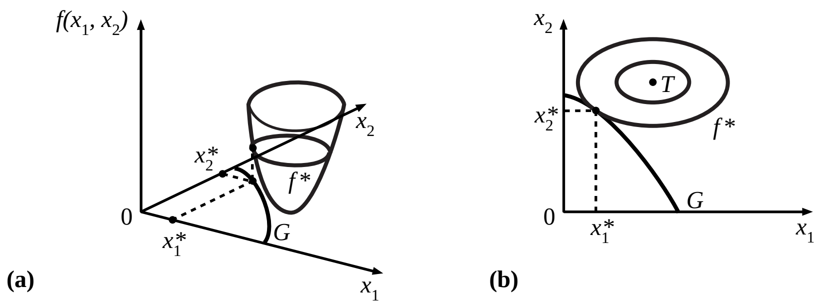


Figure 13.3 Constrained minimization

Therefore, for the case of constrained minimization, we have

Definition 13.2 The Lagrange method for finding a solution to the problem

$$\min f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = 0$$

is to derive conditions for a stationary point of the Lagrange function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

which are

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1(x_1^*, x_2^*) + \lambda^* g_1(x_1^*, x_2^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = f_2(x_1^*, x_2^*) + \lambda^* g_2(x_1^*, x_2^*) = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0$$

Since the first-order conditions for maximization and minimization problems are identical, as they were in the unconstrained case, this tells us that we need to consider second-order conditions. We study second-order conditions in section 13.2.

Example 13.2 Solve the following constrained minimization problem:

$$\min y = x_1 + x_2 \quad \text{s.t.} \quad 1 - x_1^{1/2} - x_2 = 0$$

Solution

The Lagrange function is

$$\mathcal{L} = x_1 + x_2 + \lambda\left(1 - x_1^{1/2} - x_2\right)$$

The first-order conditions are

$$1 - \left(\frac{\lambda}{2}\right) x_1^{-1/2} = 0$$
$$1 - \lambda = 0$$
$$1 - x_1^{1/2} - x_2 = 0$$

which solve to give

$$x_1^* = \frac{1}{4}, \quad x_2^* = \frac{1}{2}, \quad \lambda^* = 1$$

■

The Dual Consumer Problem with Cobb-Douglas Utility

In an earlier example of consumer-demand functions with Cobb-Douglas utility, we examined the problem of maximizing a consumer's utility function subject to a budget constraint, and derived the corresponding demand functions. We now examine a closely related problem, which is called *the dual* to the previous problem. In this, we take a *fixed value* of the utility function, and *minimize* the expenditure required to achieve it. That is, we solve

$$\min\ e = p_1x_1 + p_2x_2 \quad \text{s.t.} \quad u(x_1, x_2) = \bar{u}$$

where e is expenditure and $\bar{u}$ the required utility level. Note that we are still choosing quantities x_1 and x_2, taking prices as given.

We can interpret this problem by thinking of the value of $\bar{u}$ as representing a given *standard of living*. We are then asking: Given the prices of the goods, what is the cheapest way to achieve this standard of living?

We again assume the consumer's preferences are described by a Cobb-Douglas utility function $\bar{u} = x_1^{\alpha}x_2^{1-\alpha}$, and so the consumer's *expenditure-minimization problem* is

$$\min\ p_1x_1 + p_2x_2 \quad \text{s.t.} \quad \bar{u} - x_1^{\alpha}x_2^{1-\alpha} = 0$$

At this point, we note that we can save ourselves a little work. This problem is in fact identical to that solved in the example of long-run cost function for a firm with Cobb-Douglas production function. We just have to replace K by x_1, L by x_2, r by p_1, w by p_2, y by u and β by $1-\alpha$. We are clearly asking the same kind of question: x_1 and x_2 are "inputs" into the utility function, and we want to find the lowest-cost way of "producing" the given utility level. So we can go directly to the results of the constrained-minimization problem we obtained in that example:

$$x_1 = \left(\frac{\alpha}{1-\alpha}\right)^{(1-\alpha)}\left(\frac{p_2}{p_1}\right)^{(1-\alpha)}\bar{u}$$

$$x_2 = \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\left(\frac{p_1}{p_2}\right)^{\alpha}\bar{u}$$

These are *compensated-demand functions* for the two goods, since they show how quantities consumed vary with prices (and utility). They are clearly different from the *uncompensated-demand functions* we obtained in the earlier consumer demand example, which for ease of comparison we write out again here:

$$x_1 = \frac{\alpha m}{p_1}$$

$$x_2 = \frac{(1-\alpha)m}{p_2}$$

where m is the consumer's income. One important difference is that the demand functions in the dual problem include *both* prices, in a way that suggests that what really matters is the ratio of the prices.

To see the reason for this, consider figure 13.4. In this figure we see the effect of a fall in price of good 1, from p_1^0 to p_1^1. If we are constrained to achieve the same utility level, then after the price-change expenditure must change by just enough to allow the initial indifference curve u^0 to be reached. Therefore the budget line slides around u^0, causing demand for x_1 to change from x_1^0 to x_1^1. This is said to be a *compensated-price effect* because expenditure has changed by enough to keep utility constant. If, on the other hand, income (= expenditure) is kept unchanged, then the budget line rotates outward to B^1 as a result of the price change and a new utility level u^1 is achieved, with a change in demand to x_1^2. In this case, we have an *uncompensated-price effect*. Thus the first demand function for x_1 above gives us the compensated effect, while the second demand function for x_1 gives us the uncompensated effect.

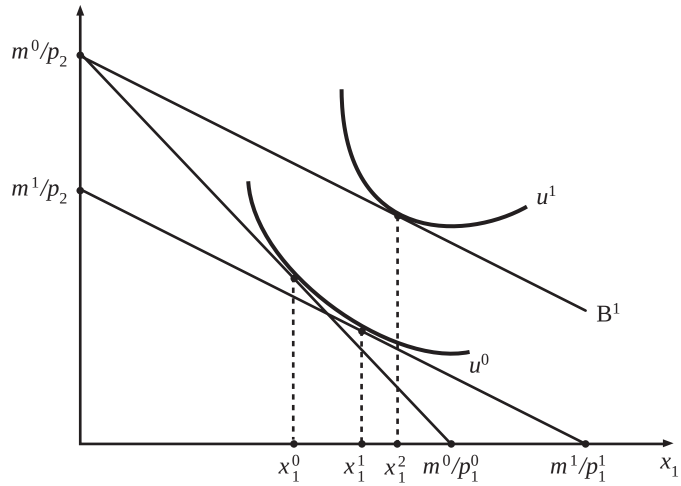


Figure 13.4 Compensated and uncompensated demand changes

Just as we derived the cost function for the firm, so here we can derive a cost function for the consumer. This is usually called the *expenditure function*, since it shows the amount of expenditure required to achieve a given utility level, at given prices, as a function of that utility level and those prices. Therefore, by substituting the compensated demand functions for x_1 and x_2 into the expression $e = p_1x_1 + p_2x_2$, and rearranging, we obtain the expenditure function

$$e = ap_1^{\alpha}p_2^{(1-\alpha)}\bar{u}$$

where $a = (\alpha/(1-\alpha))^{(1-\alpha)} + ((1-\alpha)/\alpha)^{\alpha}$.

The Interpretation of λ

We apparently introduced λ as an artifact to help us generate the conditions we know give the solution. It turns out, however, that λ has a very important and interesting economic interpretation in all constrained optimization problems. It is sufficiently important to summarize it in

Theorem 13.2 The value of the Lagrange multiplier λ at the optimal solution always tells us the effect on the optimized value of the function f of a small relaxation of the constraint.

We will prove this important theorem in section 14.3.

EXERCISES

1. Solve the following constrained maximization and minimization problems, illustrating your answer with diagrams in each case:

(a) max $y = 2x_1 + 3x_2$ subject to $2x_1^2 + 5x_2^2 = 10$

(b) max $y = x_1^{0.25}x_2^{0.75}$ subject to $2x_1^2 + 5x_2^2 = 10$

(c) min $y = 2x_1 + 4x_2$ subject to $x_1^{0.25}x_2^{0.75} = 10$

(d) max $y = (x_1 + 2)(x_2 + 1)$ subject to $x_1 + x_2 = 21$

2. **(a)** max $y = 2x_1 + 4x_2 - x^2 - 0.5x_2^2 - 2x_1x_2$ subject to $2x_1 + x_2 = 10$

(b) max $y = x_1x_2$ subject to $x_1^2 + x_2^2 = 16$

(c) max $y = x_1^2 + x_2^2$ subject to $(x_1^2/25) + (x_2^2/9) = 1$

(d) min $y = 2x_1 + x_2$ subject to $(0.2x_1^{-0.5} + 0.8x_2^{-0.5})^{-2} = 1$

3. In the example on farmer's-land allocation in this section, solve the problem in terms of outputs rather than land allocations. Draw the level curves of the net-revenue function in (y_1, y_2)-space, and illustrate the solution as a point of tangency. Show what would happen diagrammatically if the net return to output 1 rose relative to that of output 2.

4. *Short-run cost minimization.* In the model of the cost-minimizing firm in this section, suppose that the firm has a fixed amount of capital $K = K^0$ available. Solve its cost-minimization problem, derive its cost functions, discuss their properties, and compare them with the long-run functions derived in the example.

5. The consumer has the utility function

$$u = (x_1 - c_1)^{\alpha}(x_2 - c_2)^{1-\alpha}, \qquad a, b, > 0$$

where c_1 and c_2 are interpreted as minimum amounts of the good required for subsistence. Derive her demand functions, discuss their properties, and compare them with the demand functions derived in the Cobb-Douglas example.

6. Derive the compensated-demand functions and the expenditure function for the utility function in question 5 and compare to the results obtained in the Cobb-Douglas example.

7. Consider the cost-minimization problem

$$\min C = rK + wL \quad \text{s.t.} \quad \bar{y} = f(K, L)$$

Show that, at the optimum, λ^* is equal to marginal cost.

8. In the example of the student's time allocation in this section, draw the level curves of the grade-average function

$$\bar{g} = \frac{20 + 20\sqrt{t_1} - 80 + 3t_2}{2}$$

and illustrate the solution as a point of tangency.

13.2 Second-Order Conditions for Constrained Optimization

We saw in the previous section that the first-order conditions for a maximum and a minimum of a constrained problem are identical, as in the unconstrained case, and so it again becomes necessary to look at second-order conditions. One approach to these is *global:* assumptions are built into the economic model to ensure that the objective function and the constraint function(s) have the right general shape. As we will see in section 13.3, it is sufficient for a maximum (minimum) that the objective function be quasiconcave (-convex) and that the constraint function(s) defines a convex set. However, for some purposes, particularly comparative statics (discussed in the next chapter), it is useful to have the second-order conditions in *local* form, in terms of small deviations around the optimal point.

In section 12.2 we saw that the local second-order conditions in the unconstrained case could be expressed in terms of the signs of leading principal minors of the Hessian determinant of the function being optimized. In the constrained case there is a similar, though more complex, procedure. We will not derive the conditions rigorously here but simply state and explain them.

Take first the simplest possible case of a two-variable, one-constraint problem

$$\max f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = 0$$

The Lagrange function is

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

Suppose that the point $(x_1^*, x_2^*, \lambda^*)$ yields a stationary value of the Lagrange function, so we have that

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_1(x_1^*, x_2^*) + \lambda^* g_1(x_1^*, x_2^*) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = f_2(x_1^*, x_2^*) + \lambda^* g_2(x_1^*, x_2^*) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) = 0$$

We now derive the Hessian matrix of the Lagrange function

$$H^* = \begin{bmatrix} f_{11} + \lambda^* g_{11} & f_{12} + \lambda^* g_{12} & g_1 \\ f_{21} + \lambda^* g_{21} & f_{22} + \lambda^* g_{22} & g_2 \\ g_1 & g_2 & 0 \end{bmatrix}$$

It is to be understood that the partial derivatives in this matrix are all evaluated at the point $(x_1^*, x_2^*, \lambda^*)$, which is why the matrix has a '*' superscript. In the case where the constraint function is linear (i.e., the g_{ij} are zero) as in the standard consumer problem, this matrix is simply the Hessian of the objective function *bordered* by the vector $[g_1 \;\; g_2 \;\; 0]$.

Theorem 13.3 If $(x_1^*, x_2^*, \lambda^*)$ gives a stationary value of the Lagrange function $\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$, then

(i) it yields a maximum if the determinant of the bordered Hessian $|H^*| > 0$, and
(ii) it yields a minimum if the determinant of the bordered Hessian $|H^*| < 0$.

Example 13.3 Establish that the solution to example 13.1 is a true maximum.

Solution

The Hessian is

$$H = \begin{bmatrix} -0.1875x_1^{-1.75}x_2^{0.75} & 0.1875x_1^{-0.75}x_2^{-1.75} & -2 \\ 0.1875x_1^{-0.75}x_2^{-1.75} & -0.1875x_1^{0.25}x_2^{-1.25} & -4 \\ -2 & -4 & 0 \end{bmatrix}$$

At the optimal solution, $x_1^* = 600/48 = 12.5$ and $x_2^* = 300/16 = 18.75$, we have

$$H^* = \begin{bmatrix} -0.0003 & 0.0002 & -2 \\ 0.0002 & -0.0904 & -4 \\ -2 & -4 & 0 \end{bmatrix}$$

so

$$|H^*| = 0.6896 > 0$$

and we have a maximum. ■

Second-Order Conditions in the Consumer Problem

Suppose that we have the problem

$$\max u(x_1, x_2) \quad \text{s.t.} \quad m - p_1x_1 - p_2x_2 = 0$$

where u is a utility function and the budget constraint is a special, linear form of the constraint function g. We are by now familiar with the first-order conditions, so we proceed directly to the second-order conditions. The condition on the bordered Hessian determinant is

$$|H^*| = \begin{vmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = -p_2^2u_{11} + 2p_1p_2u_{12} - p_1^2u_{22} > 0$$

(where we have used the fact that $u_{12} = u_{21}$). Now recall from the first-order conditions that $p_1 = u_1/\lambda$, $p_2 = u_2/\lambda$. Then substituting into the condition above gives

$$-\frac{u_2^2u_{11} - 2u_1u_2u_{12} + u_1^2u_{22}}{\lambda^2} > 0$$

Multiplying through by $-\lambda^2$ gives the condition

$$u_2^2u_{11} - 2u_1u_2u_{12} + u_1^2u_{22} < 0$$

But this is just the condition that the utility function be strictly quasiconcave (see section 11.5). Thus we see that as the usual diagram confirms, the tangency point between an indifference curve and budget line is a true local maximum when the indifference curve is strictly convex to the origin; that is, the utility function is strictly quasiconcave.

So far in this section we have dealt only with problems having exactly two variables and one constraint. We now extend theorem 13.3 to the case of problems with $n \geq 2$ variables. Suppose that we have the problem

$$\max f(x_1, \ldots, x_n) \quad \text{s.t.} \quad g(x_1, \ldots, x_n) = 0$$

The Lagrange function is

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$$

The relevant bordered Hessian is now the $(n+1) \times (n+1)$ determinant

$$|H^*| = \begin{vmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \cdots & \mathcal{L}_{1n} & g_1 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & \mathcal{L}_{2n} & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{L}_{n1} & \mathcal{L}_{n2} & \cdots & \mathcal{L}_{nn} & g_n \\ g_1 & g_2 & \cdots & g_n & 0 \end{vmatrix}$$

We define the second-order conditions in terms of the principal minors of this determinant. Thus we have

Theorem 13.4 If the Lagrange function $f(x_1, \ldots, x_n) + \lambda g(x_1, \ldots, x_n)$ has a stationary value at $(x_1^*, \ldots, x_n^*, \lambda^*)$, then $(x_1^*, \ldots, x_n^*)$ solves

(i) $\max f(x_1, \ldots, x_n)$ s.t. $g(x_1, \ldots, x_n) = 0$ if the successive principal minors of $|H^*|$ alternate in sign in the following way

$$\begin{vmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & g_1 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} > 0, \quad \begin{vmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} & g_1 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} & g_2 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix} < 0, \ldots$$

with, $|H^*|$ itself therefore taking the same sign as $(-1)^n$

(ii) $\min f(x_1, \ldots, x_n)$ s.t. $g(x_1, \ldots, x_n) = 0$ if all the principal minors of $|H^*|$ are strictly negative.

Note that in both theorems 13.3 and 13.4 we give *sufficient* and not *necessary* conditions. A vector $(x_1^*, \ldots, x_n^*)$ may yield an optimal solution of the constrained problem and yet not satisfy these conditions. However, in applications of these theorems in economics, essentially to problems of comparative-statics, it is usually *assumed* that these sufficient conditions hold, and so we will not concern ourselves here with the pursuit of full mathematical generality.

EXERCISES

1. Take each of the optimization problems in question 1 of the exercises in section 13.1 and confirm that the answer you have derived is a true optimum.

2. Take the model of the student's time allocation in section 13.1 and confirm that the solution yields a maximum.

3. Take the general model of the farmer's land allocation in section 13.1 and confirm that the solutions yield a true constrained maximum.

13.3 Existence, Uniqueness, and Characterization of Solutions

In sections 13.1 and 13.2 we considered the problem of finding solutions to optimization problems, using the method of Lagrange multipliers, without really considering the question of whether, or under what conditions, this was really justified. In this section we answer this question.

Consider the problem of maximizing the differentiable function $f \in C^2$, and take the case where the function is strictly increasing (nothing essential in the propositions depends on this assumption, but it allows a familiar diagrammatic presentation). It is also convenient to express the idea of a constrained problem more generally than before: rather than introduce a functional constraint, we will simply talk in terms of the *feasible set* in the problem, which is the set of points from which we are allowed to choose a solution. By the definition of a constrained problem, this feasible set, denoted X, is a proper subset of the set $\mathbb{R}^n$, $n \geq 1$. We will take $n = 1$ and $n = 2$ for illustrative purposes.

The first, most fundamental issue is that of *existence:* how can we be sure that a solution to a given optimization problem exists? The answer is given by

Theorem 13.5 **(Weierstrass's theorem)** If f is a continuous function, and X is a nonempty, closed, and bounded set, then f has both a maximum and a minimum on X.

This theorem says that a solution to a maximization or a minimization problem is guaranteed to exist if the given conditions are satisfied. Note that the conditions are not *necessary:* a maximum or minimum may exist when they are not satisfied, but then again they may not exist, depending on the particular problem. Figure 13.5 illustrates the kind of problem that a discontinuity may present. In figure 13.5(a) the function has a discontinuity at x^0, and clearly no maximum or minimum exists in the set X as indicated. The requirement of continuity rules out this kind of case. Not all types of discontinuity create a problem however, as figure 13.5(b) shows. Here a maximum exists at x^* and a minimum at x_*. In most economic models, we usually assume that the function being maximized or minimized is differentiable, and since this implies continuity, this first condition is usually satisfied.

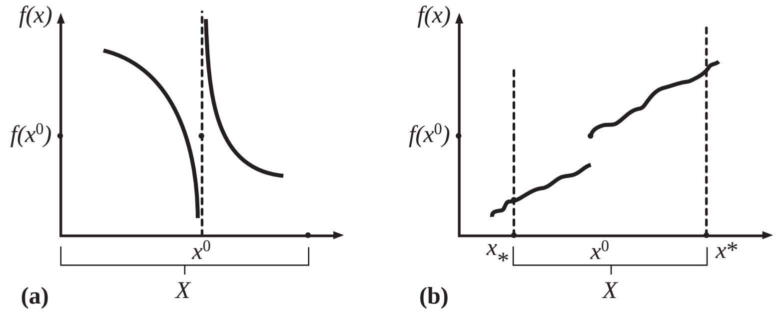


Figure 13.5 Discontinuities in f

Nonemptiness of X is a necessary condition for a solution to exist. If there are no x-values from which to choose, there *cannot* be a solution. If we think of the feasible set as being defined by some underlying constraints, then the set being empty is equivalent to these constraints not having a solution. For example, suppose that the constraints in a particular problem were

$$\begin{aligned} 3x_1 + 6x_2 &= 8 \\ x_1 + 2x_2 &= 4 \\ x_1 + x_2 &= 6 \end{aligned}$$

In this case there is no (x_1, x_2)-pair that satisfies these equations simultaneously. Diagrammatically the three lines defined by these equations do not intersect at one single point. Thus the feasible set is empty. (This incidentally illustrates the problem of having more constraints than variables, as alluded to in section 13.2.)

Recall from section 2.3 the definition of a bounded set: if X is a bounded set, then it is impossible to go to infinity in any direction while remaining within the set. To see what problems may arise when the feasible set is unbounded, consider the consumer maximization problem. Suppose that there are just two goods, and one of their prices is zero. Then an unlimited amount of that good can be "bought"—the budget set is unbounded. If f, the utility function, is increasing in the quantity of this free good, then the consumer will never be satisfied with a finite amount of the good. Therefore there is no solution to the problem.

The kind of problem that can arise when the feasible set is not closed is also easy to illustrate. Suppose that the problem is

$$\max y = 2x, \qquad 0 < x < 1$$

The feasible set X is the *open* interval (0, 1); the endpoints of the interval are not included in the set. The resulting values of y lie in the open interval (0, 2). There is no solution to the maximization problem in this case (nor to the corresponding minimization problem) because we can allow x to approach its upper bound of 1 as closely as we like without ever reaching it, and as we do so, the value of y increases toward 2 without ever reaching it. On the other hand, the problem

$$\max y = 2x, \qquad 0 \leq x \leq 1$$

has a solution at $x = 1$. In this case the feasible set contains its boundary points and is therefore closed. This is why strict inequalities are to be avoided in formulating optimization problems. In cases where the function increases as we move to the boundary of the feasible set, a solution will not exist if the boundary is not included in the set.

The formal basis for theorem 13.5 is the fact that if f is a continuous function and X a closed and bounded set, then the image set $f(X)$ is also closed and bounded. But this image set is a set of real numbers, and a closed and bounded set of real numbers possesses a maximum and a minimum.

From now on we will assume that the conditions for existence of a solution are satisfied in any problem we want to consider. The next important question is, given that a solution exists, will the Lagrange procedure actually give it, or, put differently, under what conditions will the Lagrange method work? Essentially the Lagrange method works if and only if it is possible to solve the first-order conditions for the Lagrange multipliers. This is summarized in the following:

Theorem 13.6 In the problem

$$\max f(\mathbf{x}) \quad \text{s.t.} \quad g^1(\mathbf{x}) = 0, \quad g^2(\mathbf{x}) = 0, \ldots, g^m(\mathbf{x}) = 0,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $m < n$, if $\mathbf{x}^*$ is a solution to the problem, and if the $n \times m$ matrix $G = [g_i^j(\mathbf{x}^*)]$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, has rank m, then there exist real numbers $\lambda_1, \ldots, \lambda_m$, such that $\mathbf{x}^*$ satisfies the $n + m$ conditions

$$f_i(\mathbf{x}^*) + \sum_j \lambda_j g_i^j(\mathbf{x}^*) = 0, \qquad i = 1, \ldots, n$$

$$g^j(\mathbf{x}^*) = 0, \qquad j = 1, \ldots, m$$

Given that the Lagrange multipliers exist, it is straightforward to show that the first-order conditions for the Lagrange problem do yield the optimal solution to the constrained problem. Thus, for a maximum, we have

$$f(\mathbf{x}^*) + \sum_j \lambda_j^* g^j(\mathbf{x}^*) \geq f(\mathbf{x}) + \sum_j \lambda_j^* g^j(\mathbf{x}) \qquad \text{for all } \mathbf{x}$$

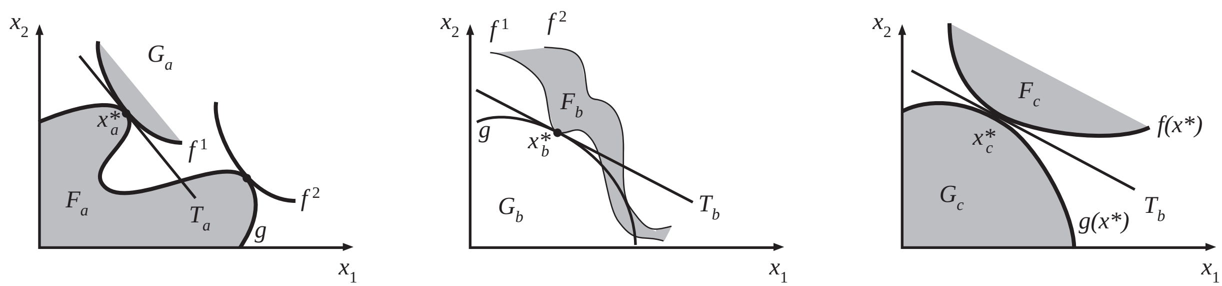
since $\mathbf{x}^*$ maximizes the Lagrange function. But then, for all *feasible* $\mathbf{x}$, $g^j(\mathbf{x}) = 0$, while we know from the first-order conditions that also $g^j(\mathbf{x}^*) = 0$, all j, and so we must have

$$f(\mathbf{x}^*) \geq f(\mathbf{x}) \qquad \text{for all feasible } \mathbf{x}$$

A similar demonstration holds in the case of a minimum.

Finally, we need to consider the point that the optimality conditions locate only *local,* and not *global, optima.* This question is distinct from that of second-order conditions, which were dealt with in the previous section. A point may satisfy both first- and second-order conditions for a maximum, say, but not be the true solution to the problem, because although it is a maximum relative to a small neighborhood of points around itself, there is some other point in the feasible set that gives a higher value of the function f. If we are using methods of solving the problem that locate only *local* optima, then we need to know conditions under which the solution(s) it finds really is (are) what we are looking for.

Figure 13.6 suggests the answer. In figure 13.6 (a) and (b) we have cases where local maxima are not global maxima. In both cases the local second-order conditions are satisfied, but moving along the constraint level curve eventually yields points that are on higher level curves of the function to be maximized.



(a) x_a^* a local but not global optimum **(b)** x_b^* a local but not global optimum **(c)** x_c^* local and global optimum

Figure 13.6 Local and global optima

In figure 13.6 (c) we have a well-behaved case: the local optimum is a unique global optimum, because no other point on the constraint level curve lies on a higher f-level curve.

We can express the differences between the cases a little differently with the help of the tangent lines T_a, T_b, and T_c shown in the figure. The key difference is that in case (c), the line T_c *separates* the shaded set F_c from the shaded set G_c. The only point these sets have in common is the maximum point x_c^*, while all other points in G_c lie below the line and all other points in F_c lie above the line. Therefore, there can be no *feasible* points, which must lie in G_c, that give higher values of the function f than $f(x_c^*)$. On the other hand, in the other two cases no such *separating line* can be drawn. The tangent lines T_a and T_b *do not separate* the sets F_a and G_a or F_b and G_b, in the sense that one set lies on one side of the line and the other set on the other side.

This suggests that we can make sure that any local optimum is also a global optimum by ensuring that the feasible set of points in a problem can be separated, in the sense just described, from the set of points (such as F_c) which is at least as good as the locally optimal point. Recall from section 2.5 the definitions of *quasiconcave* and *quasiconvex functions.* In the case in which both f and g are strictly increasing in x_1 and x_2, a quasiconcave function will have level curves that are straight lines or convex to the origin, such as $f(x^*)$ in figure 13.6 (c); and a quasiconvex function will have level curves that are straight lines or concave to the origin, such as $g(x^*)$ in figure 13.6 (c). This gives the intuitive explanation for

Theorem 13.7 In a constrained maximization problem,

$$\max f(\mathbf{x}) \quad \text{s.t.} \quad g^1(\mathbf{x}) = 0, \dots, g^m(\mathbf{x}) = 0$$

if the function f is quasiconcave, and the functions $g^1, \dots, g^m$ are all quasiconvex, then any locally optimal solution to the problem is also globally optimal.

In solving an optimization problem, it is often important to know, not only that any local optimum is also a global optimum, but also that there is a *unique* local and global optimum. The relevant conditions are obtained by a strengthening of those in theorem 13.7. As figure 13.7 (a) shows, we may have a quasiconcave function f and a quasiconvex constraint function g, but since these may have linear segments in their level curves, a local optimum, though still global, may not be unique. Figure 13.7 (b) shows that strengthening the quasiconcavity condition on f to *strict* quasiconcavity guarantees a unique solution. The same would be true if we strengthened the convexity requirement on g to *strict* quasiconvexity. Theorem 13.8 gives the sufficient conditions for uniqueness of the solution to a maximization problem in the kind of case most frequently encountered.

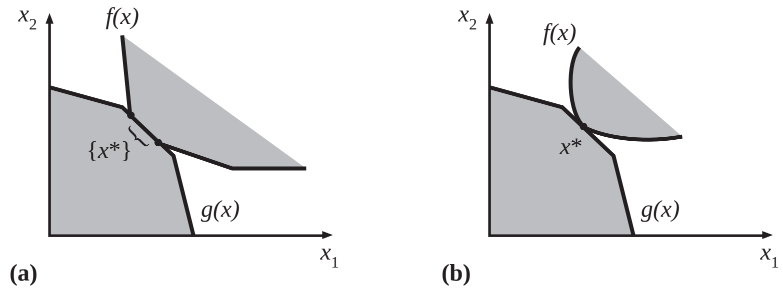


Figure 13.7 Uniqueness of the local maximum

Theorem 13.8 In a constrained maximization problem

$$\max f(\mathbf{x}) \quad \text{s.t.} \quad g^1(\mathbf{x}) = 0, \ldots, g^m(\mathbf{x}) = 0$$

where f and g are increasing functions of $\mathbf{x}$, if

(i) f is strictly quasiconcave and the functions g^j, $j = 1, \ldots, m$, are all quasiconvex, or
(ii) f is quasiconcave and the functions g^j, $j = 1, \ldots, m$, are all strictly quasiconvex,

then, a locally optimal solution is unique and also globally optimal.

EXERCISES

1. In each of the problems in question 1 of the exercises at the end of section 13.1, show that the objective function and constraint satisfy the conditions of theorem 13.7. Consider the application of theorem 13.4 in each case.

2. Draw diagrams of cases in which a feasible set is not closed, or not bounded, but a solution to the optimization problem exists.

3. Draw diagrams of cases in which the g-function is not quasiconvex, or the f-function is not quasiconcave, but a local maximum is a global maximum.

4. Use a figure similar to figure 13.6 to show that if the g-function is quasiconcave and the f-function is quasiconvex, then a local minimum is a global minimum.

5. Discuss the application of theorem 13.6 to the case where we have *two* variables and *one* constraint.

6. Formulate the equivalent of theorem 13.8 for the case of a minimization problem.

CHAPTER REVIEW

Key Concepts

Lagrangean
Lagrange function
Lagrange method
Lagrange multiplier
Weierstrass's theorem

Review Questions

1. Given a maximization or minimization problem with $n \geq 2$ variables and $m \geq 1$ constraints (with $n > m$), form the Lagrange function and write down the first-order conditions.

2. What interpretation can be placed on a Lagrange multiplier at the optimal solution?

3. Form the bordered Hessian determinant for a problem with two variables and one constraint. What sign of the determinant is sufficient to ensure a maximum? A minimum? Now generalize to $n > 2$ variables.

4. What conditions on the objective function and the feasible set are sufficient to ensure existence of a maximum and minimum solution to a constrained optimization problem? Which of these conditions are necessary?

5. What condition is sufficient to ensure the existence of the Lagrange multiplier(s) for a given constrained optimization problem?

6. What condition is sufficient to ensure that any local optimum is also a global optimum?

Review Exercises

1. A firm produces two outputs, which it sells into perfectly competitive markets. It uses one input, which is available in fixed supply. Each output is produced according to a strictly concave production function. Formulate and solve the problem of the optimal allocation of the input between the two outputs. Find an expression or value for the increase in profit that would result from a small increase in the amount of the fixed input the firm has available.

2. A firm produces a single output with two inputs according to a CES production function. It sells the output in two separate markets, in both of which it has a monopoly. It buys its inputs in competitive markets. Formulate and solve the

problem of the profit-maximizing choice of inputs, total output, and sales to the two markets. Interpret the Lagrange multiplier in this problem.

3. An entrepreneur has a fixed sum of money to invest. He can invest it in a productive investment, which will yield a profit in one year's time that is an increasing, strictly concave function of the amount invested. Or, he can put it in the bank, which will pay him a fixed rate of interest in one year's time. He can also borrow from the bank at the same interest rate, against the promise to repay out of future profits. Formulate and solve the problem of choosing the amount of productive investment and borrowing or lending from the bank, to maximize total income available in one year's time.

4. Two single-output firms have identical cost and demand functions. One maximizes profit subject to a constraint that its sales revenue equal a certain amount. The other maximizes sales revenue subject to the constraint that its profit equal a certain amount. Under what conditions would they produce the same output?

Chapter 14 Comparative Statics

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Comparative Statics Examples
- Government in the Simple Keynesian Model: Example
- Effects of a Change in Income on Price and Quantity: Example
- Effect of a Tax on Monopoly Output: Example
- Effect of a Change in the Discount Factor on Investment: Example
- A Linear IS-LM Model: Example
- Effects of a Wage Change in a Cobb-Douglas Model: Example
- Slutsky Equation for Cobb-Douglas Preferences: Example
- The Profit Function
- Profit Function for a Competitive Firm: Example
- The Indirect Utility Function
- The Expenditure Function
- Expenditure Function for a Consumer: Example

As we discussed in chapter 1, economic models have two types of variables: endogenous variables, whose values the model is designed to explain, and exogenous variables, whose values are taken as given from outside the model. The solution values we obtain for the endogenous variables will typically depend on the values of the exogenous variables, and a central part of the analysis will often be to show how the solution values of the endogenous variables change with changes in the exogenous variables. This is the problem of comparative-static equilibrium analysis or **comparative statics.**

In the first section of this chapter we illustrate comparative-statics analysis with some simple examples. In the second section we consider in some depth the standard methods of comparative statics. In the last section we consider some more recent developments which take the form of applications of the envelope theorem.

14.1 Introduction to Comparative Statics

We start with four economic models.

The Simple Keynesian Model of Income Determination

Let Y denote the value of the aggregate supply of goods and services in the economy. Since this accrues as sales revenue to firms who then pay it out as incomes to suppliers of inputs, including labor, we also refer to Y as *national income*. The aggregate demand for goods and services has two components: consumption demand C and investment demand I. We take I as exogenous, but C is determined by the *consumption function*

$$C = cY, \qquad 0 < c < 1$$

where the constant c is the *marginal propensity to consume*. The equilibrium condition is that aggregate supply must equal aggregate demand, or

$$Y = C + I$$

implying that when we substitute cY for C and solve for Y,

$$Y^* = \frac{I}{1 - c}$$

This is illustrated in figure 14.1. Along the horizontal axis we measure Y, along the vertical, C, I, and aggregate demand $C + I$. Thus the 45° line OE shows the set of points at which $C + I = Y$, that is, the set of possible equilibrium points.

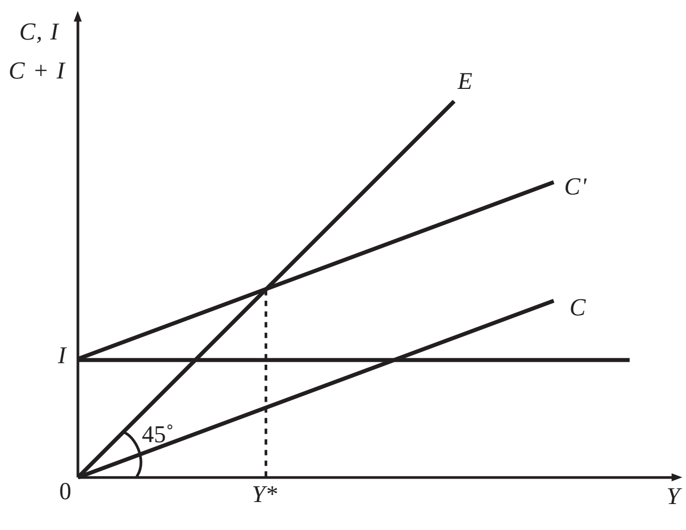


Figure 14.1 Equilibrium in the simple Keynesian model

The line OC, with slope c, denotes the consumption function $C = cY$, and the horizontal line gives the exogenous investment level I. The line IC', also with slope c, is therefore the aggregate-demand function $C + I = cY + I$, and the point at which this intersects the 45° line gives the equilibrium income level Y^* at which aggregate demand is equal to aggregate supply.

The comparative-statics question we ask in this model is: How does a change in exogenous investment I affect the equilibrium income level Y^*?

Algebraically, the answer is found simply by regarding Y^* as a function of I, either *implicitly,* through the equilibrium condition

$$(1 - c)Y^* - I = 0$$

or *explicitly,* through the solution

$$Y^* = \frac{I}{1 - c} \tag{14.1}$$

In either case, by differentiation, it follows that

$$\frac{dY^*}{dI} = \frac{1}{1 - c} > 0 \tag{14.2}$$

Therefore an increase in investment increases equilibrium national income by a multiple $1/(1 - c)$—the multiplier (see section 3.3 for a discussion of the multiplier).

Diagrammatically we illustrate the comparative statics in figure 14.2. An increase in exogenous investment from I to I' shifts the aggregate-demand line up vertically by the same amount, to the new line $I'C''$, thus giving a higher equilibrium income Y^{**}.

A Linear Market Model

The demand for a good is given by the linear demand function

$$D = a - bp + cy, \qquad a, b, c > 0$$

where D is the quantity demanded, p is its price, and y is the aggregate consumers' income.

The supply of the good is given by the linear supply function

$$S = \alpha + \beta p, \qquad \alpha, \beta > 0; \ \alpha < a + cy$$

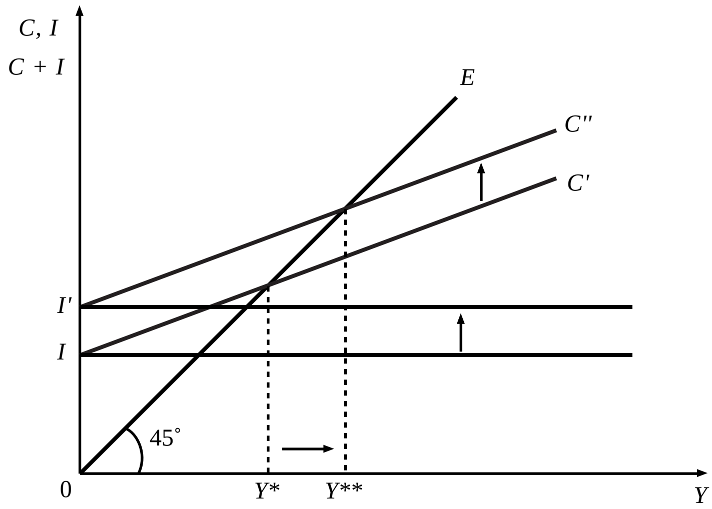


Figure 14.2 An increase in exogenous investment increases equilibrium income

where S is the quantity supplied. For equilibrium in this market we require that supply equals demand, implying that

$$p^* = \frac{a - \alpha + cy}{b + \beta} \tag{14.3}$$

and equilibrium supply and demand follow from substituting for p^* into the supply and demand functions.

We illustrate this solution in figure 14.3. Note that to ensure a solution inside the positive quadrant, we require the condition $a + cy > \alpha$. The comparative-statics question in this model is: How does a change in consumers' income y affect the equilibrium price p^*?

We can regard p^* as a function of y, either *implicitly* through

$$a - \alpha - (b + \beta)p^* + cy = 0$$

or *explicitly* by

$$p^* = \frac{a - \alpha}{b - \beta} + \frac{c}{b + \beta}y$$

In either case, we obtain by differentiation

$$\frac{dp^*}{dy} = \frac{c}{b + \beta} > 0 \tag{14.4}$$

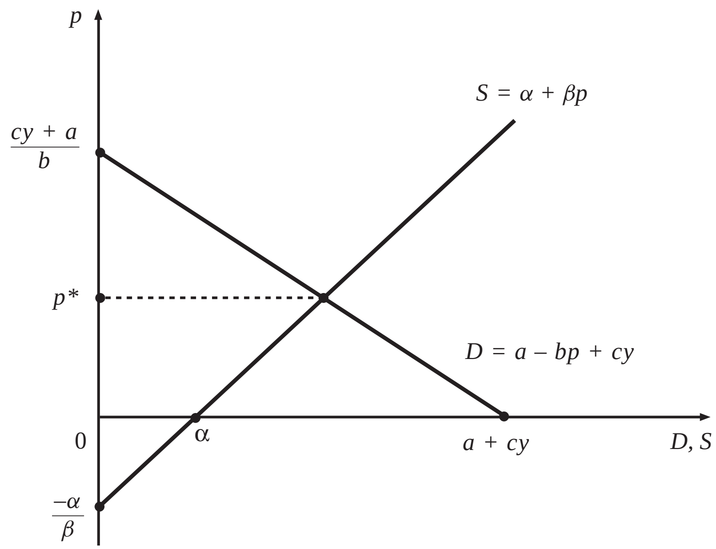


Figure 14.3 Equilibrium in a linear market

Thus the effect of an increase in income in this model is to increase the equilibrium price by an amount which is

1. higher, the greater is c, which gives the effect of a \$1 increase in consumers' income on the quantity demanded;
2. lower, the higher are b and β, the slope coefficients (with respect to the *price* axis) of the demand and supply curves.

Diagrammatically we show this effect in figure 14.4. An increase in consumers' income from y to y' shifts the demand curve outward in a parallel fashion, giving the new point of intersection with the supply curve at p^{**}. Thus the new equilibrium price has increased.

Output Tax on a Monopoly

A monopoly firm faces the (inverse) linear demand function

$$p = a - bq, \qquad a, b > 0$$

and has the total-cost function

$$C = cq^2, \qquad c > 0$$

The government imposes a tax of t per unit of output, and the firm's net of tax-profit function is

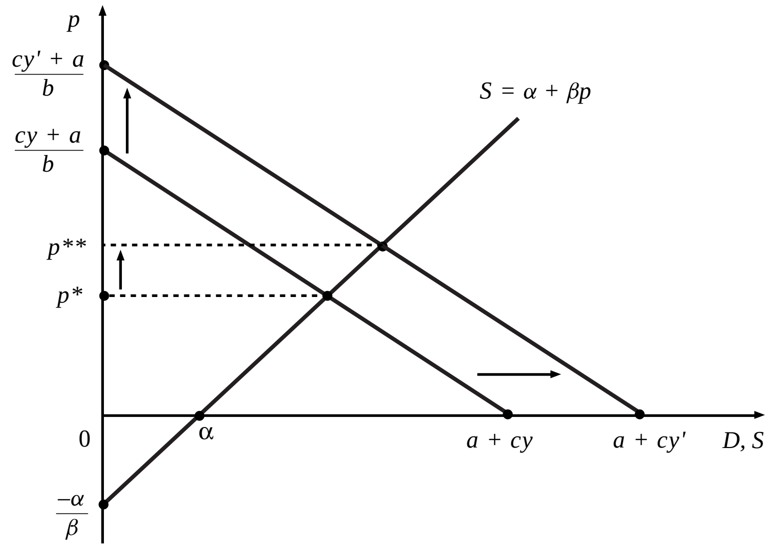


Figure 14.4 An increase in consumers' income raises the equilibrium price

$$\begin{aligned}\pi(q,t) &= aq - bq^2 - cq^2 - tq \\ &= (a-t)q - (b+c)q^2\end{aligned}$$

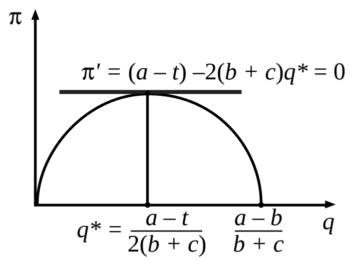


Figure 14.5 Profit-maximizing monopoly with tax

Its profit-maximizing output must satisfy the condition (assuming that $q > 0$ at the solution)

$$a - t - 2(b+c)q^* = 0$$

which gives the solution

$$q^* = \frac{a-t}{2(b+c)}$$

This is illustrated in figure 14.5. Note that we must assume that $a > t$ to be sure of a solution with $q^* > 0$. The linear demand function and quadratic cost function give a quadratic profit function.

In figure 14.6 we show the *marginal-profit function*

$$\frac{d\pi}{dq} = a - t - 2(b+c)q$$

which is linear with a negative slope and cuts the horizontal axis at q^*.

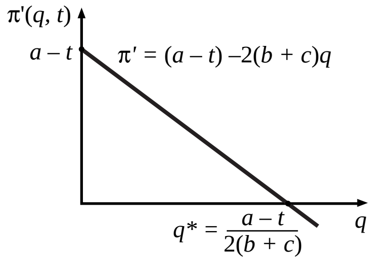


Figure 14.6 The marginal-profit function

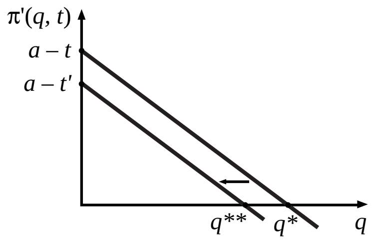


Figure 14.7 An increase in the tax rate reduces profit-maximizing output

The comparative-statics question in this model is: How does a change in the tax rate t affect the profit-maximizing output level q^*? We answer the question by noting that q^* is defined as a function of t, either *implicitly* by

$$a - t - 2(b + c)q^* = 0$$

or *explicitly* by

$$q^* = \frac{a}{2(b + c)} - \frac{t}{2(b + c)} \tag{14.5}$$

In each case differentiation gives

$$\frac{dq^*}{dt} = -\frac{1}{2(b + c)} < 0 \tag{14.6}$$

Therefore an increase in the tax rate reduces profit-maximizing output, by an amount that is smaller, the greater the slopes (with respect to the *quantity* axis) of the demand and marginal-cost curves.

We illustrate this result in figure 14.7, in terms of the marginal-profit function. The increase in the tax rate from t to t' shifts the marginal-profit curve down in a parallel fashion, giving an intersection point with the horizontal axis, namely a profit-maximizing output, at $q^{**} < q^*$.

Optimal Growth

A macroeconomic planner can control the allocation of resources between consumption and investment. The economy exists for only two periods, and the planner wishes to maximize the utility function

$$u(C_1, C_2) = \ln C_1 + \beta \ln C_2, \qquad 0 < \beta < 1$$

where C_t is consumption in period $t = 1, 2$ and β is a discount factor.

There is an exogenously given total income, Y_1^0, in period 1, that must be allocated between consumption and investment:

$$C_1 + I = Y_1^0$$

Since the economy no longer exists after period 2, all income available in that period will be consumed then. The relationship between income (= consumption) in period 2 and investment in period 1 is given by the function

$$C_2 = I^{\alpha}, \qquad 0 < \alpha < 1$$

which is increasing and strictly concave. We want to determine optimal investment in this economy. We can formulate the problem as one involving I only by writing

$$C_1 = Y_1^0 - I$$

and so substituting for C_1 and C_2 in the objective function gives

$$\begin{aligned} v(I) &= \ln\left(Y_1^0 - I\right) + \beta \ln(I^{\alpha}) \\ &= \ln\left(Y_1^0 - I\right) + \alpha\beta \ln I \end{aligned}$$

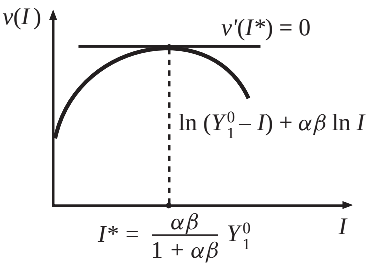


Figure 14.8 Optimal investment: Total utility

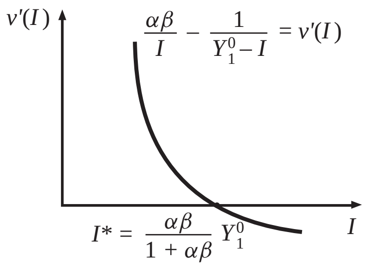


Figure 14.9 Optimal investment: Marginal utility

The first-order condition for a maximum of v with respect to I is

$$v'(I^*) = -\frac{1}{Y_1^0 - I^*} + \frac{\alpha\beta}{I^*} = 0$$

which implies that

$$\alpha\beta Y_1^0 - [1 + \alpha\beta]I^* = 0$$

Thus the optimal solution is to devote the fraction $\alpha\beta/(1+\alpha\beta)$ of current income to investment, and the rest to current consumption.

This is illustrated in figures 14.8 and 14.9. Again, note that we have a strictly concave function (which can be checked by differentiating v a second time) and the marginal-utility function $v'(I)$ has a negative slope, though, of course, it is not linear in this case.

The comparative-statics question in this model is: How does a change in the initial income Y_1^0 affect the optimal investment, I^*? We answer this by noting that I^* can be regarded as a function of Y_1^0, either *implicitly* through

$$\alpha\beta Y_1^0 - (1 + \alpha\beta)I^* = 0$$

or *explicitly* by

$$I^* = \frac{\alpha\beta}{1 + \alpha\beta} Y_1^0 \tag{14.7}$$

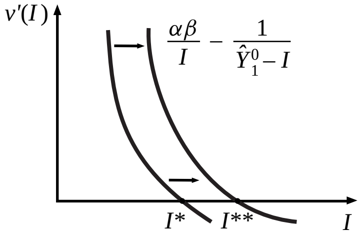


Figure 14.10 An increase in first-period income increases optimal investment

Then in either case differentiation gives

$$\frac{dI^*}{dY_1^0} = \frac{\alpha\beta}{1+\alpha\beta} > 0 \tag{14.8}$$

Therefore an increase in initial income increases optimal investment. Since $0 < \alpha$, $\beta < 1$, investment always increases by the same fraction of initial income. Moreover, since $\alpha\beta/(1+\alpha\beta)$ is *increasing* in both α and β (see question 4 of the exercises in this section), we can say that the higher is α (the more productive is investment) and the higher is β (the less heavily future utility is discounted), then the greater is the effect of a change in income on optimal investment, essentially because the higher then is the share of income that is invested.

This is illustrated in figure 14.10. An increase in Y_1^0 to $\hat{Y}_1^0$ shifts the $v'(I)$ curve rightward, since the negative part of the function defining that curve decreases. Then the intercept on the I-axis increases, implying that optimal investment increases from I^* to I^{**}.

The analysis of the four models we have worked through is typical of the kind of analysis we carry out in economics and also simple enough that we can see clearly *how* the analysis is carried out. We used the basic relationships in the model to derive a **fundamental equation** containing only the endogenous variable, whose solution we are seeking, the exogenous variable, and the parameters of the model. The next step is to solve the equation for the endogenous variable *as a function of* the exogenous variable (equations 14.1, 14.3, 14.5, and 14.7 in these examples).

The question comparative statics seeks to answer is: What is the effect of a change in an exogenous variable on the solution value of the endogenous variable? We answer this by finding the derivative of the equilibrium value of the endogenous variable with respect to the exogenous variable (equations 14.2, 14.4, 14.6, and 14.8 in these examples.) We are interested in the signs of these expressions because they give us *qualitative* information about the direction of change in the equilibrium value following a change in the exogenous variable. Such qualitative information can usually also be obtained from a purely diagrammatic analysis, as we saw. The advantage of the algebraic approach is that it shows very clearly how the parameters of the model determine the strength and direction of the comparative-statics effects.

There are two directions in which we now have to generalize the analysis. In the four models of this section, we assumed specific and quite special functional forms. We have to consider how the methods can be extended to cover cases in which the relationships in a model are specified only as general functions, though possibly with certain restrictions on their shapes, such as concavity or convexity. Second, the models contained only one endogenous and one exogenous variable. We want to generalize this to the case of any number of variables of either type. We look at both of these generalizations in section 14.2.

EXERCISES

1. In the simple Keynesian model of income determination, assume that $c = 0.8$, that $I = 1{,}000$ initially, and then that I increases to 1,200. Draw the counterparts of figure 14.2 and solve for the two levels of national income. What is the value of the multiplier in this case?

2. In the linear market model, take the functions

$$D = 100 - p + 2y$$
$$S = 50 + 2p$$

with $y = 10$ initially and then $y = 20$. Draw the counterpart of figure 14.4 and solve for the equilibrium prices.

3. In the model of a profit-maximizing monopoly with tax, take the functions

$$p = 100 - q$$
$$C = 0.5q^2$$

and with $t = \$1$ initially, and then $t = \$2$, draw the counterparts of figures 14.5 and 14.7 and solve for profit-maximizing outputs in each case. Also restate the comparative-statics analysis in terms of the usual diagram with marginal-revenue and marginal-cost curves. Explain the relation between this diagram and figures 14.5 and 14.7.

4. In the model of optimal growth assume

$$\alpha = 0.5, \quad \beta = 0.8, \quad Y_1^0 = 1{,}000, \quad \hat{Y}_1^0 = 1{,}500$$

Draw the counterparts of figures 14.8 and 14.10 and solve for the two investment levels. Also, show that the ratio I^*/Y_1^0 *increases* with both α and β. [Hint: Differentiate $\alpha\beta/(1 + \alpha\beta)$ with respect to α or β.] Give an economic (as opposed to mathematical) explanation of why this happens.

5. Determine the comparative-statics effects of changes in the following parameters holding exogenous variables fixed:

(a) In the Keynesian model, show the effect on equilibrium national income of a fall in c, both diagrammatically and algebraically. Give an explanation of this effect in economic terms.

(b) In the linear market model, show the effect of changes in b, β, and c, with y given and explain the results.

(c) In the monopoly model with tax, show the effects of changes in a and c and explain the results.

6. The demand for housing D is given by the function

$$D = 100p^{-1}r^{-2}$$

where p is the price of housing and r is the mortgage interest rate. Treat r as exogenous. The supply of housing is given by

$$S = \bar{S}$$

where $\bar{S}$ is exogenous. Solve for the equilibrium housing price and then carry out the comparative-statics analysis with respect to the mortgage interest rate and the housing supply. Illustrate and explain your answers.

7. The demand for a country's exports is given by

$$X = a - br + cY_R, \qquad a, b, c > 0$$

where X is export demand, r is the rate of exchange (measured as the value of the country's currency in terms of a basket of other countries' currencies), and Y_R is income in the rest of the world. The country's demand for imports is given by

$$Q = \alpha + \beta r + \gamma Y_D, \qquad \alpha, \beta, \gamma > 0$$

where Q is import demand and Y_D is the country's national income.

Taking first all income as exogenous, solve for the exchange rate that achieves equilibrium of exports and imports (equilibrium in the balance of trade). Give the reasons for the signs on the coefficients of r in the X and Q equations. Then carry out the comparative-statics analysis with respect to the income variables. Illustrate and explain your results.

Now assume that the exchange rate and income in the rest of the world are exogenous, and that equilibrium of the balance of trade must be brought about by adjustments in the country's income. Solve for the equilibrium income level and carry out the comparative-statics analysis with respect to the exchange rate, and the rest of the world income. Illustrate and explain your answer.

What problems arise if, in this model, both the exchange rate and the country's income are treated as endogenous?

8. A perfectly competitive firm faces the market price p and produces output y according to the production function

$$y = aL^b, \qquad a > 0, 0 < b < 1$$

where L is labor input. It hires labor on a perfectly competitive market at wage rate w. Formulate its profit function in terms only of the endogenous variable labor, find the profit-maximizing labor demand, and carry out the comparative-statics analysis with respect to the output price and wage rate. Illustrate and explain your answer.

14.2 General Comparative-Statics Analysis

We now want to generalize the discussion of the previous section. First, we consider the case in which there is still only one endogenous variable, x, and one exogenous variable, α, but we do not restrict ourselves to specific functional forms. We assume only that we have an economic model, the equilibrium solution of which is given by an equation of the form

$$f(x^*, \alpha) = 0$$

where x^* is the equilibrium value of the endogenous variable x. The function f is assumed differentiable. Its specific interpretation will depend on the economic model we are working with. We want to know the effect of a change in α on x^*, and we interpret this in mathematical terms as wanting to say as much as we can about the derivative $dx^*/d\alpha$, assuming it exists. In particular, we want to identify the *sign* of this derivative, since that tells us the direction of change in the equilibrium value x^* following a change in α.

We proceed as follows: Assume that it is possible to solve the above equation for x^* as a *differentiable* function of α (we will shortly consider the conditions under which it is possible to do this), and write the resulting solution as $x^*(\alpha)$. Then insert this into the equilibrium condition to obtain

$$f(x^*(\alpha), \alpha) = 0$$

Now differentiate through this equation, which we treat as an identity, with respect to α, to obtain

$$f_x \frac{dx^*}{d\alpha} + f_\alpha = 0$$

where $f_x \equiv \partial f/\partial x$ and $f_\alpha \equiv \partial f/\partial \alpha$. Then solving this equation, on the assumption that $f_x \neq 0$ gives

$$\frac{dx^*}{d\alpha} = -\frac{f_\alpha(x^*, \alpha)}{f_x(x^*, \alpha)} \tag{14.9}$$

which is the derivative we seek. Note that this is simply an application of theorem 11.3, the implicit function theorem, and hence note that

- it *must* be assumed that $f_x \neq 0$;
- the partial derivatives f_x and f_α are evaluated at the equilibrium point $(x^*(\alpha), \alpha)$, and so can be regarded as given numbers.

Example 14.1 Suppose that we have the implicit function

$$f(x^*(\alpha), \alpha) = \ln x^* - 2\alpha^2 = 0$$

Find the value $dx^*/d\alpha$.

Solution

Applying the result in equation (14.9) gives

$$\frac{dx^*}{d\alpha} = -\frac{f_\alpha}{f_x} = -\frac{(-4\,\alpha)}{(1/x^*)} = 4\,\alpha x^*$$

■

Effect of Income Change in the Market for a Good

The market-demand function for a good is $D(p, y)$, where p is price and y is aggregate consumers' income. The market-supply function for the good is $S(p)$. Here p corresponds to the endogenous variable x, and y to the exogenous variable α. Then the equilibrium value of the price p^* is given by the equality of demand and supply

$$D(p^*, y) - S(p^*) = 0$$

and so the function f corresponds to $D - S$, or *excess demand*. The relevant derivatives are $D_p - S_p$ and D_y, and so we have the solution

$$\frac{dp^*}{dy} = -\frac{D_y}{D_p - S_p}$$

Suppose that $D_p < 0$ (the good is not a Giffen good) and $S_p > 0$. Then the effect of an increase in income on equilibrium price depends on whether the good is a normal good ($D_y > 0$) or an inferior good ($D_y \leq 0$). If $D_p > 0$ (the good is a Giffen good), then the good is also necessarily inferior ($D_y < 0$) and so the effect of an increase in income on equilibrium price depends on the sign of $D_p - S_p$.

In general, this analysis tells us that anything could happen, and this is probably the rule, rather than the exception, in comparative-statics problems.

The reason for the inability to give a unique sign to the effect dp^*/dy is the very general nature of the information on which the analysis is based. All we have are assumptions about the slopes of the various functions, that is, about the signs of the derivatives D_p, D_y, and S_p. We should not be too surprised therefore if, based on so little information, nothing very definite comes out of the analysis.

That is not to say that the analysis is useless—far from it. The method leads us to work systematically through *all the logically possible cases* and define the *economic assumptions* upon which each possible case will arise. In any particular application, we can then try to find out which particular assumptions are appropriate—whether the good is normal or not, for example—and then apply the results for that case. Indeed the generality of the analysis is a strength—we see that if any *particular result* is being asserted, then that must rest explicitly or implicitly on an assumption about the form of the functions in the model. For example, if a manufacturer claims that an increase in consumers' incomes will always result in an increase in the price of the good he produces, he must be implicitly assuming that his good is a normal good.

Comparative Statics with Several Endogenous and Exogenous Variables

We have seen that we can extend the comparative-statics method readily to problems in which only general functional specifications are given. We now go further and consider the generalization to problems with more than one endogenous and one exogenous variable.

We begin by considering the case of just two variables of each type. As we will see, we require one equilibrium or first-order condition—an f-function—for each *endogenous* variable. Therefore in a model with two endogenous and two exogenous variables, the solution will be given by the conditions

$$f^1(x_1^*, x_2^*, \alpha_1, \alpha_2) = 0$$
$$f^2(x_1^*, x_2^*, \alpha_1, \alpha_2) = 0$$

Now the equilibrium solutions for the endogenous variables will depend on both exogenous variables. We want to derive and sign, if possible, the four derivatives $\partial x_i^*/\partial \alpha_j$, $i, j = 1, 2$.

Assume that it is possible to solve these equations for x_1^* and x_2^* as differentiable functions of α_1 and α_2, $x_1^*(\alpha_1, \alpha_2)$ and $x_2^*(\alpha_1, \alpha_2)$. Then inserting these into the above conditions gives

$$f^1(x_1^*(\alpha_1, \alpha_2), x_2^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2) = 0$$
$$f^2(x_1^*(\alpha_1, \alpha_2), x_2^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2) = 0$$

Regarding these now as identities, we differentiate through with respect to α_1, to obtain

$$f_1^1 \frac{\partial x_1^*}{\partial \alpha_1} + f_2^1 \frac{\partial x_2^*}{\partial \alpha_1} + f_{\alpha_1}^1 = 0$$
$$f_1^2 \frac{\partial x_1^*}{\partial \alpha_1} + f_2^2 \frac{\partial x_2^*}{\partial \alpha_1} + f_{\alpha_1}^2 = 0$$

where $f_{\alpha_1}^i \equiv \partial f^i / \partial \alpha_1$, $i = 1, 2$. Again, the partial derivatives are all evaluated at the given equilibrium point and so can be regarded as given numbers. We can therefore write these equations as the linear system

$$\begin{bmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{bmatrix} \begin{bmatrix} \partial x_1^* / \partial \alpha_1 \\ \partial x_2^* / \partial \alpha_1 \end{bmatrix} = \begin{bmatrix} -f_{\alpha_1}^1 \\ -f_{\alpha_1}^2 \end{bmatrix}$$

In order to solve this system, we require that the determinant

$$|D| = f_1^1 f_2^2 - f_1^2 f_2^1 \neq 0$$

Assuming that this condition holds and applying Cramer's rule, we have the solutions

$$\frac{\partial x_1^*}{\partial \alpha_1} = \frac{\begin{vmatrix} -f_{\alpha_1}^1 & f_2^1 \\ -f_{\alpha_1}^2 & f_2^2 \end{vmatrix}}{|D|} = \frac{-\left(f_{\alpha_1}^1 f_2^2 - f_{\alpha_1}^2 f_2^1\right)}{|D|}$$
$$\frac{\partial x_2^*}{\partial \alpha_1} = \frac{\begin{vmatrix} f_1^1 & -f_{\alpha_1}^1 \\ f_1^2 & -f_{\alpha_1}^2 \end{vmatrix}}{|D|} = \frac{-\left(f_1^1 f_{\alpha_1}^2 - f_1^2 f_{\alpha_1}^1\right)}{|D|}$$

To sign these derivatives, we then need to know the signs of all the partial derivatives involved. Moreover notice that the numerators and denominators involve differences between two terms, and so we may also have to know or assume something about the relative magnitudes of these terms in order to put signs to the solutions.

In exactly the same way, we can derive the comparative-statics effects of changes in α_2, though the details of this are left to the reader. We now consider some economic applications.

The IS-LM Model

Consider now an IS-LM model in which aggregate expenditure E is a function of aggregate income Y and the interest rate R. We will assume that there is an exogenous component to expenditure, $\bar{E}$, so we write

$$E = \bar{E} + E(Y, R), \qquad 0 < E_Y < 1,\ E_R < 0$$

where the partial derivatives indicate that we expect expenditure to increase when income increases and fall when the interest rate rises.

The equilibrium condition is that the aggregate supply and demand for goods and services must be equal

$$Y = \bar{E} + E(Y, R)$$

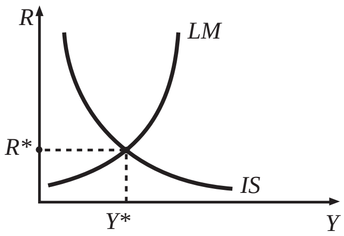


Figure 14.11 Equilibrium in the IS-LM model

We have two endogenous variables, Y and R, with one exogenous variable $\bar{E}$, but only one equation, and so we cannot solve for the equilibrium values of the endogenous variables. What this means is that given a value for the rate of interest, we can find a value of income that equates the demand and supply of goods and services, or conversely, given a level of income, we can find a rate of interest that achieves this equilibrium. But the single condition itself is insufficient to determine unique equilibrium levels of both variables.

This is illustrated in figure 14.11 as the IS curve. The slope is negative because, given $\bar{E}$, there is an implicit function relating Y and R:

$$Y - \bar{E} - E(Y, R) = 0 \tag{14.10}$$

Then, from the rule for differentiating implicit functions, we have

$$\frac{dR}{dY} = \frac{1 - E_Y}{E_R} < 0$$

The sign follows from the assumptions on the partial derivatives. An increase in $\bar{E}$ shifts the IS curve upward. Thus, if we hold Y fixed and treat R and $\bar{E}$ as variable, then we have

$$\frac{dR}{d\bar{E}} = -\frac{1}{E_R} > 0$$

The LM curve represents equilibrium in the money market in which the demand for money is

$$L = L(Y, R), \qquad L_Y > 0,\ \ L_R < 0$$

Equilibrium is where the demand for money is equal to money supply, $\bar{M}$, an exogenous constant, giving

$$L(Y, R) - \bar{M} = 0 \tag{14.11}$$

This condition determines a second relationship between Y and R, graphed as the LM curve in figure 14.11. This curve shows the set of (Y, R)-pairs at which the supply and demand for money are equal. For any given interest rate it shows the level of income required to achieve equilibrium in the money market, or, for any given income level it shows the interest rate required for the equilibrium. The positive slope of the curve is explained by keeping $\bar{M}$ fixed and Y and R variable in the money-market equilibrium condition. By differentiation, we then have

$$\frac{dR}{dY} = -\frac{L_Y}{L_R} > 0$$

where the sign is implied by the assumptions on the partial derivatives.

The exogenous money supply is a parameter in the LM relationship. To show how changes in $\bar{M}$ affect the LM curve, treat Y as fixed and R and $\bar{M}$ as variable in the money-market equilibrium condition. Then differentiating gives

$$\frac{dR}{d\bar{M}} = \frac{1}{L_R} < 0$$

Therefore an increase in the money supply reduces the interest rate required for money-market equilibrium at each level of income.

Then, putting equations (14.10) and (14.11) together determines overall equilibrium values of income Y^* and the interest rate R^*:

$$Y^* - \bar{E} - E(Y^*, R^*) = 0 \tag{14.12}$$

$$L(Y^*, R^*) - \bar{M} = 0 \tag{14.13}$$

That is, the values of Y^* and R^* achieve *simultaneous equilibrium* on the market for goods and services and the market for money. As figure 14.11 shows, this corresponds to the intersection of the IS and LM curves. These two conditions are the counterparts of the functions f^1 and f^2 in the general discussion earlier, with Y and R corresponding to x_1 and x_2, and $\bar{E}$ and $\bar{M}$ corresponding to α_1 and α_2.

In the comparative-statics analysis, we want to find the effects of changes in the two exogenous variables, $\bar{E}$ and $\bar{M}$, on the equilibrium values of the endogenous variables Y^* and R^*.

Then, applying the general results for the case of a change in $\bar{E}$, we have the linear system

$$\begin{bmatrix} (1 - E_Y) & -(E_R) \\ L_Y & L_R \end{bmatrix} \begin{bmatrix} \partial Y^*/\partial \bar{E} \\ \partial R^*/\partial \bar{E} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We denote, by $|D|$, the determinant

$$|D| \equiv (1 - E_Y)L_R + L_Y(E_R) < 0$$

The sign follows unambiguously from the restrictions on the partial derivatives. The comparative-statics effects are

$$\frac{\partial Y^*}{\partial \bar{E}} = \frac{\begin{vmatrix} 1 & -(E_R) \\ 0 & L_R \end{vmatrix}}{|D|} = \frac{L_R}{|D|} > 0$$

$$\frac{\partial R^*}{\partial \bar{E}} = \frac{\begin{vmatrix} (1 - E_Y) & 1 \\ L_Y & 0 \end{vmatrix}}{D} = -\frac{L_Y}{D} > 0$$

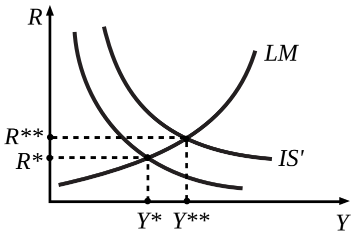


Figure 14.12 An increase in exogenous investment increases equilibrium income and the interest rate

Thus an increase in investment causes an increase in both national income and the rate of interest. The increase in demand for investment goods increases income directly; this increases the demand for money and so raises the rate of interest with the money supply fixed. This is illustrated in figure 14.12. Although this model is quite general, the assumptions on the signs of the various partial derivatives, together with the assumption that $E_Y < 1$, are enough to determine the comparative-statics effects unambiguously.

The comparative-statics effects of a change in $\bar{M}$ are derived in a similar way, and can also be shown to be unambiguously determined on the given assumptions. This is left as an exercise (see question 1 of the exercises at the end of this section).

Competitive Firm's Input Demands

A firm sells its output into a perfectly competitive market and so faces a fixed price p. It also hires labor in a competitive labor market at a wage w and rents capital on a competitive capital market at rental rate r. It has the production function $f(L, K)$, which is strictly concave, and it seeks to maximize profit

$$\pi = pf(L, K) - wL - rK$$

Its first-order conditions are

$$pf_L(L^*, K^*) - w = 0$$
$$pf_K(L^*, K^*) - r = 0$$

We wish to determine the effects on input demands, L^* and K^*, of changes in the input prices. Applying the earlier general results for the effects of a change in the wage w gives the linear system

$$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Defining the determinant

$$|D| = p^2(f_{LL}f_{KK} - f_{KL}f_{LK})$$

we note that if the second-order sufficient conditions for the problem are satisfied, then $D > 0$. That is, the determinant D is precisely the Hessian determinant we obtain when we formulate the second-order conditions for the profit-maximization problem. This is an important point that we will take up further below. We then have the comparative-statics results

$$\frac{\partial L^*}{\partial w} = \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{D} = \frac{pf_{KK}}{|D|} < 0$$

$$\frac{\partial K^*}{\partial w} = \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{KL} & 0 \end{vmatrix}}{D} = -\frac{pf_{KL}}{|D|}$$

The sign of the first of these derivatives follows from $f_{KK} < 0$. This in turn follows from the second-order conditions. Thus the demand curve for labor has a negative slope. However, in order to sign the effect of a change in the wage rate on the demand for capital, we need to know the sign of f_{KL}, the effect of a change in the labor input on the marginal product of capital. It is plausible to assume that this is positive (though it may not be) and so an increase in the wage rate would also decrease the demand for capital. We can derive the effects of a change in the rental rate of capital in a similar way, and this is left as an exercise.

Comparative Statics for Constrained Optimization Problems

The comparative-statics methods we have developed and illustrated so far do not allow us to handle the comparative-statics analysis of constrained optimization problems. This is because the smallest such problem would involve two choice variables and one constraint, giving a system of *three* first-order conditions in *three* endogenous variables, the two choice variables and the Lagrange multiplier. We could handle such a problem by substituting from the constraint into the objective function to give an unconstrained problem. However, to aim for generality, we now need to extend the method of comparative statics a little further to deal with the constrained problem directly.

Thus suppose that the solution of a model is given by three conditions in three endogenous variables, with any number m of exogenous variables

$$f^1(x_1^*, x_2^*, x_3^*; \alpha_1, \ldots, \alpha_m) = 0 \tag{14.14}$$

$$f^2(x_1^*, x_2^*, x_3^*; \alpha_1, \ldots, \alpha_m) = 0 \tag{14.15}$$

$$f^3(x_1^*, x_2^*, x_3^*; \alpha_1, \ldots, \alpha_m) = 0 \tag{14.16}$$

where the functions $f^k, k = 1, 2, 3$, are all differentiable. Assume that it is possible to solve for each endogenous variable as a differentiable function of the exogenous variables

$$x_i^* = x_i(\alpha_1, \ldots, \alpha_m), \qquad i = 1, 2, 3$$

Substituting these back into the functions f^k and differentiating with respect to any α_j, $j = 1, \ldots, m$, gives the linear system

$$\begin{bmatrix} f_1^1 & f_2^1 & f_3^1 \\ f_1^2 & f_2^2 & f_3^2 \\ f_1^3 & f_2^3 & f_3^3 \end{bmatrix} \begin{bmatrix} \partial x_1^*/\partial \alpha_j \\ \partial x_2^*/\partial \alpha_j \\ \partial x_3^*/\partial \alpha_j \end{bmatrix} = \begin{bmatrix} -f_{\alpha_j}^1 \\ -f_{\alpha_j}^2 \\ -f_{\alpha_j}^3 \end{bmatrix}$$

where all the partial derivatives are evaluated at the point

$$(x_1^*, x_2^*, x_3^*; \alpha_1, \ldots, \alpha_m)$$

and so are given numbers. Given that the determinant $|F|$ of the left-hand matrix is nonzero, we can solve for $\partial x_i^*/\partial \alpha_j$ by using Cramer's rule. This involves forming the determinant $|F_{ij}|$ by replacing the ith column of $|F|$ with the column

$$\begin{bmatrix} -f_{\alpha_j}^1 \\ -f_{\alpha_j}^2 \\ -f_{\alpha_j}^3 \end{bmatrix}$$

and evaluating

$$\frac{\partial x_i^*}{\partial \alpha_j} = \frac{|F_{ij}|}{|F|}, \qquad i = 1, 2, 3; \; j = 1, \ldots, m$$

When the model is one of constrained optimization, the functions f^k will be the first-order partial derivatives of the Lagrange function and the equilibrium conditions are the first-order conditions. In that case, the determinant $|F|$ is the Hessian determinant of the system. Recall that the sufficient second-order conditions are expressed in terms of the sign of $|F|$. These conditions can be used to help sign the comparative statics. We see how this works out in some examples.

The Slutsky Equation

In chapter 13 we considered the problem of a consumer's optimal choice of consumption quantities. The solution to this problem takes the form of a set of *demand functions*, one for each good. A major aim of the analysis is to show how the consumer's demand for a good varies with prices, and so this is a problem in comparative-statics analysis. In this example, we consider this problem for the case of two goods, x_1 and x_2. The consumer's demands for these goods are given by the solution to the problem

$$\max\ u(x_1, x_2) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 = m$$

Here the endogenous variables are the demands x_1 and x_2, and the exogenous variables are the prices p_1 and p_2, and income m. Applying the Lagrangean method gives the first-order conditions

$$\begin{aligned} u_1(x_1^*, x_2^*) - \lambda^* p_1 &= 0 \\ u_2(x_1^*, x_2^*) - \lambda^* p_2 &= 0 \\ m - p_1x_1^* - p_2x_2^* &= 0 \end{aligned}$$

We now look at the effects of changes in prices and income on the demands. Applying the standard method leads to

$$\frac{\partial x_1}{\partial p_1} = \frac{\begin{vmatrix} \lambda^* & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ x_1^* & -p_2 & 0 \end{vmatrix}}{|D|} = -\frac{\lambda^* p_2^2}{D} + \frac{x_1^*(p_1u_{22} - p_2u_{12})}{D} \tag{14.17}$$

$$\frac{\partial x_2}{\partial p_2} = \frac{\begin{vmatrix} u_{11} & 0 & -p_1 \\ u_{21} & \lambda^* & -p_2 \\ -p_2 & x_2^* & 0 \end{vmatrix}}{|D|} = -\frac{\lambda^* p_1^2}{D} + \frac{x_2^*(p_2u_{11} - p_1u_{21})}{D} \tag{14.18}$$

$$\frac{\partial x_1}{\partial m} = \frac{\begin{vmatrix} 0 & u_{12} & -p_1 \\ 0 & u_{22} & -p_2 \\ -1 & -p_2 & 0 \end{vmatrix}}{|D|} = -\frac{(p_1 u_{22} - p_2 u_{12})}{D} \tag{14.19}$$

$$\frac{\partial x_2}{\partial m} = \frac{\begin{vmatrix} u_{11} & 0 & -p_1 \\ u_{21} & 0 & -p_2 \\ -p_1 & -1 & 0 \end{vmatrix}}{|D|} = -\frac{(p_2 u_{11} - p_1 u_{21})}{D} \tag{14.20}$$

where the determinant $|D|$ is given by

$$|D| = \begin{vmatrix} u_{11} & u_{12} & -p_1 \\ u_{21} & u_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix}$$

Now the condition for a maximum to this problem is that the determinant of the Hessian is positive. This is exactly the determinant $|D|$, so $|D| > 0$ by the second-order condition.

Next, consider the sign of the partial with respect to income, say $\partial x_1^*/\partial m$. The theory places no restrictions on the signs of u_{22} and u_{12}. Thus we have no way of saying whether the numerator is positive or negative. Likewise we have no way of saying whether the numerator in the expression for $\partial x_2^*/\partial m$ is positive or negative. Therefore the effects on demands of changes in income cannot be determined unambiguously. This lack of a definite sign for the effect of an increase in income on the demand for a good is a strength of the theory, rather than a weakness, and gives rise to a classification of goods into one of three types: normal goods, for which demand increases when income increases; strictly inferior goods, for which demand falls as income increases; and weakly inferior goods, for which the demand remains unchanged as income increases.

This discussion helps in determining the effect of price on demand. For example, using equation (14.19) in equation (14.17) we may write

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{\lambda^* p_2^2}{|D|} - x_1^* \frac{\partial x_1}{\partial m} \tag{14.21}$$

This is referred to as the **Slutsky equation** for the effect of a change in p_1 on the demand for x_1. The first term, or *substitution effect*, is clearly negative, since $|D| > 0$ and $\lambda^* = u_1/p_1 > 0$ also. The second term, or the *income effect*, depends on the sign of $\partial x_1^*/\partial m$. If x_1 is *normal*, then $\partial x_1^*/\partial m > 0$ and so $\partial x_1^*/\partial p_1 < 0$. If x_1 is weakly inferior, then $\partial x_1^*/\partial m = 0$ and again $\partial x_1^*/\partial p_1 < 0$. If x_1 is strictly inferior so that $\partial x_1^*/\partial m < 0$, then the two effects work against each other. The substitution effect may dominate so that again $\partial x_1^*/\partial p_1 < 0$, or the two effects may cancel (so the demand curve at that point is vertical), or the income effect may dominate so that $\partial x_1^*/\partial p_1 > 0$ and the demand curve is *positively sloped*. This is the case of the *Giffen good*.

The general method may be summarized in

Definition 14.1

The general method of comparative statics Given that the n equations

$$\begin{aligned} f^1(x_1^*, x_2^*, \dots, x_n^*; \alpha_1, \dots, \alpha_m) &= 0 \\ f^2(x_1^*, x_2^*, \dots, x_n^*; \alpha_1, \dots, \alpha_m) &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ f^n(x_1^*, x_2^*, \dots, x_n^*; \alpha_1, \dots, \alpha_m) &= 0 \end{aligned}$$

can be solved to give the equilibrium values of the n endogenous variables x_1^*, $x_2^*, \dots, x_n^*$ as differentiable functions of the m exogenous variables $\alpha_1, \dots, \alpha_m$, we have that

$$\frac{\partial x_i^*}{\partial \alpha_j} = \frac{|F_{ij}|}{|F|}, \qquad i = 1, \dots n;\ j = 1, \dots m$$

where $|F| \neq 0$ is the determinant

$$|F| = \begin{vmatrix} f_1^1 & f_2^1 & \cdots & f_n^1 \\ f_1^2 & f_2^2 & \cdots & f_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n & f_2^n & \cdots & f_n^n \end{vmatrix}$$

and $|F_{ij}|$ is formed by replacing the ith column of $|F|$ by the jth column of the $n \times m$ matrix

$$\begin{bmatrix} -f_{\alpha_1}^1 & -f_{\alpha_2}^1 & \cdots & -f_{\alpha_m}^1 \\ -f_{\alpha_1}^2 & -f_{\alpha_2}^2 & \cdots & -f_{\alpha_m}^2 \\ \vdots & \vdots & \ddots & \vdots \\ -f_{\alpha_1}^n & -f_{\alpha_2}^n & \cdots & -f_{\alpha_m}^n \end{bmatrix}$$

Note that

- It must be assumed that $|F| \neq 0$.
- The partial derivatives are evaluated at the initial equilibrium point and so are given numbers.

- Where the system of equations represents the first-order conditions of a maximization or minimization problem, the second-order conditions for this problem will give the sign of the determinant $|F|$, since this is the bordered Hessian determinant found in the analysis of second-order conditions.

You should check that all the examples considered are particular applications of this method.

We have stressed that the system of equations in definition 14.1 are *assumed* to yield solutions for the n endogenous variables as differentiable functions of the m exogenous variables. The key theorem that gives the conditions under which this may be done is simply an extension of theorem 11.3.

Theorem 14.1 **(Implicit function theorem)** Given an equation system

$$\begin{aligned} f^1(x_1^*, x_2^*, \ldots, x_n^*; \alpha_1, \ldots, \alpha_m) &= 0 \\ f^2(x_1^*, x_2^*, \ldots, x_n^*; \alpha_1, \ldots, \alpha_m) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ f^n(x_1^*, x_2^*, \ldots, x_n^*; \alpha_1, \ldots, \alpha_m) &= 0 \end{aligned}$$

let $(x_1^*, \ldots, x_n^*; \alpha_1^0, \ldots, \alpha_m^0)$ be a point satisfying these equations, and let the functions have continuous partial derivatives up to the rth order, over some open set of points in $\mathbb{R}^{n+m}$ around $(x_1^*, \ldots, x_n^*; \alpha_1^0, \ldots, \alpha_m^0)$. If the determinant

$$|F| = \begin{vmatrix} f_1^1 & f_2^1 & \cdots & f_n^1 \\ f_1^2 & f_2^2 & \cdots & f_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n & f_2^n & \cdots & f_n^n \end{vmatrix} \neq 0$$

where $f_i^k \equiv \partial f^k / \partial x_i$, and these partial derivatives are all evaluated at $(x_1^*, \ldots, x_n^*; \alpha_1^0, \ldots, \alpha_m^0)$, then the equation system defines $x_1, \ldots, x_n$ as functions of the α_js in some neighborhood of $(x_1^*, \ldots, x_n^*; \alpha_1^0, \ldots, \alpha_m^0)$. These functions also have continuous partial derivatives up to the rth order. In that neighborhood we have

$$\begin{bmatrix} f_1^1 & f_2^1 & \cdots & f_n^1 \\ f_1^2 & f_2^2 & \cdots & f_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ f_1^n & f_2^n & \cdots & f_n^n \end{bmatrix} \begin{bmatrix} \partial x_1^*/\partial \alpha_j \\ \partial x_2^*/\partial \alpha_j \\ \vdots \\ \partial x_n^*/\partial \alpha_j \end{bmatrix} = \begin{bmatrix} -f_{\alpha_j}^1 \\ -f_{\alpha_j}^2 \\ \vdots \\ -f_{\alpha_j}^n \end{bmatrix}$$

for each $j = 1, \ldots, m$.

This theorem provides the basis for the general method of comparative statics set out in definition 14.1. For that purpose all we need is for the functions f^i to possess continuous first derivatives.

EXERCISES

1. In the general IS-LM model, what are the comparative-statics effects of a change in the money supply?

2. In the model of the competitive firm's input demands, what are the comparative-statics effects of a change in the price of capital r?

3. Derive and interpret the Slutsky equations for a consumer with utility functions **(a)** $u = x_1x_2$, **(b)** $u = \sqrt{x_1} + x_2$.

4. In the general IS-LM model, assume that the partial derivative E_R is *positive* rather than negative. Derive the implications of this for the comparative statics of the model.

5. Derive the Slutsky equations for the case in which the consumer buys three goods, where the consumer's problem is

$$\max u(x_1, x_2, x_3) \quad \text{s.t.} \quad p_1x_1 + p_2x_2 + p_3x_3 = m$$

and u is strictly quasi-concave.

6. The demand function for coffee is given by

$$D_c = 100 - 2p_c + 0.5p_t$$

and that for tea is given by

$$D_t = 120 - p_t + 0.75p_c$$

where p_c is the price of coffee and p_t is the price of tea. The respective supply functions are

$$S_c = 10 + p_c + 5w_c$$
$$S_t = 5 + 2p_t + 2w_t$$

where w_c and w_t are the indexes of weather conditions affecting production of coffee and tea respectively. Interpret these supply and demand functions. Give the comparative-statics effects on equilibrium prices of changes in the weather conditions variables.

7. Countries 1 and 2 trade with each other, under fixed exchange rates. The relevant functions are

	Country 1	Country 2
Consumption functions	$C_1 = 0.8Y_1$	$C_2 = 0.7Y_2$
Investment	$I_1 = I_1^0$	$I_2 = I_2^0$
Imports	$Q_1 = 0.3Y_1$	$Q_2 = 0.5Y_2$

Find the effects of a change in each country's exogenous investment on the equilibrium income levels in both countries. Explain your results. [Note: One country's imports are the other country's exports. Aggregate demand in country 1 is $C_1 + I_1^0 + Q_2$ while aggregate supply is $Y_1 + Q_1$ and similarly for country 2. Equilibrium in each country requires aggregate demand to equal aggregate supply.]

14.3 The Envelope Theorem

In the comparative-statics analysis of constrained maximization and minimization problems, it is often helpful to use an approach based on the **envelope theorem,** instead of, or as well as that based on the implicit function theorem. Thus suppose that we have the problem

$$\max_{x_1, x_2} f(x_1, x_2; \alpha) \quad \text{s.t.} \quad g(x_1, x_2; \alpha) = 0$$

where α is an exogenous variable. The Lagrange function for this problem is

$$\mathcal{L}(x_1, x_2, \lambda; \alpha) = f(x_1, x_2; \alpha) + \lambda g(x_1, x_2; \alpha)$$

and the first-order conditions are

$$\begin{aligned} f_1(x_1^*, x_2^*; \alpha) + \lambda^* g_1(x_1^*, x_2^*; \alpha) &= 0 \\ f_2(x_1^*, x_2^*; \alpha) + \lambda^* g_2(x_1^*, x_2^*; \alpha) &= 0 \\ g(x_1^*, x_2^*; \alpha) &= 0 \end{aligned}$$

If we assume that the functions f and g possess continuous first and second derivatives, and that the determinant

$$|D| = \begin{vmatrix} f_{11} + \lambda^* g_{11} & f_{12} + \lambda^* g_{12} & g_1 \\ f_{21} + \lambda^* g_{21} & f_{22} + \lambda^* g_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} \neq 0$$

then we can apply the implicit function theorem. This amounts to saying we can solve for the endogenous variables as differentiable functions of the exogenous variable in the neighborhood of the optimal point, so the value of the function f in the same neighborhood is

$$f(x_1(\alpha), x_2(\alpha); \alpha) \equiv V(\alpha)$$

and V is known as the **value function** for the maximization problem. The value function expresses directly the idea that the maximized value of the function f depends, *via* the maximization procedure, only on the exogenous variable in the problem.

In the same way we can write the Lagrange function as a function of the parameter

$$\mathcal{L} = f(x_1(\alpha), x_2(\alpha); \alpha) + \lambda(\alpha) g(x_1(\alpha), x_2(\alpha); \alpha)$$

We now notice an interesting fact. Consider the total derivative of the Lagrange function with respect to α. This is

$$\frac{d\mathcal{L}}{d\alpha} = (f_1 + \lambda g_1)\frac{dx_1}{d\alpha} + (f_2 + \lambda g_2)\frac{dx_2}{d\alpha} - g\frac{d\lambda}{d\alpha} + (f_\alpha + \lambda g_\alpha)$$

Now, at the optimal point, we have $f_i + \lambda g_i = 0$, $i = 1, 2$, and $g = 0$, and so if we evaluate $d\mathcal{L}/d\alpha$ at the optimal point, the first three terms vanish, and we are left with

$$\frac{d\mathcal{L}}{d\alpha} = f_\alpha + \lambda^* g_\alpha = \frac{\partial \mathcal{L}}{\partial \alpha}$$

That is, although a change in α induces changes in the values of the endogenous variables, for small enough changes *at the optimal point,* the effects of these changes on the Lagrange function can be ignored because the partial derivatives of the Lagrange function with respect to the endogenous variables are zero at that point.

The **envelope theorem** establishes a connection between the derivatives of the value function and the derivatives of the Lagrange function, with respect to the parameter α, at the optimal point. Thus for the value function, using the chain rule of differentiation, we have

$$\frac{dV}{d\alpha} = f_1\frac{dx_1}{d\alpha} + f_2\frac{dx_2}{d\alpha} + f_\alpha$$

Substituting for f_1 and f_2 from the first-order conditions gives

$$\frac{dV}{d\alpha} = -\lambda^*\left(g_1\frac{dx_1}{d\alpha} + g_2\frac{dx_2}{d\alpha}\right) + f_\alpha$$

at the optimal point. We can write the constraint as

$$g(x_1(\alpha), x_2(\alpha), \alpha) = 0$$

where, because the x_i are optimal solutions to the constrained problem and therefore satisfy the constraint, this equality holds identically. Then differentiating with respect to α we have

$$g_1\frac{dx_1}{d\alpha} + g_2\frac{dx_2}{d\alpha} = -g_\alpha$$

Substituting into the expression for $dV/d\alpha$ gives

$$\frac{dV}{d\alpha} = f_\alpha + \lambda^* g_\alpha = \frac{\partial\mathcal{L}}{\partial\alpha}$$

which is the form the envelope theorem takes for this example.

The envelope theorem tells us that we can find the effect of a change in the exogenous variable on the optimized value of the objective function simply by taking the *partial derivative* of the Lagrange function with respect to the exogenous variable at the optimal solution to the problem.

Theorem 14.2 **(Envelope theorem)** Given the problem

$$\begin{aligned} &\max f(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) \\ &\quad g^1(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) = 0 \\ \text{s.t.} &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad g^K(x_1, \ldots, x_n; \alpha_1, \ldots, \alpha_m) = 0 \end{aligned}$$

and the corresponding value function $V(\alpha_1, \ldots, \alpha_m)$, and Lagrange function $\mathcal{L} = f + \sum \lambda_k g^k$, we have that

$$\frac{\partial V}{\partial \alpha_j} = \frac{\partial \mathcal{L}}{\partial \alpha_j} = f_{\alpha_j} + \sum_{k=1}^{K} \lambda_j g^k_{\alpha_j}, \qquad j = 1, \ldots m$$

A very simple, but important, application of the envelope theorem gives the interpretation of a Lagrange multiplier in an optimization problem. Suppose that we have the problem

$$\max f(x_1, \ldots, x_n) \quad \text{s.t.} \quad \begin{array}{r} g^1(x_1, \ldots, x_n) + \alpha_1 \ = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g^K(x_1, \ldots, x_n) + \alpha_K = 0 \end{array}$$

where the exogenous variables α_k $(k = 1, \ldots, K)$, only enter in the constraints as "constraint constants," the Lagrange function becomes

$$\mathcal{L} = f(x_1, \ldots, x_n) + \sum_{k=1}^{K} \lambda_k [g^k(x_1, \ldots, x_n) + \alpha_k]$$

Applying the envelope theorem directly gives

$$\frac{\partial V}{\partial \alpha_k} = \frac{\partial \mathcal{L}}{\partial \alpha_k} = \lambda_k^*$$

Thus the Lagrange multiplier measures the rate at which the value function changes when the corresponding constraint is tightened or relaxed slightly. This interpretation of the Lagrange multiplier is of fundamental importance in economic applications of methods of constrained optimization. One implication is immediate: if a constraint is nonbinding at the optimum, so that a small tightening or relaxing of it has no effect on the solution, then the associated Lagrange multiplier will take the value zero at this optimum.

Long-Run and Short-Run Cost Curves

It is not obvious from the statement of the envelope theorem why it should have been given this name. The reason is that the theorem was discovered as a result of the investigation of the relationship between short-run and long-run cost curves. This relationship can be described in the following terms:

> *The long-run total-cost curve is the envelope of the family of short-run cost curves generated by varying the level of the fixed input.*

The envelope theorem allows this relationship to be established rigorously. To illustrate, we consider the following example.

The long-run cost function for a firm was derived in chapter 13. In general, the problem is

$$\min C = wL + rK \quad \text{s.t.} \quad y = f(L, K)$$

for a given y. Given that the conditions for the implicit function theorem are met, the first-order conditions yield functions $L(w, r, y)$ and $K(w, r, y)$, and substituting for L and K in the objective function gives the value function, which in this case is the *long-run cost function*

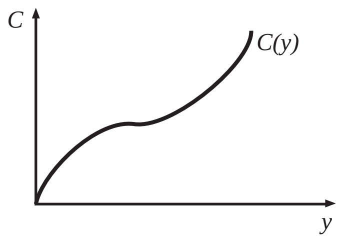


Figure 14.13 Long-run total-cost curve

$$C = wL(w, r, y) + rK(w, r, y) = C(w, r, y)$$

This gives the minimized production cost as a function of the exogenous variables, input prices and output. However, in all of what follows we hold input prices constant and so simply write the long-run cost as $C(y)$. In figure 14.13 we draw the long-run total-cost curve, which shows how minimized costs vary with output holding input prices constant.

We define the *short-run cost minimization* problem of the firm by assuming that one of the inputs, typically K, is fixed in amount and that the cost associated with K is a fixed cost. Denote the fixed amount of capital by K_a, and let $F_a = rK_a$ denote the corresponding fixed cost. Then the firm's short-run cost-minimization problem is

$$\min_L c = wL + F_a \quad \text{s.t.} \quad y = f(L, K_a)$$

The solution is that cost is minimized by using as little of the variable input as possible, subject to producing the required output level, since the lower is L the lower is cost. But, since K is given, this amounts simply to finding the value of L that satisfies the constraint. We write this solution value as

$$L = \phi(y, K_a)$$

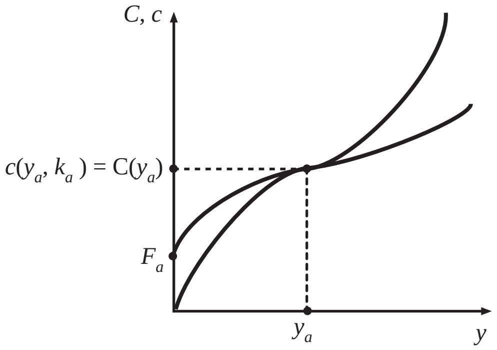
and the minimized short-run cost is then given by the short-run total-cost function

$$c = w\phi(y, K_a) + F_a = c(y, K_a)$$

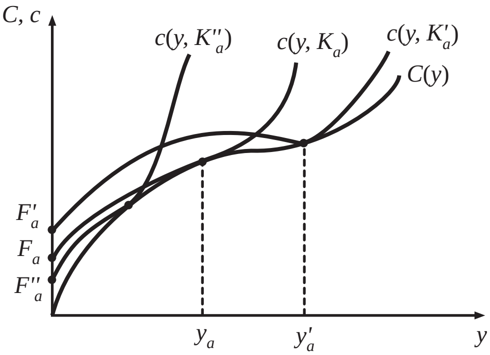
which is the value function for this short-run problem. (Again, we suppress input prices because they are assumed constant.)

Now define y_a as the value of y for which

$$C(y) = c(y, K_a)$$



(a) Short-run and long-run cost functions



(b) Envelope relation

Figure 14.14

The relation between the long-run and short-run costs is illustrated in figure 14.14 (a). The short-run cost curve has a vertical intercept of F_a and is everywhere above the long-run cost curve except at y_a, where they just touch.

Now letting output y vary, and holding K fixed at K_a in the first problem, while allowing K to vary in whatever way is required for optimality in the second problem, implies the inequality

$$c(y, K_a) \geq C(y)$$

since imposing a constraint—that $K = K_a$—*cannot* improve the outcome of an optimization problem. In fact we know that for this cost-minimization problem the inequality will hold strictly when $y \neq y^a$.

The choice of the capital level K_a here is arbitrary. By changing it, say, to K'_a, there is another input level, y'_a at which long-run and short-run total costs are just equal. Repeating this for each possible level of capital generates a family of short-run cost curves, each of which touches the long-run curve at only one point, as in figure 14.14 (b). Thus the long-run curve is said to be the *envelope* of the short-run curves (see also question 7 of the exercises at the end of this section).

The Shadow Wage Rate

A central planner controls an economy with two sectors, producing outputs x_1 and x_2 respectively. This economy is very small relative to the rest of the world, and so takes as given the world prices p_1 and p_2 for the goods. The planner wishes to maximize the value of national output Y at these world prices. Labor is the only input and this is available in fixed total amount L^0. The production functions in the two sectors are

$$x_1 = a_1 L_1^b, \quad x_2 = a_2 L_2^b, \qquad a_i > 0;\ 0 < b < 1;\ i = 1, 2$$

where L_i is the amount of labor used in sector i. The planner's problem is

$$\max Y = p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad x_i = a_i L_i^b, \qquad i = 1, 2$$
$$L_1 + L_2 = L^0$$

We can use this information to write the Lagrange function as

$$\mathcal{L} = p_1 a_1 L_1^b + p_2 a_2 L_2^b + \lambda(L^0 - L_1 - L_2)$$

The first-order conditions are

$$bp_i a_i L_i^{b-1} - \lambda^* = 0, \qquad i = 1, 2$$
$$L^0 - L_1 - L_2 = 0$$

Solving these conditions then gives

$$L_1 = c_1(p_1, p_2)L^0, \quad L_2 = c_2(p_1, p_2)L^0$$

where

$$c_1(p_1, p_2) = \left[1 + \left(\frac{p_1 a_1}{p_2 a_2}\right)^{1/(b-1)}\right]^{-1}$$

and

$$c_2(p_1, p_2) = 1 - c_1(p_1, p_2)$$

Thus sector i receives a share c_i of total labor input, which is independent of the total scale of the economy and depends only on the relative output prices.

The optimized value of national output is then

$$\begin{aligned} Y^* &= p_1 x_1^* + p_2 x_2^* \\ &= p_1 a_1 [c_1(p_1, p_2)L^0]^b + p_2 a_2 [c_2(p_1, p_2)L^0]^b \\ &= V(p_1, p_2, L^0) \end{aligned}$$

where V is the value function. We are now interested in the derivatives of this function, namely the effects on national income of changes in the available labor L^0 and the world prices p_i.

Notice that L^0 is a constraint constant. Therefore we can apply the envelope theorem immediately to obtain

$$\frac{\partial V}{\partial L^0} = \lambda^*$$

That is, the value of the Lagrange multiplier given by the solution to the problem measures the effect on the value of national output of a small change in labor supply. We could interpret this as saying that the social planner would be prepared to pay a maximum of λ^* to obtain a small amount of extra labor. Note that λ^* has the dimension "\$/unit of labor," just like the wage rate, and so we refer to this as the **shadow wage rate.**

Consider now the effect of a change in a world price, say p_1, on the value of national output. On the face of it this could seem to involve some tricky differentiation, but notice that the prices appear *only in the objective function* of the problem and not in the constraint. Then using the envelope theorem gives

$$\frac{\partial V}{\partial p_1} = x_1^* = a_1[c_1(p_1, p_2)L^0]^b$$

Therefore, if we know the level of output in a sector at the optimal solution, we also know the effect on the value of national output of a small change in its price.

EXERCISES

1. For the shadow-wage-rate model, let $p_1 = 1$, $p_2 = 2$, $a_1 = 100$, $a_2 = 50$, $b = 0.5$, and $L^0 = 1000$. Calculate the optimal labor allocations, outputs, and the shadow wage rate. Write out the value function and confirm the results for the derivatives of this function numerically.

2. Show that the indirect utility function derived in the expenditure function example on the website is strictly quasiconvex in prices. Explain why it cannot be restricted to be convex in prices, nor convex or concave in income.

3. A competitive firm has the production function

$$x = L^{0.5}K^{0.3}$$

Derive its profit function and confirm this chapter's results on its derivatives.

4. Derive the indirect utility and expenditure functions for a consumer with the utility function

$$u = x_1^{0.6}x_2^{0.4}$$

with prices $p_1 = 1$, $p_2 = 2$, and income $m = 100$. Then find:

(a) The amount of extra income we would have to give the consumer to allow her to maintain the utility level initially reached, when we increase the price of good 1 to $p_1 = 2$

(b) The maximum amount of income the consumer would be prepared to pay to induce us *not* to increase the price of good 1 to $p_1 = 2$

Show that these two amounts can be expressed in terms of differences in values of the expenditure function.

5. A consumer has utility function

$$u = (x_1 - c_1)^a (x_2 - c_2)^{1-a}$$

where $c_1, c_2 > 0$ are interpreted as subsistence levels. Derive her indirect utility and expenditure functions. (Note: The demand functions for this problem were derived in the answer to exercise 5 for section 13.1.)

6. For the constrained optimization problems exercise 1 at the end of section 13.1, derive the value function as a function of the *constraint constant* in each case.

7. Given the envelope relation between the long-run and short-run total-cost curves shown in figure 14.14 (b), show that this implies that:

(a) At an output such as y_a in figure 14.14 (b), short-run marginal cost $\partial c(y_a, k_a)/\partial y$ equals long-run marginal cost $C'(y_a)$.

(b) Short-run average cost $c(y_a, k_a)/y_a$ equals long-run average cost $C(y_a)/y_a$.

Sketch the implied average and marginal cost curves.

CHAPTER REVIEW

Key Concepts

comparative statics
envelope theorem
fundamental equation
implicit function
implicit function theorem
shadow wage rate
Slutsky equation
value function

Review Questions

1. **(a)** What does comparative-statics analysis do?

(b) How does it do it?

2. What are the sufficient conditions under which we can carry out the comparative-statics analysis of a general economic model?

3. In a model based on an optimization problem, how may the second-order conditions help us in the comparative-statics analysis?

4. How do we proceed if the sign of the comparative-statics effect cannot be determined unambiguously?

5. What is the value function in a constrained optimization problem? Name some economic examples of value functions.

6. State, prove, and explain the envelope theorem.

7. What does the envelope theorem tell us about the interpretation of a Lagrange multiplier in a constrained optimization problem? Give some examples of such Lagrange multipliers and their interpretations.

Review Exercises

1. For each of the following constrained optimization problems, find the comparative-statics effects of a change in the α-variables, and derive and sketch the value functions:

   (a) $\max y = x_1^{0.25}x_2^{0.75}$ subject to $2x_1 + 4x_2 = \alpha$

   (b) $\max y = 2x_1 + 3x_2$ subject to $\alpha_1 x_1^2 + 5x_2^2 = \alpha_2$

   (c) $\max y = x_1^{0.25}x_2^{0.75}$ subject to $2x_1^2 + 5x_2^2 = \alpha$

   (d) $\min y = 2x_1 + 4x_2$ subject to $x_1^{0.25}x_2^{0.75} = \alpha$

   (e) $\max y = (x_1 + 2)(x_2 + 1)$ subject to $x_1 + x_2 = \alpha$

   (f) $\min y = \alpha_1 x_1 + x_2$ subject to $(x_1 + 2)(x_2 + 1) = \alpha_2$

2. Compare and explain the difference in the comparative-statics results for the following IS-LM models:

   (a) Consumption function: $C = 0.8Y + 30r$
   Investment function: $I = I^0 + 0.1Y - 10r$
   Demand for money: $30 + 0.2Y - 10.5r$
   $M^0 = 100,\ I^0 = 20$

   (b) Consumption function: $C = 0.8Y - 30r$
   Investment function: $I = I^0 + 0.1Y - 10r$
   Demand for money: $30 + 0.06Y - 60r$
   $M^0 = 100,\ I^0 = 20$

(c) Consumption function: $C = 0.8Y - 30r$
Investment function: $I = I^0 + 0.1Y - 10r$
Demand for money: $50 + 0.5Y - 0.11r$
$M^0 = 100,\ I^0 = 20$

3. A consumer has the utility function

$$u = u(x_{11}, x_{12}) + \beta u(x_{21}, x_{22}), \qquad 0 < \beta < 1$$

where x_{ti} is the amount of good $i = 1, 2$ consumed in period $t = 1, 2$. The prices of the goods are p_1 and p_2, and are the same in each period. The consumer's income in period t is m_t and not necessarily equal in both periods.

(a) Assume first that the consumer has separate budget constraints in each period. Derive the indirect utility function and comment on its form. Interpret the Lagrange multipliers in this problem. Under what conditions are they equal?

(b) Now assume it is possible to borrow or lend income between the two periods at a fixed interest rate r. Show that the consumer cannot be worse off as a result of this. Give conditions under which she is strictly better off. Obtain the indirect utility function in this case.

4. The demand functions for two goods are given by

$$D_i = D_i(p_1, p_2, y), \qquad i = 1, 2$$

where p_1 and p_2 are prices and y is consumers' income. The supply functions are

$$S_i = S_i(p_i), \qquad i = 1, 2$$

Carry out a general comparative-statics analysis of the effect of a change in consumers' income on the equilibrium prices of the goods.

5. A consumer has the strictly quasiconcave indirect utility function

$$u = v(m_1) + \beta v(m_2), \qquad 0 < \beta < 1$$

where m_t is income available for expenditure on consumption in period

$t = 1, 2$. Her *wealth constraint* is

$$m_1 + \frac{m_2}{1+r} = \bar{m}_1 + \frac{\bar{m}_2}{1+r}$$

where r is the interest rate at which she can borrow or lend and $\bar{m}_t$ is the exogenously *endowed income* in period t. Derive and interpret the Slutsky equations for the effect of changes in the interest rate on the choice of income in period $t = 1, 2$.

6. A firm produces two outputs and wants to maximize sales revenue, subject to the constraint that its net of tax profit $(1 - \tau)\pi$ must be no less than a given value, π^0. Here, τ is the rate of profit tax. Analyze the comparative-statics effects of a change in the profit's tax rate on the firm's output choices and compare these to the case of a profit-maximizing firm.

Chapter 15 Concave Programming and the Kuhn-Tucker Conditions

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Cost Minimization
- The Linear-Programming Problem

In the constrained optimization problems of chapter 13, we used the case where the function constraints are always *equalities*. This is usually referred to as the "classical optimization problem." However, sometimes this is not the most sensible formulation of a problem from the point of view of economics, and problems can arise that require us to set the constraints as *weak inequalities*. In this chapter we develop the necessary conditions for solutions of this type of problem. Because it is assumed that the objective and constraint functions are all concave, it is generally referred to as the *concave-programming problem*.

15.1 The Concave-Programming Problem

We can write the concave-programming problem in a simple form as

$$\max_{x_1, x_2} f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) \geq 0, \qquad x_1, x_2 \geq 0 \tag{15.1}$$

The name **concave programming** was used to distinguish this type of problem from that of *linear programming*, which is a special case of concave programming. The word "concave" appears because the functions f and g are assumed to be concave. We discuss the reasons for this below. We also assume the functions to be differentiable. Note also the non-negativity conditions on the variables. More general forms of the problem are obtained by increasing the numbers of variables and constraints. Also problems in which the f or g functions are convex can be handled by noting that if a function is convex, then its negative is concave (see section 2.4).

As with optimization with equality constraints, there are first-order, necessary conditions for the solution (x_1^*, x_2^*) to the problem in equation (15.1). These conditions are known as the **Kuhn-Tucker (K-T) conditions.** We proceed by first stating the rule for deriving the K-T conditions, and then provide the argument to justify this rule. The reader who is prepared to accept the justification can proceed directly to the applications in the problem solutions.

Definition 15.1

(Kuhn-Tucker conditions) To derive the K-T conditions for solution of the problem

$$\max f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) \geq 0, \qquad x_1, x_2 \geq 0$$

where both functions are concave and differentiable, we first form the Lagrange function

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda g(x_1, x_2)$$

and then *maximize* it with respect to the variables x_1 and x_2, and *minimize* it with respect to the variable λ, subject to the constraints $x_1, x_2, \lambda \geq 0$. This yields the K-T conditions

$$\frac{\partial \mathcal{L}}{\partial x_i} = f_i(x_1^*, x_2^*) + \lambda^* g_i(x_1^*, x_2^*) \leq 0, \qquad x_i^* \geq 0$$

$$x_i^* \frac{\partial \mathcal{L}}{\partial x_i} = 0, \qquad i = 1, 2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = g(x_1^*, x_2^*) \geq 0, \qquad \lambda^* \geq 0$$

$$\lambda^* \frac{\partial L}{\partial \lambda} = 0$$

You should refer to section 12.3, where we develop the necessary conditions for maximization and minimization subject to nonnegativity conditions on the variables.

We will explore the interpretation of the K-T conditions in the context of some economic applications. Note that we have introduced a kind of optimization procedure that we have not encountered earlier in this book, that of finding a maximum of a function with respect to some variables and a minimum of the function with respect to one or more others. This procedure is called finding a **saddle point** of a function. We have already discussed the general idea of a saddle point of a function in section 12.1. Essentially then, solving the concave-programming problem involves finding a saddle point of the Lagrange function.

The justification for this procedure is provided by the Kuhn-Tucker theorem. Here we will state this theorem and prove a version of it for the simple case of one constraint and two variables. Later we will give a more general statement, but without proof.

Theorem 15.1 **(Kuhn-Tucker theorem)** Given the problem

$$\max f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) \geq 0, \qquad x_1, x_2 \geq 0$$

if f and g are concave and differentiable, and if there exists a point (x_1^0, x_2^0) such that $g(x_1^0, x_2^0) > 0$ (this is called **Slater's condition**), then there exists a Lagrange multiplier λ^* such that the K-T conditions given in definition 15.1 are both necessary and sufficient for the point (x_1^*, x_2^*) to be a solution to this problem.

Proof

First we show that if f and g are concave and differentiable, if Slater's condition is met, and if (x_1^*, x_2^*) is a solution to the concave-programming problem, then there exists a λ^* such that $(x_1^*, x_2^*, \lambda^*)$ satisfies the K-T conditions, and these conditions are *necessary conditions* for a solution.

We first make use of an important fact about concave functions. Suppose that we have two concave functions $h_1(x_1, x_2)$ and $h_2(x_1, x_2)$ defined for all $x_1, x_2 \geq 0$. Suppose also that there is *no point* (x_1^0, x_2^0) *in this domain such that* $h_1(x_1^0, x_2^0) > 0$ *and* $h_2(x_1^0, x_2^0) > 0$. Then it is always possible to find numbers $p_1, p_2 \geq 0$, *not both zero*, such that for all (x_1, x_2) in the domain

$$p_1 h_1(x_1, x_2) + p_2 h_2(x_1, x_2) \leq 0$$

In words, as long as there is no point in the domain at which the function values are simultaneously positive, it is always possible to find nonnegative weights such that the weighted sum of the function values is nonpositive—the negative value of one of the functions can be made to at least offset the positive value of the other function (if one exists) for *fixed weights* p_1 and p_2. Figure 15.1 illustrates for a number of pairs of concave functions.

This fact can be used directly to establish the proposition we want to prove. Consider the following pair of inequalities, involving the constraint function and objective function of the concave-programming problem,

$$g(x_1, x_2) > 0$$
$$f(x_1, x_2) - f(x_1^*, x_2^*) > 0$$

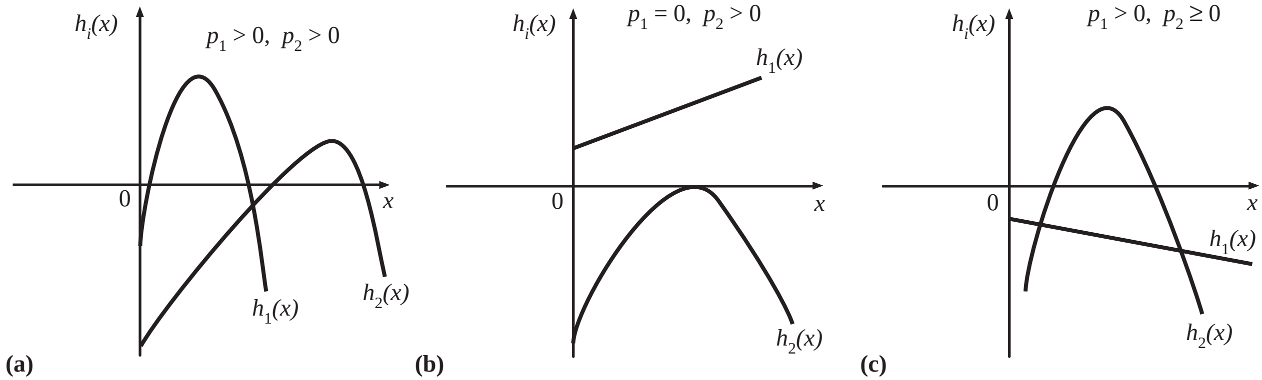


Figure 15.1 Fundamental property of concave functions

for $x_1, x_2 \geq 0$, with (x_1^*, x_2^*) a solution to the concave-programming problem. By the definition of a solution, no pair of x-values in the domain can satisfy these strict inequalities, since that would imply that there exists a feasible point that is better than the optimal point. Therefore there exist $p_1, p_2 \geq 0$, not both zero, such that

$$p_1[f(x_1, x_2) - f(x_1^*, x_2^*)] + p_2 g(x_1, x_2) \leq 0$$

implying that

$$p_1 f(x_1, x_2) + p_2 g(x_1, x_2) \leq p_1 f(x_1^*, x_2^*)$$

Since $p_2 \geq 0$ and $g(x_1^*, x_2^*) \geq 0$, we must have $p_2 g(x_1^*, x_2^*) \geq 0$. But if we set $x_1 = x_1^*$ and $x_2 = x_2^*$ in the first inequality, we also have $p_2 g(x_1^*, x_2^*) \leq 0$. Thus we must have $p_2 g(x_1^*, x_2^*) = 0$, and so we can add this to the right-hand side of the second inequality without affecting it. Also for *any* $p \geq 0$, we must have $pg(x_1^*, x_2^*) \geq 0$, and finally we can write

$$\begin{aligned} p_1 f(x_1, x_2) + p_2 g(x_1, x_2) &\leq p_1 f(x_1^*, x_2^*) + p_2 g(x_1^*, x_2^*) \\ &\leq p_1 f(x_1^*, x_2^*) + pg(x_1^*, x_2^*) \end{aligned}$$

for *all* $p \geq 0, x_1, x_2 \geq 0$. This is almost, but not quite, the result we are after. If we now define $\lambda^* = p_2/p_1$ and $\lambda = p/p_1$, we can write

$$f(x_1, x_2) + \lambda^* g(x_1, x_2) \leq f(x_1^*, x_2^*) + \lambda^* g(x_1^*, x_2^*) \leq f(x_1^*, x_2^*) + \lambda g(x_1^*, x_2^*)$$

for *all* $\lambda \geq 0, x_1, x_2 \geq 0$. Thus we *appear to have* the result that under the condition of concavity of f and g, with an optimal solution to the concave-programming problem (x_1^*, x_2^*) we can always associate a Lagrange multiplier $\lambda^* \geq 0$ such that

(x_1^*, x_2^*) *maximizes* and λ^* *minimizes* the Lagrange function subject to $x_1 \geq 0$, $x_2 \geq 0, \lambda \geq 0$.

In defining $\lambda^*$and λ, we implicitly assumed that $p_1 \neq 0$, so that we could divide through by it. However, this may not always be warranted. Thus suppose that $g(x_1, x_2) \leq 0$ for all $x_1, x_2 \geq 0$. Then we could choose $p_1 = 0$, $p_2 > 0$ as our weights. In that case it would not be possible to define λ and λ^* (refer, e.g., to figure 15.1 (b)). For the result to hold, we therefore need a further condition known as a *constraint qualification,* which is Slater's condition in definition 15.1: there exist $x_1^0, x_2^0 \geq 0$ such that $g(x_1^0, x_2^0) > 0$.

Intuitively this condition implies that p_1 can never be set at 0, since if it were, p_2 must be positive and the inequality could not then be satisfied (see figure 15.1 (c)). Thus, for a problem satisfying Slater's condition, with the constraint function taking on a strictly positive value somewhere in the domain, the Lagrange procedure always works. (See exercise 1 in this section for a problem in which Slater's condition is not satisfied and the Lagrange procedure does not work.)

The inequalities we have just derived define $(x_1^*, x_2^*, \lambda^*)$ as a saddle point of the Lagrange function and so they are referred to as the **saddle-point conditions.** Given the differentiability of f and g, they are then equivalent to the K-T conditions, because both are equivalent to the statements

(x_1^*, x_2^*) maximizes $f(x_1, x_2) + \lambda^* g(x_1, x_2)$ for all $x_1, x_2 \geq 0$
λ^* minimizes $f(x_1^*, x_2^*) + \lambda g(x_1^*, x_2^*)$ for all $\lambda \geq 0$

To complete the proof, we now have to show that a point (x_1^*, x_2^*) that satisfies the saddle-point conditions, or equivalently the K-T conditions, is a solution to the concave-programing problem, so these conditions are sufficient as well as necessary. From the saddle-point condition we have the inequality

$$f(x_1^*, x_2^*) + \lambda^* g(x_1^*, x_2^*) \geq f(x_1, x_2) + \lambda^* g(x_1, x_2) \qquad \text{for all } x_1, x_2 \geq 0$$

From the second K-T condition in definition 15.1, we have that if $\lambda^* > 0$, then $g(x_1^*, x_2^*) = 0$, and that if $g(x_1^*, x_2^*) > 0$, then $\lambda^* = 0$. In the first case, the inequality becomes

$$f(x_1^*, x_2^*) \geq f(x_1, x_2) + \lambda^* g(x_1, x_2)$$

Since for *feasible* (x_1, x_2) we must have $g(x_1, x_2) \geq 0$, it follows that

$$f(x_1^*, x_2^*) \geq f(x_1, x_2)$$

In the second case, if $\lambda^* = 0$, the same inequality follows immediately. ■

This concludes our discussion of the theoretical basis for the K-T conditions. We now turn to applications of the conditions.

The Consumer's Problem

The consumer's problem, when we allow for the consumer to spend less than the maximum available and when we explicitly allow for nonnegativity constraints, is

$$\max u(x_1, x_2) \quad \text{s.t.} \quad m - p_1x_1 - p_2x_2 \geq 0 \quad \text{and} \quad x_1, x_2 \geq 0$$

The Lagrange function is

$$\mathcal{L} = u(x_1, x_2) + \lambda(m - p_1x_1 - p_2x_2)$$

and the K-T conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = u_i - \lambda^* p_i \leq 0, \quad x_i^* \geq 0, \quad x_i^*(u_i - \lambda^* p_i) = 0, \qquad i = 1, 2 \tag{15.2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1x_1^* - p_2x_2^* \geq 0, \quad \lambda^* \geq 0, \quad \lambda^*(m - p_1x_1^* - p_2x_2^*) = 0 \tag{15.3}$$

We need only consider cases in which at least one of the demands is positive at the optimum. Then from the first condition we see that if $x_i^* > 0$ and its marginal utility u_i is positive, we must also have $\lambda^* > 0$. It follows from the second condition that the budget constraint must be satisfied as an equality; that is, it is a binding constraint. Since it is usually assumed in consumer theory that $u_i > 0$—this is known as the "non-satiation" assumption—we have the justification for always taking the budget constraint as an equality rather than a weak inequality.

Next suppose that $x_1^* > 0$, $x_2^* > 0$. Then condition (15.2) becomes $u_i = \lambda^* p_i$, $i = 1, 2$, giving the condition

$$\frac{u_1}{u_2} = \frac{p_1}{p_2}$$

This is the condition that the indifference curve be tangent to the budget constraint, as illustrated in figure 15.2 (a).

There are, however, two further cases to consider, namely those in which one of the goods has a zero value at the optimum. Take the case

$$x_1^* > 0, \quad x_2^* = 0$$

From condition (15.3) we then have $x_1^* = m/p_1$, while condition (15.2) implies that

$$u_1 = \lambda^* p_1$$
$$u_2 \leq \lambda^* p_2$$

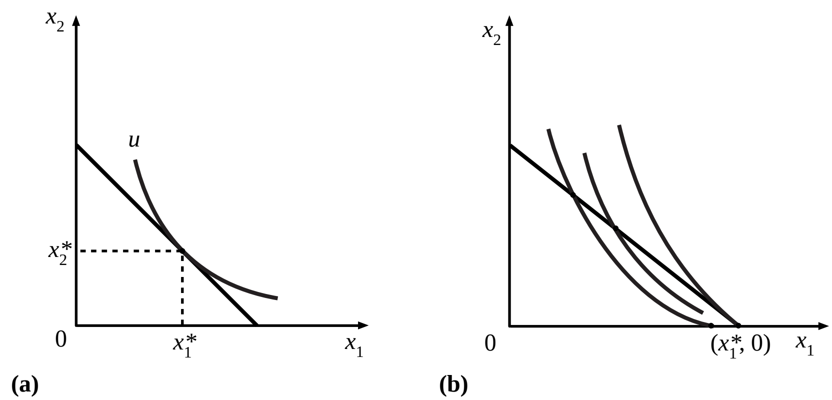


Figure 15.2 Solution possibilities for the consumer's problem

giving

$$\frac{u_1}{u_2} \geq \frac{p_1}{p_2}$$

(Be sure you can explain the direction of this inequality.) This solution is illustrated in figure 15.2 (b). There is not an interior tangency solution, because along the budget constraint the indifference curves are everywhere steeper than the budget line. At the corner solution, the slope of the indifference curve is either equal to, or greater in absolute value than, that of the budget line, as the condition above states. The reader should now derive the condition and draw the diagram for the case in which $x_1^* = 0$, $x_2^* > 0$.

In the usual textbook treatment of the consumer's problem, a case is presented in which both goods are positive. However, if we consider a world with many goods and not just two, it is clear that the corner solution is in fact the rule since no one consumes something of every good in existence. The intuitive explanation of the equilibrium can be put as follows: at the point $(x_1^*, 0)$ in figure 15.2 (b), the consumer can exchange one unit of good 1 for p_1/p_2, units of good 2 but would require u_1/u_2 units of good 2 to be just as well off, and since $u_1/u_2 \geq p_1/p_2$, she will not want to make the trade.

EXERCISES

1. Consider the problem

$$\max x_1 + x_2 \quad \text{s.t.} \quad -\left(x_1^2 + x_2^2\right) \geq 0 \quad \text{and} \quad x_1, x_2 \geq 0$$

Try solving it using the K-T conditions. What problems arise? Why? [Hint: Check Slater's condition].

2. In the cost-minimization problem, what consequences follow from assuming the following cases?

(a) It is possible to produce output using only labor

(b) One of the input prices may be zero

3. Solve the problem

$$\max u = (x_1 + a)x_2^b \quad \text{s.t.} \quad p_1x_1 + p_2x_2 \leq m \quad \text{and} \quad x_1, x_2 \geq 0$$

where $a, b \geq 0$. Discuss the economics of the solution.

4. Solve the following problems:

(a) $\max y = 3x_1 + 2x_2$ subject to $4x_1 + x_2 \leq 10$ $x_1, x_2 \geq 10$

(b) $\max y = 8x_1 + 2x_2$ subject to $4x_1 + x_2 \leq 10$ $x_1, x_2 \geq 10$

(c) $\max y = 10x_1 + 2x_2$ subject to $4x_1 + x_2 \leq 10$ $x_1, x_2 \geq 10$

5. A central planner uses labor L to produce two outputs, x_1 and x_2, according to the production functions

$$x_1 = 10L_1 - 0.5L_1^2$$
$$x_2 = 8L_2 - 0.75L_2^2$$

where L_i is labor allocated to sector $i = 1, 2$. She wishes to maximize the value of total output

$$V = 4x_1 + 5x_2$$

where 4 and 5 are the world prices of x_1 and x_2 respectively.

(a) Assuming that there are 12 units of labor available, find the optimal labor allocation and the shadow price of labor.

(b) Assume now that there are 20 units of labor available. What is the new optimal labor allocation and the shadow price of labor?

(c) Illustrate and explain your results in each case.

(Note: Formulate the problem as having just one constraint by substituting for x_1 and x_2 in the V function.)

6. Solve the cost-minimization problem

$$\min wL + rK + vR \quad \text{s.t.} \quad R^a(L + bK)^c - y \geq 0 \quad \text{and} \quad R, L, K \geq 0$$

where R is raw material, L labor, and K capital, and $a, b, c > 0$, with $a + c < 1$. Interpret the production function in this problem.

15.2 Many Variables and Constraints

In the previous section, we developed and applied the K-T conditions for the case of one constraint and two variables. We now extend the discussion to the case of any number of variables and constraints. Note that it is not necessary to make any assumption about the relation between the number of variables and number of constraints, as we had to do in the classical case, because the constraints are now in the form of inequalities and we do not have to worry that more constraints than variables cause an empty feasible set.

By using exactly the same method of proof as we used in the previous section, it is possible to prove

Theorem 15.2 **(Kuhn-Tucker theorem)** Given the problem

$$\max f(x_1, x_2, \ldots, x_n) \quad \text{s.t.} \quad \begin{aligned} g^1(x_1, x_2, \ldots, x_n) &\geq 0 \\ g^2(x_1, x_2, \ldots, x_n) &\geq 0 \\ \ldots\ldots\ldots&\ldots\ldots \\ g^m(x_1, x_2, \ldots, x_n) &\geq 0 \\ x_1, x_2, \ldots, x_n &\geq 0 \end{aligned}$$

if all the functions f and g^j, $j = 1, \ldots, m$ are concave and differentiable, and there exists a point $(x_1^0, x_2^0, \ldots, x_n^0)$ such that $g^j(x_1^0, x_2^0, \ldots, x_n^0) > 0$, all $j = 1, \ldots, m$, then there exist m Lagrange multipliers λ_j^* such that the following conditions are necessary and sufficient for the point $(x_1^*, x_2^*, \ldots, x_n^*)$ to be a solution for this problem:

$$\begin{aligned} f_i(x_1^*, x_2^*, \ldots, x_n^*) - \sum \lambda_j^* g_i^j(x_1^*, x_2^*, \ldots, x_n^*) &\leq 0, & x_i^* &\geq 0 \\ x_i^* \left(f_i - \sum \lambda_j^* g_i^j \right) &= 0, & i &= 1, \ldots, n \\ g^j(x_1^*, x_2^*, \ldots, x_n^*) &\geq 0, & \lambda_j^* &\geq 0 \\ \lambda_j^* g^j(x_1^*, x_2^*, \ldots, x_n^*) &= 0, & j &= 1, \ldots, m \end{aligned}$$

We note that the K-T conditions for the general problem are derived by forming the Lagrange function

$$\mathcal{L} = f(x_1, x_2, \ldots, x_n) + \sum \lambda_j g^j(x_1, x_2, \ldots, x_n)$$

and then *maximizing* it with respect to the x_i and *minimizing* it with respect to the λ_j subject to the nonnegativity conditions $x_i \geq 0, \lambda_j \geq 0$.

We now use this in an economic application.

Time Constraint

Time is generally a scarce resource, as is income. Moreover it generally takes time to buy and consume goods. Let us take the simple case of two goods, and suppose that one unit of good 1 takes t_1 units of time to consume and that one unit of good 2 takes t_2 units of time. The total time available for consumption is T. The consumer has a utility function defined on goods alone, $u(x_1, x_2)$. Then her problem is

$$\max u(x_1, x_2) \quad \text{s.t.} \quad \begin{aligned} p_1x_1 + p_2x_2 &\leq m \\ t_1x_1 + t_2x_2 &\leq T \\ x_1, x_2 &\geq 0 \end{aligned}$$

If the first constraint is a money-budget constraint; the second could be thought of as a time-budget constraint. It clearly makes sense to express these as inequalities since to make them equalities would imply, in general, that at most one consumption bundle can be bought. When we write them as inequalities, we allow for the possibility that the consumer may have too much time and not enough money or the converse.

The Lagrange function for the problem is

$$\mathcal{L} = u(x_1, x_2) + \lambda_1(m - p_1x_1 - p_2x_2) + \lambda_2(T - t_1x_1 - t_2x_2)$$

and the K-T conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = u_i - \lambda_1^* p_i - \lambda_2^* t_i \leq 0, \quad x_i^* \geq 0, \quad x_i^*(u_i - \lambda_1^* p_i - \lambda_2^* t_i) = 0, \qquad i = 1, 2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = m - p_1x_1^* - p_2x_2^* \geq 0, \quad \lambda_1^* \geq 0, \quad \lambda_1^*(m - p_1x_1^* - p_2x_2^*) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = T - t_1x_1^* - t_2x_2^* \geq 0, \quad \lambda_2^* \geq 0, \quad \lambda_2^*(T - t_1x_1^* - t_2x_2^*) = 0$$

These conditions define five possible cases of interest, but two of these involve the possibility that one of the x-values is zero. Since we discussed this kind of case earlier, we will not do that here but simply assume both variables are strictly positive at the optimum. Then the first condition becomes a strict equality. Given nonsatiation, so that $u_i > 0$, the first condition implies immediately that both λ-values cannot be zero—at least one constraint must bind. Thus we have three cases to consider:

Case 1 $\lambda_1^* > 0$, $\lambda_2^* > 0$. The second and third conditions then become equalities and as a result we can solve them for the optimal x-values. Cramer's rule yields

$$x_1^* = \frac{mt_2 - Tp_2}{p_1t_2 - p_2t_1}$$
$$x_2^* = \frac{Tp_1 - mt_1}{p_1t_2 - p_2t_1}$$

Diagrammatically, as figure 15.3 (a) shows, this solution occurs at the intersection of the two lines corresponding to the two constraints. The highest possible indifference curve touches the kink formed by the intersection of the lines. The feasible set in the problem is, of course, the shaded area. We can use the first condition to obtain the condition for this case

$$\frac{u_1}{u_2} = \frac{\lambda_1^* p_1 + \lambda_2^* t_1}{\lambda_1^* p_2 + \lambda_2^* t_2}$$

This says that the marginal rate of substitution lies between the values p_1/p_2 and t_1/t_2, as is readily confirmed in the diagram.

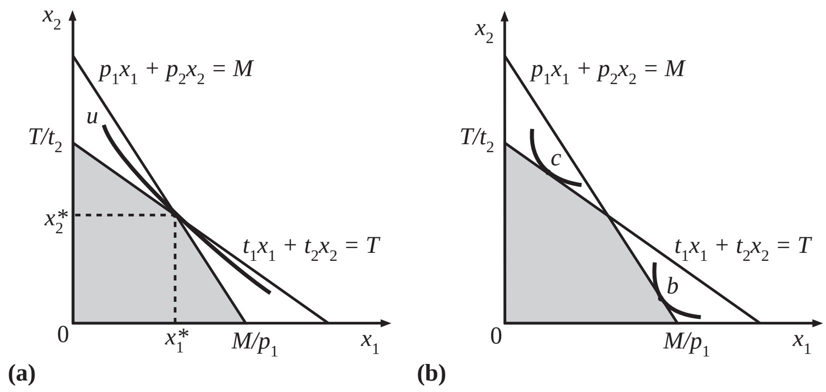


Figure 15.3 Solution possibilities in the time-constraint problem

Case 2 $\lambda_1^* > 0, \lambda_2^* = 0$. Only the money-budget constraint is binding. In that case the first condition yields simply

$$\frac{u_1}{u_2} = \frac{p_1}{p_2}$$

Thus we have a standard tangency solution with the money-budget constraint. This case may occur because the time-budget constraint lies everywhere above the money constraint, or because the preferences of the consumer are such that a tangency occurs only on the part of the boundary of the feasible set formed by the money-budget constraint. This latter case is illustrated in figure 15.3(b). This is a case in which the consumer has time to spare, while money is relatively scarce—the case usually considered in models of the consumer.

Case 3 $\lambda_1^* = 0, \lambda_2^* > 0$. Only the time constraint is binding. In that case the first condition yields

$$\frac{u_1}{u_2} = \frac{t_1}{t_2}$$

This is a tangency with the time constraint, as illustrated in figure 15.3 (b). Such a consumer has money to spare but not enough time in which to spend it all!

EXERCISES

1. A consumer has the utility function

$$u = x_1 x_2$$

and she faces the money-income constraint

$$2x_1 + 3x_2 \leq 100$$

and the time constraint

$$x_1 + 4x_2 \leq 80$$

Solve for her utility-maximizing consumption bundle and the values of the shadow prices of the constraints.

2. Suppose that it is possible to exchange money for time at a fixed wage rate w (e.g., you could hire someone to buy and prepare food, or be hired for that purpose). Show that the two budget constraints in the "time-constraint problem" collapse to one. How does this budget line relate to those drawn in figure 15.4? Explain why the existence of this market cannot make anyone worse off and makes some people better off as compared to the situation shown in figure 15.4.

3. A planner in a small, open economy has the utility function

$$u = x_1 x_2$$

where x_i is the aggregate consumption of good $i = 1, 2$. Production in the economy takes place according to the production functions

$$y_1 = 10 l_1^{0.5}, \quad y_2 = 5 l_2^{0.5}$$

where y_i is output and l_i is labor in sector i. The world prices of the goods are $p_1 = \$10$ and $p_2 = \$8$. The total amount of labor available in the economy is 1,000 units. Find the utility-maximizing outputs, consumptions, and labor allocations. What is the shadow wage rate at the solution? Illustrate and explain your answers.

4. A planner in a small, open economy has a utility function

$$u = x_1 x_2$$

and production takes place according to the production functions

$$y_1 = 10L_1 - 0.5L_1^2, \quad y_2 = 10L_2 - L_2^2$$

The world prices of the goods are $p_1 = \$10$ and $p_2 = \$5$. The total amount of labor available in the economy is five units. Find the utility-maximizing outputs, consumptions, and labor allocations, and the solution value of the shadow wage rate. Compare your solution to that of the previous problem. Now assume that the economy has 20 units of labor. Solve the problem again and compare this solution to the previous two.

CHAPTER REVIEW

Key Concepts

concave programming
Kuhn-Tucker conditions
saddle point
saddlepoint conditions
shadow price
Slater's condition

Review Questions

1. What are the characteristic features of a concave-programming problem?
2. What is meant by a saddle point of the Lagrange function?
3. State the Kuhn-Tucker (K-T) conditions for both a maximization problem and a minimization problem.
4. The point that satisfies the K-T conditions for a concave-programming problem must be an optimal solution for that problem. Why is this so?
5. Why is any linear-programming problem a special case of the concave-programming problem?

Review Exercises

1. A worker has the utility function

$$u = x_1 x_2$$

where x_1 is income and x_2 is leisure. Her budget constraint is given by

$$x_1 = m + w(T - x_2)$$

where m is nonwage income, w is the given wage rate, and T is total time available to be divided between work and leisure. There is, however, a maximum number of hours she can work, given by $H \leq T$. Formulate and solve the problem of optimal choice of income and leisure (or work) given this constraint. Illustrate your results.

2. A consumer has the utility function $u(x_1, x_2)$ defined on two goods. The budget constraint is of the usual kind. However, she is rationed in the market for each good in that there is a maximum, $\bar{x}_i$ of each good $i = 1, 2$, that she is able to buy. Solve the consumer's optimization problem and discuss the form of her demand functions for the goods.

3. An investor has a given income of $1,000 this period and is certain also to receive an income of $1,000 next period. He can lend money for one period at an interest rate $r_L = 0.05$, or he can borrow money for one period at an interest rate of $r_B = 0.15$. His utility function is

$$u = \log x_1 + 0.8 \log x_2$$

where x_t is his consumption in period $t = 1, 2$. Find his utility-maximizing borrowing or lending. [Hint: First sketch the feasible set.]

4. A central planner wants to allocate three resources to the production of two goods. The production technology is linear, and the following table gives the amount of each input required to produce one unit of the given output:

		Output	
		x_1	x_2
Input	L	2	3
	K	2.5	2
	R	3	0.5

The planner wants to maximize the value of output at world prices, which are \$10 for x_1 and \$8 for x_2. The resource availabilities are

$$L^0 = 100, \quad K^0 = 100, \quad R^0 = 90$$

Find the optimal resource allocation and the associated shadow prices of the resources.

5. A consumer has the utility function

$$u = x_1 x_2$$

where x_1 is meat and x_2 potatoes. The price of meat is \$10 per pound and that of potatoes is \$1 per pound. She has an income of \$80. In addition to her budget constraint, she has a *subsistence-calorie constraint:* she must consume at least 1,000 calories. One pound of potatoes yields 20 calories, one pound of meat yields 60 calories. Find her optimal consumption bundle. Now suppose that the price of potatoes rises to \$1.60 per pound. Find the new optimal bundle. Explain your results, and discuss their significance for the concept of a Giffen good.

6. A monopolist supplies a market with the inverse-demand function

$$p = 100 - (q_1 + q_2)$$

where q_1 is output produced at plant 1 and q_2 output produced at plant 2. The total-cost functions at the plants are

$$C_1 = q_1^2, \quad C_2 = 1.25q_2^2$$

Each plant has a *fixed-maximum capacity* of 12 units of output.

(a) Suppose that the firm is able to expand capacity by one unit *at only one plant*. Which plant should it choose?

(b) Suppose now that the firm is able to invest in expanding capacity by four units in total. How should that be allocated between the two plants?

(c) If instead the firm is able to invest 10 units of capacity output, how should that be allocated?

Illustrate and explain your answers in each case.

Part V Integration and Dynamic Methods

Chapter 16 Integration

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- More on Consumer Surplus Measurement

In this chapter, we address the question of whether knowing the derivative of a function, $f'(x)$, allows one to determine, or recover, the original function $f(x)$. Since this process is the reverse of differentiation it is referred to as antidifferentiation, although it is also referred to as finding the indefinite integral. Related to this concept is the definite integral of a function, which is the area beneath a curve between two points. The process of integration is very useful in economics as it reflects the relationship between stocks and flows (e.g., investment and capital stock) and marginal and total concepts (e.g., marginal and total cost). It also provides the basic mathematics required to do the dynamic analysis of chapters 17 to 25.

16.1 The Indefinite Integral

Given the function $f(x) = x^2$, we know from the rules of differentiation that its derivative is $f'(x) = 2x$. Therefore, having been told that the derivative of some function is $f'(x) = 2x$, one might expect that we can determine, by reasoning in reverse, that the function that has this derivative is $f(x) = x^2$. However, since for any constant value C, the function $f(x) = x^2 + C$ has this same derivative, it is clear that we cannot recover *entirely* the form of the original function simply by knowing its derivative. In general, since the derivative of a constant is zero, it follows that if the function $f(x)$ has derivative function $f'(x)$, then any function $g(x) = f(x) + C$ also has derivative function $f'(x)$. Therefore, knowing the derivative of a function allows one to determine the function itself only up to some unknown constant term that must be added to it. This process is called **antidifferentiation** for the obvious reason that it is the inverse operation of differentiation. We say that the derivative of the function $f(x) = x^2 + C$ is $f'(x) = 2x$ and that the antiderivative of $f'(x) = 2x$ is the function $x^2 + C$.

Although the term antidifferentiation is perhaps a more obvious one to use for the process that is the reverse of differentiation, this operation is more usually referred to as **integration.** Also the antiderivative of a function is referred to as the **indefinite integral.** Later in this chapter we introduce the **definite integral.** We usually drop the qualifier (i.e., definite vs. indefinite) if no confusion is likely.

Although the process of integration is just the inverse of differentiation, there is some added notation to be learned. Rather than referring to the function by $f(x)$ and its derivative by $f'(x)$, it is more common to use $F(x)$ to refer to the function and $f(x)$ as its derivative (i.e., if $f(x) = dF(x)/dx$, then $F(x)$ is the antiderivative or integral of $f(x)$). Since knowing the derivative function $f(x)$ only allows us to recover the function itself up to some arbitrary constant, we say that the (indefinite) integral of $f(x)$ is $F(x) + C$, where C is referred to as the **constant of integration.** Symbolically we write

$$\int f(x)\,dx = F(x) + C$$

In this notation $f(x)$ is called the **integrand.** Since

$$\frac{dF(x)}{dx} = f(x)$$

an alternative notation for the integral of $f(x)$ is $\int dF(x)$ where

$$\int f(x)\,dx \equiv \int \left(\frac{dF(x)}{dx}\right) dx \quad \text{or} \quad \int dF(x)$$

Since the process of integration is just the inverse of differentiation, each rule of differentiation implies a corresponding rule of integration, and so the rules of integration given below have their counterparts in section 5.4.

Rules of Integration

Rule 1 Power rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

Rule 2 Integral of a sum:

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Rule 3 Integral of a constant multiple:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Rule 4 Exponential rule:

$$\int e^x\,dx = e^x + C$$

Rule 5 Logarithmic rule:

$$\int \frac{1}{x}\,dx = \ln(x) + C, \qquad x > 0$$

Example 16.1 Integrate each of the following:

(i) $\int x^3\,dx$
(ii) $\int (x^2 + x^5)\,dx$
(iii) $\int 15x^2\,dx$

Solution

(i) Using rule 1,

$$\int x^3\,dx = \frac{x^4}{4} + C$$

(ii) Using rule 2,

$$\begin{aligned}\int (x^2 + x^5)\,dx &= \int x^2\,dx + \int x^5\,dx \\ &= \frac{x^3}{3} + C_1 + \frac{x^6}{6} + C_2 \\ &= \frac{x^3}{3} + \frac{x^6}{6} + C, \qquad \text{where } C = C_1 + C_2\end{aligned}$$

(Note that each of the integrals in this example has a constant of integration, which we can combine into a single constant.)

(iii) Using rule 3,

$$\begin{aligned}\int 15x^2\,dx &= 15\int x^2\,dx \\ &= 15\left[\left(\frac{x^3}{3}\right)+C_1\right] \\ &= 5x^3 + C, \qquad \text{where } C = 15C_1\end{aligned}$$

(Again, note the combining of constants 15 and C_1.) Each of these may be checked by differentiating the solution in each case to arrive at the integrand. ■

The rules of integration given above follow directly from the rules of differentiation. For example, to prove rule 1, we simply need to note that

$$\frac{d[x^{n+1}/(n+1)+C]}{dx} = \frac{1}{(n+1)}(n+1)x^n = x^n$$

Just as for the rules of differentiation, the rules of integration can be combined to obtain more general results. For example, given a function made up of a sum of an arbitrary number of functions,

$$h(x) = f^1(x) + f^2(x) + \cdots + f^n(x) = \sum_{i=1}^{n} f^i(x)$$

successive application of rule 2 implies that

$$\int h(x)\,dx = \int\left(\sum_{i=1}^{n} f^i(x)\right)dx = \sum_{i=1}^{n}\int f^i(x)\,dx$$

Therefore, for a particular example such as the one below, we can integrate an expression made up of a sum of terms by integrating each term separately and then summing the result

$$\int (x^2 + 3x^4 + 6x^5)\,dx = x^3/3 + 3x^5/5 + x^6 + C$$

(Check by differentiating the right side to obtain the integrand.)

Also, there are straightforward generalizations of rules 4 and 5. For example, since by the chain rule of differentiation

$$\frac{d\left[e^{f(x)}\right]}{dx} = f'(x)e^{f(x)}$$

it follows that $e^{f(x)}$ is the antiderivative of $f'(x)e^{f(x)}$; that is,

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

Therefore, if the integrand is composed of the function $e^{f(x)}$ multiplied by the derivative of $f(x)$, that is, is of the form $f'(x)e^{f(x)}$, then one can immediately write down the integral. For example

$$\int 2xe^{x^2}\,dx = e^{x^2} + C$$

A similar effect resulting from the chain rule applies to functions involving the logarithm. That is, since

$$\frac{d[\ln(f(x))]}{dx} = \frac{f'(x)}{f(x)}$$

it follows that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + C$$

The chain rule procedure is not always useful. For example, to find the integral of a function involving a term like $e^{f(x)}$, it requires that this term just happens to be multiplied by $f'(x)$. Thus, for example, the chain rule cannot be used to evaluate $\int 2x^3e^{x^2}\,dx$. Nevertheless, there are some important applications for this rule.

The examples above will probably convince you that finding the integral of an arbitrarily chosen function is not always a straightforward task. It is generally easier to check that a specified function is the integral of some *original* function if the result is provided. For example, it is not so easy to *guess* a result such as

$$\int (x^3 + e^x)(3x^2 + e^x)\,dx = \frac{(x^3 + e^x)^2}{2} + C$$

although it is easy to check its validity (i.e., just differentiate $(x^3 + e^x)^2/2$ to obtain the integrand). With a little practice, however, one can recognize certain types or patterns of integration problems that initially appear quite complex but are actually fairly easy to solve, including the example above. In section 16.5 we present techniques of integration that substantially extend the range of integrals that can be easily evaluated.

The simple rules of integration given above make it possible to solve many simple problems and to generate several economic applications of integration.

The rate of change of an economic variable, for example, is the derivative of that variable, and so integration allows one to obtain a variable upon knowing its derivative. The following examples illustrate this type of relationship.

The Relationship between Marginal- and Total-Product Functions

Let $Q(L)$ represent the total-product function and MP(L) the marginal-product function, where L is the single input labor. Since the marginal-product function is the derivative of the total-product function, we can write $\text{MP}(L) = dQ(L)/dL$. Since integration is the reverse of differentiation, then the total-product function is simply the integral of the marginal-product function, $Q(L) = \int \text{MP}(L)\, dL$. Thus, if one is told that the marginal product of an input is constant, $\text{MP}(L) = a$, a a positive constant, then we know that the total-product function is linear since

$$Q(L) = \int a dL = a \int dL = aL + C$$

Economic intuition tells us that if $L = 0$, then $Q = 0$, and so

$$Q(0) = a0 + C = 0 \Leftrightarrow C = 0$$

which means we can write simply that $Q(L) = aL$. In fact it often turns out to be the case for this sort of problem that some intuitive economic condition can be invoked to solve for the constant of integration, C.

The Relationship between the Growth in the Money Supply and the Stock of Money in the Economy

Suppose that a central bank has decided to increase the money supply by a constant amount, \$$k$ per year, for the foreseeable future. Letting $M(t)$ represent the stock of money in the economy at time t, it follows that

$$\frac{dM(t)}{dt} = k$$

and so

$$M(t) = \int k\, dt = k \int dt = kt + C$$

Suppose that we let M^0 represent the current stock of money (i.e., $M(0) = M^0$). This implies that

$$M(0) = k(0) + C = M^0 \Rightarrow C = M^0$$

and so we can rewrite $M(t)$ as

$$M(t) = kt + M^0$$

Now, if the consumer price level is a constant function of the money supply, we can write

$$P(t) = \beta M(t) = \beta kt + \beta M^0, \qquad \beta > 0$$

and deduce that the rate of increase in prices will be the constant fraction βk (since $dP(t)/dt = \beta k$).

Note that $M(t)$ is an example of a **stock variable** in economics, while the rate at which it changes over time, $dM(t)/dt$, is the related **flow variable.**

Generally speaking, the process of finding integrals involves being able to *determine* what function, when differentiated, will produce the integrand. For cases in which this is not so easy to do, one can refer to tables in such publications as the *CRC Standard Mathematical Tables* or use a symbolic software program such as *Mathematica* or *Maple*. All of the exercises for section 16.1, however, require only the application of the simple rules given earlier in this section.

EXERCISES

1. Evaluate the following integrals:

(a) $\int (x^4 + 2x^3 + 4x + 10)\,dx$

(b) $\int x^{2/3}\,dx$

(c) $\int 10e^x\,dx$

(d) $\int 6xe^{x^2}\,dx$

(e) $\int \frac{3x^2 + 2}{(x^3 + 2x)}\,dx$

2. Evaluate the following integrals:

(a) $\int (6x^3 + 10x^2 + 5)\,dx$

(b) $\int (x^{1/2} + 5x^{-2/3})\,dx$

(c) $\int 2e^x \, dx$

(d) $\int 8x^2 e^{x^3} \, dx$

(e) $\int \frac{2x^3 + 1}{x^4 + 2x} \, dx$

3. Evaluate the following integrals. Use the information provided in order to determine the constant of integration.

(a) $F(x) = \int 2 \, dx, \; F(0) = 0$

(b) $F(x) = \int 6x \, dx, \; F(0) = 5$

(c) $F(x) = \int (5x^3 + 2x + 6) \, dx, \; F(0) = 0$

(d) $F(x) = \int 2x \, dx, \; F(3) = 10$

4. Evaluate the following integrals. Use the information provided in order to determine the constant of integration.

(a) $F(x) = \int dx, \; F(0) = 0$

(b) $F(x) = \int x^{1/2} \, dx, \; F(0) = 5$

(c) $F(x) = \int (2x^3 + 4x) \, dx, \; F(0) = 0$

(d) $F(x) = \int x^{-1/2} \, dx, \; F(4) = 8$

5. A firm uses one input, labor (L), to produce output (Q). The marginal-product function for the input is $\text{MP}(L) = 10L^{1/2}$. Find the production function, $Q(L)$. Assume that $Q = 0$ if $L = O$.

6. Let $\text{MP}(L) = aL^b$ be the marginal product of labor for a firm using a single input (L) to produce output (Q). Assume that $Q = 0$ if $L = 0$. Find the production function $Q(L)$. Discuss how the value of b determines whether or not there is diminishing, increasing, or constant marginal productivity of labor. Use diagrams to explain the relationship between the appropriate conditions that must be imposed on the marginal-product function and the total-product function in order to give rise to diminishing, increasing, or constant marginal productivity of labor.

7. Prove the second rule of integration

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

by using the corresponding rule of differentiation.

16.2 The Riemann (Definite) Integral

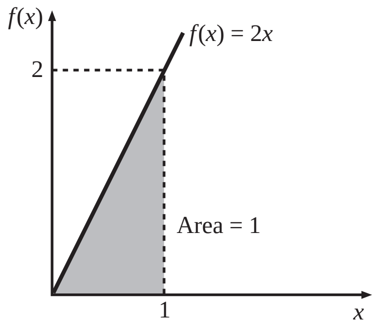


Figure 16.1 Area under the curve $f(x) = 2x$ on the interval [0, 1]

The **Riemann integral,** or definite integral, of a function defined on some interval is the area underneath the curve over that interval. For example, consider the function $f(x) = 2x$ over the interval $0 \leq x \leq 1$ drawn below. Since the area of a triangle is one-half times the base times the height, it is clear that the area beneath the curve defined by $f(x)$ over this interval is 1 (see figure 16.1). We will use this example to introduce the mathematical apparatus used to define formally the *area* under a curve and show how to compute it using the process of integration.

We first need to motivate the idea of a **partition** over an interval. Let $[a, b]$ be a closed interval, and for convenience, let us use $x_0 = a$ and $x_n = b$ as equivalent notation for the endpoints. A partition for the interval $[a, b]$ or $[x_0, x_n]$ is a set of subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

determined by values

$$x = x_0, x_1, x_2, \ldots, x_n$$

that satisfy the condition that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

For our example, $[a, b] = [0, 1]$, and so the x_i values satisfy the condition that

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$$

We will let $I_i = [x_{i-1}, x_i]$ refer to the ith interval or subinterval of the partition. Thus the partition is composed of a set of closed subintervals that *cover* the interval [0, 1] and that overlap only at the endpoints.

For example, by choosing the points $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.7$, and $x_3 = 1$, we obtain the following partition of [0, 1] (see figure 16.2):

$$\{[0, 0.2],\ [0.2, 0.7],\ [0.7, 1]\}$$

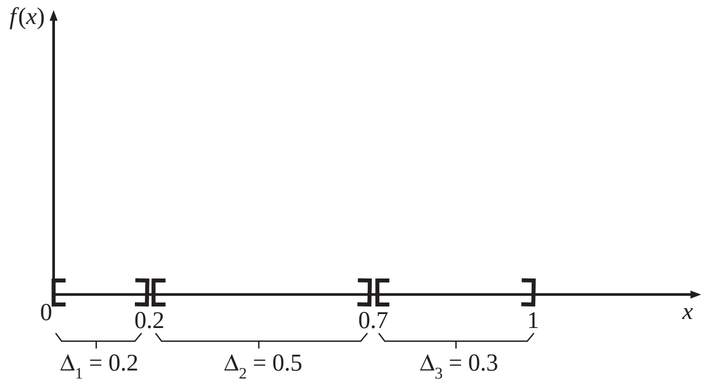


Figure 16.2 Partition {[0, 0.2],[0.2, 0.7],[0.7, 1]} over the interval [0, 1]

with corresponding lengths

$$\Delta_1 = 0.2, \quad \Delta_2 = 0.5, \quad \Delta_3 = 0.3$$

Definition 16.1

A set of points $x_0, x_1, x_2, \ldots, x_n$ satisfying the properties $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ generates a set of n subintervals, $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$, which induces a **partition** of the closed interval $[a, b]$. The length of each subinterval is $\Delta_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$.

Definition 16.2

Suppose that the set of subintervals $I_i = [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ is a partition on the closed interval $[a, b]$. Let $\omega_i \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$ be an arbitrary set of points from the set of subintervals. Then

$$\mathcal{S} = \sum_{i=1}^{n} f(\omega_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(\omega_i)\Delta_i$$

is called a **Riemann sum** for the function $f(x)$ over the subinterval $[a, b]$.

If we arbitrarily choose a point ω_i in each subinterval, namely $\omega_i \in I_i \equiv [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can then define a set of rectangles with dimensions determined by the length of each subinterval, Δ_i, and height equal to $f(\omega_i)$. The sum of the areas of such a set of rectangles is called a Riemann sum (definition 16.2).

For our example we choose $\omega_1 = 0.2$, $\omega_2 = 0.6$, and $\omega_3 = 0.8$, implying that $f(\omega_1) = 0.4$, $f(\omega_2) = 1.2$, and $f(\omega_3) = 1.6$. The result (see figure 16.3) is

$$\mathcal{S} = (0.4 \times 0.2) + (1.2 \times 0.5) + (1.6 \times 0.3) = 1.16$$

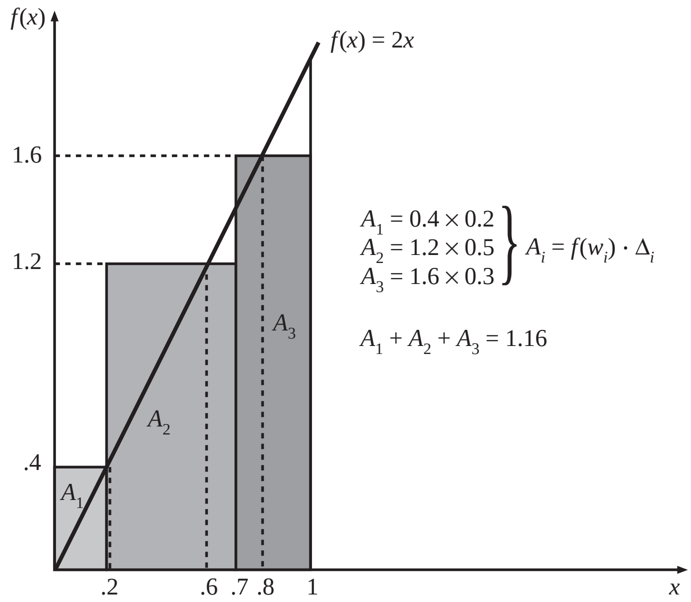


Figure 16.3 Example of a Riemann sum for the function $f(x) = 2x$ over the subinterval [0, 1] using the same partition of [0, 1] as in figure 16.2

Now suppose that we think of such a sum $\mathcal{S}$ as an approximation of the area under the curve $f(x) = 2x$ over the interval [0, 1]. This may not seem a very intuitively pleasing idea, since both the subintervals (partition) to use and the particular points within each subinterval are chosen in an arbitrary manner. The area that is computed, then, depends on these choices and so is not well-defined. However, if one finds that upon making finer and finer partitions (such that the length of the *widest* subinterval approaches zero) a *convergent* sequence of values for $\mathcal{S}$ is generated, then it is sensible to think of the limit of such a sequence as being the area under the curve. To illustrate such a process consider the partition defined by

$$x_i = \frac{i}{n}, \qquad i = 0, 1, 2, \ldots, n$$

which generates subintervals

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$$

This set of subintervals satisfies the requirements of a partition on [0, 1], and there is a natural sense in which the partition is *finer* for larger values of n. (Note that the width of each subinterval is $1/n$.) For example, with $n = 4$ we get the partition

$$[0, 0.25], \quad [0.25, 0.5], \quad [0.5, 0.75], \quad [0.75, 1]$$

which is illustrated in figure 16.4.

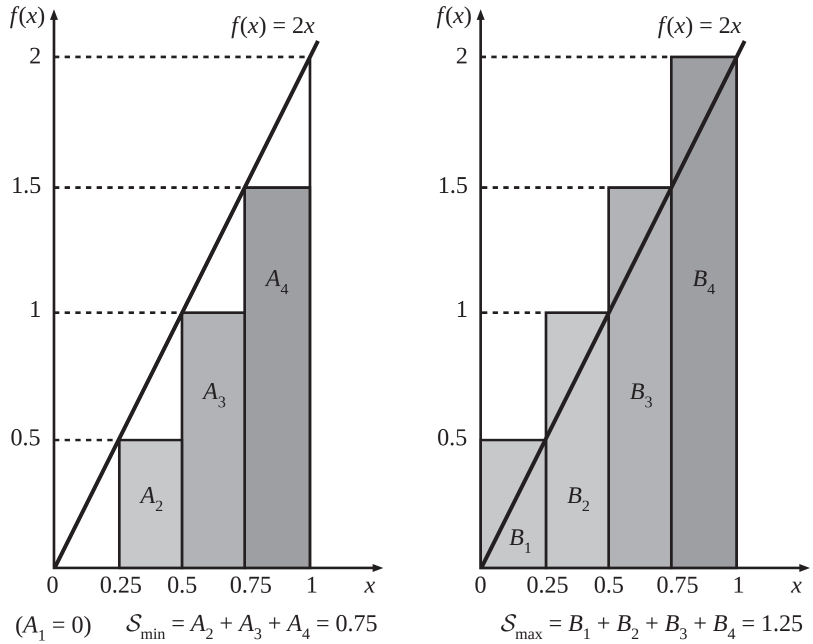


Figure 16.4 Examples of a lower sum ($\mathcal{S}_{\min}$) and an upper sum ($\mathcal{S}_{\max}$)

Next consider the following two extreme ways of choosing the points ω_i within each of the subintervals. First, choose the ω_i values to generate the smallest possible value for the Riemann sum $\mathcal{S}$; that is, choose $\omega_i \in [x_{i-1}, x_i]$ such that $f(\omega_i) \leq f(x)$ for all $x \in [x_{i-1}, x_i]$. Refer to these values as $\underline{\omega}_i$ and the sum they generate as the **lower sum** $\mathcal{S}_{\min}$, where

$$\mathcal{S}_{\min} = \sum_{i=1}^{n} f(\underline{\omega}_i)\Delta_i$$

Next, choose a set of $\omega_i \in [x_{i-1}, x_i]$ values to generate the largest possible value for the Riemann Sum $\mathcal{S}$; that is, choose ω_i such that $f(\omega_i) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$. Let us refer to these values as $\bar{\omega}_i$ and the sum they generate as the **upper sum** $\mathcal{S}_{\max}$, where

$$\mathcal{S}_{\max} = \sum_{i=1}^{n} f(\bar{\omega}_i)\Delta_i$$

For our example, it is easy to see that the values $\underline{\omega}_i$ are found by choosing the leftmost point of each subinterval, while the values $\bar{\omega}_i$ are found by choosing the rightmost point of each subinterval (see figure 16.4). The result is that

$$\mathcal{S}_{\min} = 0(0.25) + 0.5(0.25) + 1(0.25) + 1.5(0.25) = 0.75$$

while

$$\mathcal{S}_{\max} = 0.5(0.25) + 1(0.25) + 1.5(0.25) + 2(0.25) = 1.25$$

Now it is clear that $\mathcal{S}_{\min}$ is always an *underestimate* of the area under the curve, while $\mathcal{S}_{\max}$ is always an *overestimate* of the area under the curve. If, upon using finer and finer partitions, we discover that the values for $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ both converge to the same number, $\mathcal{S}^*$, then it makes sense intuitively to say that the area under the curve is well defined and that $\mathcal{S}^*$ is in fact the value of this area. For our example an increase in n leads to a finer partition ($\Delta_i = 1/n\ \forall i$) and as $n \to \infty$ the values for $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ do in fact converge to the same value.

To see this formally, note that we choose the $\underline{\omega}_i$ values to generate $\mathcal{S}_{\min}$ by choosing the leftmost points of the subintervals $[x_{i-1}, x_i]$, namely $\underline{\omega}_i = x_{i-1}$. For the partition $x_i = i/n$, where $i = 0, 1, 2, 3, \ldots, n$, this gives

$$\begin{aligned}\mathcal{S}_{\min} &= \sum_{i=1}^{n} f(\underline{\omega}_i)\Delta_i = \sum_{i=1}^{n} f(x_{i-1})\Delta_i \\ &= \sum_{i=1}^{n} 2\left(\frac{i-1}{n}\right)\left(\frac{1}{n}\right) = \frac{2}{n^2}\sum_{i=1}^{n}(i-1)\end{aligned}$$

Now, since

$$\sum_{i=1}^{n}(i-1) = 0 + 1 + 2 + \cdots + (n-1) = \left(\frac{n}{2}\right)(n-1)$$

we get

$$\mathcal{S}_{\min} = \left(\frac{2}{n^2}\right)\left[\left(\frac{n}{2}\right)(n-1)\right] = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n}$$

Similarly we choose the $\bar{\omega}_i$ values to generate $\mathcal{S}_{\max}$ by choosing the rightmost points of the subintervals $[x_{i-1}, x_i]$, namely $\bar{\omega}_i = x_i$. For the partition $x_i = i/n$, where $i = 0, 1, 2, 3, \ldots, n$, this gives

$$\mathcal{S}_{\max} = \sum_{i=1}^{n} f(\bar{\omega}_i)\Delta_i = \sum_{i=1}^{n} f(x_i)\Delta_i$$
$$= \sum_{i=1}^{n} 2\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \frac{2}{n^2}\sum_{i=1}^{n} i$$

Now, since

$$\sum_{i=1}^{n} i = [1 + 2 + \cdots + n] = \frac{n+1}{2}n$$

it follows that

$$\mathcal{S}_{\max} = \left(\frac{2}{n^2}\right)\left(\frac{n+1}{2}n\right) = \frac{n^2+n}{n^2} = 1 + \frac{1}{n}$$

Notice that, as expected, $\mathcal{S}_{\max} > \mathcal{S}_{\min}$. However, as the partition becomes arbitrarily fine (i.e., as $n \to \infty$) the two values converge to the same limit:

$$\lim_{n\to\infty} \mathcal{S}_{\max} = \lim_{n\to\infty} \mathcal{S}_{\min} = 1$$

If this is the case, we say that the function is integrable and that this limit is the area under the curve.

Definition 16.3

A function is said to be **integrable** on the closed interval $[a, b]$ if for every $\epsilon > 0$, there is some value $\delta > 0$ such that

$$\left|\sum_{i=1}^{n} f(\omega_i)\Delta_i - L\right| < \epsilon$$

for any partition on $[a, b]$ such that $\max \Delta_i < \delta$ (i.e., the length of the largest subinterval of the partition is less than δ) and for any selection of points $\omega_i \in [x_{i-1}, x_i]$. We call this value the definite integral of $f(x)$ over the interval $[a, b]$ and write

$$\int_a^b f(x)\,dx = L$$

Notice that for a function to be integrable on an interval $[a, b]$ it must be the case that as one takes finer and finer partitions of the interval $[a, b]$, using *any* set of points ω_i within the subintervals, the Riemann sum must become arbitrarily close (converge) to some value L. In our example, as $n \to \infty$, the choices of ω_i leading to either the largest possible or smallest possible values of $\mathcal{S}$ converge to the same value. Thus the Riemann sum converges to this same value for *any* arbitrary choice of ω_i values.

In regards to the notation $\int_a^b f(x)\,dx$, the sign $\int$ resembles the letter S and implies a *summation* process, the value a is called the lower limit and b the upper limit of the integral, referring to the end points of the interval $[a, b]$, and dx refers to the interval widths, Δ_i, in the expression for $\mathcal{S}$.

The method of finding the area under a curve using this technique is tedious, as even the simple example above shows. However, as long as a function is continuous, we can apply the **fundamental theorem of integral calculus,** which is presented below.

Theorem 16.1 **(Fundamental theorem of integral calculus)** If the function $f(x)$ is continuous on the closed interval $[a, b]$ and if $F(x)$ is any antiderivative (indefinite integral) of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(b)$ is the antiderivative of $f(x)$ evaluated at the point $x = b$ and $F(a)$ is the antiderivative of $f(x)$ evaluated at the point $x = a$. The expression $[F(b) - F(a)]$ is often denoted by $[F(x)]_a^b$.

Notice in this theorem that we ignore the constant of integration since, if included, it would in any case be *subtracted out;* that is, $[F(b) + C] - [F(a) + C] = F(b) - F(a)$.

For our example of finding the area under the curve $f(x) = 2x$ over the interval $[0, 1]$, the fundamental theorem of integral calculus clearly “works.” That is, since $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, we can compute the area according to the expression

$$\int_0^1 2x\,dx = F(1) - F(0) = 1^2 - 0^2 = 1$$

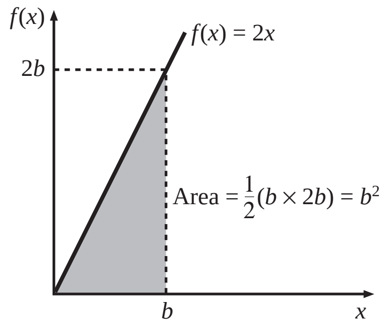


Figure 16.5 Area under the curve $f(x) = 2x$ over the interval [0, b]

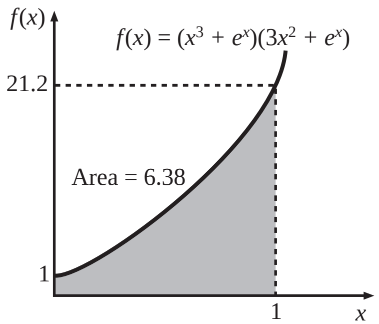


Figure 16.6 Area under the curve $f(x) = (x^3 + e^x)(3x^2 + e^x)$ over the interval [0, 1]

More generally, for the interval [0, b] of this function, we get that

$$\int_0^b 2x\,dx = [x^2]_0^b = b^2 - 0^2 = b^2$$

This is clearly correct since the area defined by the interval [0, b] below the curve $f(x) = 2x$ is $(1/2) \times b \times 2b = b^2$ (see figure 16.5).

For functions that generate substantially more complicated geometric shapes than that created by the function $f(x) = 2x$, the fundamental theorem of integral calculus is immensely useful. Consider the example mentioned earlier of the function $f(x) = (x^3 + e^x)(3x^2 + e^x)$. It would not be a simple matter to determine the area between the points $x = 0$ and $x = 1$ under this curve using only one's knowledge of geometric shapes. However, the antiderivative of this function was earlier shown to be

$$F(x) = \int (x^3 + e^x)(3x^2 + e^x)\,dx = \frac{(x^3 + e^x)^2}{2}$$

and so the relevant area is

$$\begin{aligned} \int_0^1 (x^3 + e^x)(3x^2 + e^x)\,dx &= \left[\frac{(x^3 + e^x)^2}{2}\right]_0^1 \\ &= \frac{(1^3 + e^1)^2}{2} - \frac{(0^3 + e^0)^2}{2} \\ &= \frac{(1 + e)^2}{2} - \frac{1}{2} \doteq 6.38 \qquad (\text{using } e \doteq 2.71) \end{aligned}$$

This result is illustrated in figure 16.6.

Although at first glance it may seem to be a remarkable coincidence that the area under a curve over some interval can be computed by simply subtracting the values of the antiderivative of the function evaluated at the two end points of the interval, one can gain an intuitive understanding of the fundamental theorem of integral calculus using the economic relationship between stocks and flows, such as the relationship between capital stock and net investment.

Let $K(t)$ be the function that describes the level of capital stock at time t and $I(t)$ be the rate of net investment (i.e., gross investment less depreciation). Then $K(t + \Delta t) - K(t)$ is the addition to capital stock that takes place over time period $[t, t + \Delta t]$ and

$$\frac{K(t + \Delta t) - K(t)}{\Delta t}$$

is the (average) rate at which capital stock changes over this time period. By definition,

$$\lim_{\Delta t \to 0} \left(\frac{K(t + \Delta t) - K(t)}{\Delta t} \right) = \frac{dK}{dt} = I(t)$$

where $I(t)$ is the *instantaneous* rate of net investment at time period t. Thus, since $I(t)$ is the derivative of $K(t)$, it follows that $K(t)$ is the antiderivative of $I(t)$, meaning that $K(t) = \int I(t)\,dt$. Moreover the sum of net investments made between two points in time represents the change in the level of capital stock between these two points in time. Over the interval $[a, b]$ we get

$$K(b) - K(a) = \sum_{i=1}^{n} I(\omega_i)\Delta_i = \int_a^b I(t)\,dt$$

where $\sum_{i=1}^{n} I(\omega_i)\Delta_i$ is a Riemann sum taken over a *fine partition* (max $\Delta_i \to 0$) of the interval $[a, b]$. This general relationship is illustrated by figure 16.7 and the following simple example.

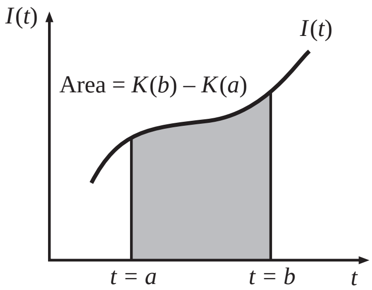


Figure 16.7 The change in the level of capital stock, $K(t)$, between time periods $t = a$ and $t = b$; is the area under the investment function over the interval $[a, b]$

Example 16.2 In this economic example we look at the relationship between net investment and capital stock when the rate of net investment changes *discretely* over time.

We consider a firm that, at the beginning of time period $t = 1$, has an initial stock of capital valued at \$10,000,000, and so we write $K(1) = 10{,}000{,}000$. For simplicity we assume that capital does not depreciate and that the firm invests in new capital at a constant rate of \$1,500,000 per year for three years (so we write $I(t) = 1{,}500{,}000$ for $t = 1, 2, 3$). The firm's stock of capital at the end of three years will then be its initial capital stock plus the sum of net investments made over the three years. That is,

$$K(3) = K(1) + I(1) + I(2) + I(3) = 10 + 1.5 + 1.5 + 1.5 = 14.5\,\text{mil}$$

Rewriting this gives

$$K(3) - K(1) = I(1) + I(2) + I(3) = \sum_{t=1}^{3} I(t)$$

which is a Riemann sum with $\Delta_i = 1$ for each i. Note that in this example we can use a partition with $\Delta_i = 1$ rather than choose a *fine partition* with the maximum $\Delta_i \to 0$ because the rate of investment is constant within each subinterval. ■

Example 16.3 In this economic example we look at the relationship between net investment and capital stock when the rate of net investment changes *continuously* over time.

Suppose that a firm begins at time $t = 0$ with a capital stock of $K(0) = \$500{,}000$ and, in addition to replacing any depreciated capital, is planning to invest in new capital at the rate $I(t) = 600t^2$ over the next ten years. The planned level of capital stock ten years from now is computed according to

$$K(10) = K(0) + \int_0^{10} I(t)\,dt = 500{,}000 + \int_0^{10} (600t^2)\,dt$$

The indefinite integral for this function is $200t^3 + C$, implying that

$$\int_0^{10} (600t^2)\,dt = [200t^3]_0^{10} = 200(10)^3 - 200(0)^3 = 200{,}000$$

and so

$$K(10) = 500{,}000 + 200{,}000 = 700{,}000$$

is the planned level of capital stock at the end of ten years. ■

The examples above illustrate that the sum of net investment made by a firm between two periods represents the increase in the firms capital stock. This result illustrates the general mathematical relationship that

$$F(b) - F(a) = \int_a^b f(x)\,dx = \sum_{i=1}^{n} f(\omega_i)\Delta_i$$

where $f(x)$ is a flow variable like investment and $F(x)$ is the related stock variable like capital (i.e., $dF(x)/dx = f(x)$). The following proposition provides a basis for this general relationship. Its proof is left as an exercise (question 7) at the end of this section.

Theorem 16.2 If f is a continuous function on the closed interval $[a, b]$, then for any $x \in [a, b]$ the function $F(x)$ defined by $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of f; that is, $F'(x) = f(x)$.

The relationship of this theorem to theorem 16.1 can be illustrated for the simple example of $f(t) = 2t$. For this function we get, according to theorem 16.1,

$$F(x) = \int_a^x f(t)\,dt = F(x) - F(a) = x^2 - a^2$$

Since a is a constant, $F'(x) = f(x)$, and so the definite integral with variable x as upper limit does in fact generate an antiderivative of the integrand.

Strictly speaking, theorem 16.2 should precede theorem 16.1 and be used to prove it. To see this, let $x = b$ and $x = a$ successively in theorem 16.2 to get

$$F(b) = \int_a^b f(t)\,dt + C$$

and

$$F(a) = \int_a^a f(t)\,dt + C$$

The area under a curve over an interval of width zero is zero:

$$\int_a^a f(t)\,dt = 0$$

As a result we get

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - C$$

or

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Besides explaining the fundamental theorem of integral calculus, theorem 16.2 can be usefully applied in some economic problems. Returning to the capital stock/investment example, we get

$$K(t) = \int_{t_0}^t I(x)\,dx \qquad \text{with } K'(t) = I(t)$$

Therefore the rate of change in capital stock at time t is just the value of the integrand (rate of investment) at time period t.

EXERCISES

1. The following computations are to be made for the function $f(x) = 3x + 1$ over the interval [0, 1].

(a) Use the formula for finding the area of a triangle and a rectangle to find the area under the function $f(x) = 3x + 1$ over the interval [0, 1].

(b) Find $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ for the partition $\{[0, 1/3], [1/3, 2/3], [2/3, 1]\}$. Compare to the area found in part (a). Illustrate with a graph of the function.

(c) Find $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ for the partition

$$\left\{\left[0, \frac{1}{5}\right], \left[\frac{1}{5}, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{3}{5}\right], \left[\frac{3}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, 1\right]\right\}$$

Compare to the area found in part (a). Illustrate with a graph. Also compare your answer to part (b).

(d) Find $\mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$ for the partition

$$\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, \frac{n}{n}\right]\right\}$$

Compare to the area found in part (a). What happens if you take the limit as $n \to \infty$? Discuss in relation to definition 16.3.

2. Repeat parts (b) and (c) of exercise 1 for the function $f(x) = x^2$.

3. Evaluate the following definite integrals. (Note: This exercise is taken from exercise 1 of exercises 16.1.)

(a) $\displaystyle\int_0^1 (x^4 + 2x^3 + 4x + 10)\, dx$

(b) $\displaystyle\int_0^8 x^{2/3}\, dx$

(c) $\displaystyle\int_{-1}^0 10e^x\, dx$

(d) $\displaystyle\int_1^2 6xe^{x^2}\, dx$

(e) $\displaystyle\int_1^2 \frac{3x^2 + 2}{(x^3 + 2x)}\, dx$

4. Evaluate the following definite integrals. (Note: This exercise is taken from exercise 2 of exercises 16.1.)

(a) $\int_2^5 (6x^3 + 10x^2 + 5)\,dx$

(b) $\int_0^{64} (x^{1/2} + 5x^{-2/3})\,dx$

(c) $\int_{-1}^{1} 2e^x\,dx$

(d) $\int_0^1 8x^2e^{x^3}\,dx$

(e) $\int_1^5 \frac{2x^3 + 1}{x^4 + 2x}\,dx$

5. Suppose that an economy's net investment flow is $I(t) = 10t^{1/2}$. Letting $K(0) = K^0$ represent the current stock of capital, use the definite integral to find the level of capital five years from now.

6*. Suppose that an economy's net investment flow is $I(t) = at^b$. Letting $K(1) = \bar{K}$ represent the current stock of capital, find the function representing the level of capital T years from now. Is it the case that if $b < 1$, $K(T)$ will be finite? Discuss. In section 16.4 we study more formally integrals whose limits are not finite.

7*. Prove theorem 16.2. (You will need to use property 1 from section 16.3.)

16.3 Properties of Integrals

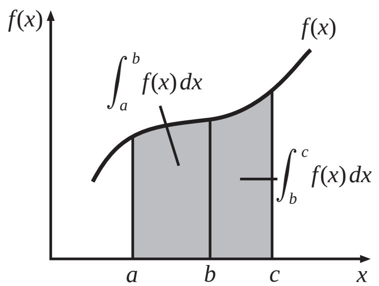


Figure 16.8 Illustration of property 1

In this section we will state some simple but useful properties of definite integrals. We will generally avoid the qualifying term *definite* and just refer to the *properties of integrals*. All of the properties presented here are fairly obvious if you think of the definite integral as the area beneath a curve. Therefore we only provide abbreviated proofs of the results.

For example, since $\int_a^c f(x)\,dx = F(c) - F(a)$ is the area under the curve $f(x)$ between the points $x = a$ and $x = c$, it follows that if $x = b$ is some point between a and c, then $\int_a^c f(x)\,dx$ is equal to the area under $f(x)$ between the points a and b, $\int_a^b f(x)\,dx$, plus the area between the points b and c, $\int_b^c f(x)\,dx$. This property is summarized below and illustrated in figure 16.8.

Property 1 If a, b, and c are points in $\mathbb{R}$ such that $a < b < c$, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

The proof of property 1 is trivial as the left side of the equation, which is equal to $F(c) - F(a)$, is clearly equal to the right side of the equation $[F(b) - F(a)] + [F(c) - F(b)] = F(c) - F(a)$.

Property 2

$$\int_a^a f(x)\,dx \equiv \lim_{c \to a} \int_a^c f(x)\,dx = 0$$

Property 2, that $\int_a^a f(x)\,dx = 0$, is also obvious from the notion of areas. It can be derived formally by considering what happens when taking the limit of the integral $\int_a^c f(x)\,dx$ as the point c approaches the point a:

$$\lim_{c \to a} \int_a^c f(x)\,dx = \lim_{c \to a} [F(c) - F(a)] = F(a) - F(a) = 0$$

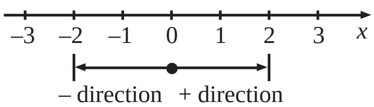


Figure 16.9 Illustration of property 3

When first learning about negative and positive numbers we are generally told that, for example, the negative number -2 is the same distance from the origin as is the positive number $+2$ except one moves in the opposite direction ($\leftarrow$ for negative numbers), as illustrated in figure 16.9. If we integrate a function $f(x)$ from point c to a, rather than a to c, then we measure the same area but do so in the opposite direction. This provides us with the intuition underlying property 3.

Property 3 Reversing the direction of integration changes the sign of the integral. That is,

$$\int_c^a f(x)\,dx = -\int_a^c f(x)\,dx$$

This property is also easy to prove, since

$$\int_c^a f(x)\,dx = F(a) - F(c) = -[F(c) - F(a)] = -\int_a^c f(x)\,dx$$

Up to now we have been treating the integral $\int_a^c f(x)\,dx$ as the area beneath the curve $f(x)$ and, implicitly, above the horizontal or x-axis. However, if a function $f(x)$ is negatively valued on the interval $[a, c]$, then the integral $\int_a^c f(x)\,dx$ refers to the area *above* the curve $f(x)$ and *below* the x-axis. In general then, we say that the integral $\int_a^c f(x)\,dx$ is the area between the function $f(x)$ and the x-axis over the interval $[a, c]$. Any area *beneath* the x-axis is measured as a negative value. This is clear from the following example, which is illustrated in figure 16.10.

$$\int_0^2 (-2x - 1)\,dx = [-x^2 - x]_0^2 = [-4 - 2] - [0 - 0] = -6$$

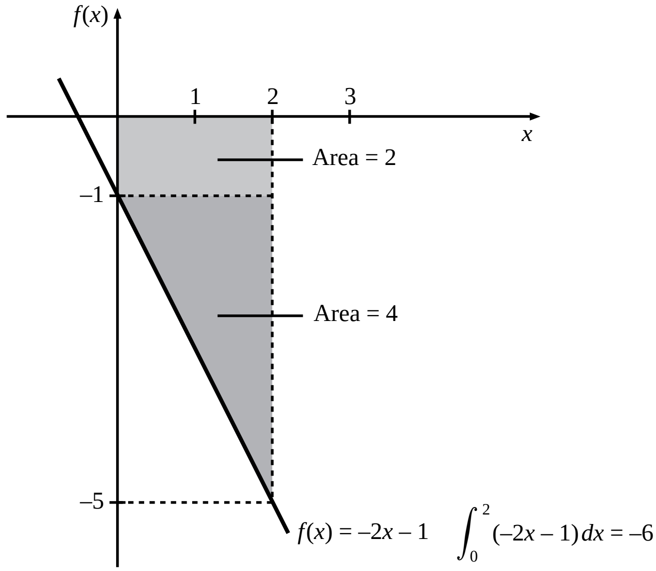


Figure 16.10 Illustration of property 4

Property 4 If a function $f(x)$ is negatively valued on the interval $[a, c]$, $a < c$, then $\int_a^c f(x)\,dx < 0$ where $|\int_a^c f(x)\,dx|$ is the area of the region between the curve, $f(x)$, and the x-axis between the points a and c.

As an economic application of property 4, consider the following scenario of a firm's net investment over a period from $t = a$ to $t = c$, $c > a$. Suppose that there is positive net investment $[I(t) > 0]$ during some subintervals of time and negative net investment $[I(t) < 0]$ over other subintervals of time as indicated in figure 16.11. Overall net investment (ΔI) over the interval $t = a$ to $t = c$ is the sum of the *positive* areas less the sum of the *negative* areas as indicated below.

$$\Delta I = A_1 - A_2 + A_3 - A_4 + A_5$$

The equivalent integral statement is

$$\int_a^c I(t)\,dt = \int_a^{b_1} I(t)\,dt + \int_{b_1}^{b_2} I(t)\,dt + \int_{b_2}^{b_3} I(t)\,dt + \int_{b_3}^{b_4} I(t)\,dt + \int_{b_4}^{c} I(t)\,dt$$

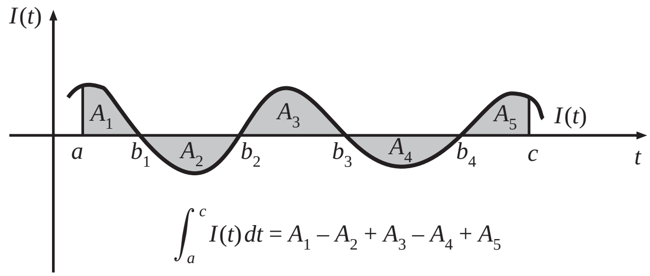


Figure 16.11 To compute the integral of a function that alternates between positive and negative values, add the areas formed above the horizontal axis and then subtract those areas formed below the horizontal axis

Consumer and Producer Surplus Measures

Measuring the impact that a change in the economic environment has on producers and consumers is an important exercise in microeconomics. The change under consideration may be a price change or a change in costs owing to technological innovation or government policy. We will consider a single price change, or the elimination of a product's availability altogether.

If the price of a product were to rise, then the producers of this product would be better off, while consumers would be worse off. Producer surplus and consumer surplus, respectively, are measures of these impacts. We first define mathematically and then explain intuitively the concept of producer surplus.

Definition 16.4 Let $\text{MC}(q)$, $q \in R_+$ be a continuous marginal-cost function for a firm producing output level q, and let $p = p_0$ be the price of its product. If $q = q_0$ is the profit-maximizing output level for this firm (i.e., q_0 is determined by $p_0 = \text{MC}(q_0)$), then its **producer surplus,** PS, is

$$\text{PS} = p_0 q_0 - \int_0^{q_0} \text{MC}(q)\, dq$$

provided that this value is nonnegative.

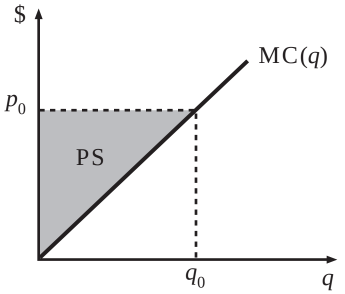


Figure 16.12 Producer surplus (PS) for a firm at price $= p_0$

Since the marginal-cost function is the derivative of the total variable cost function, $\text{TVC}(q)$, it follows that

$$\int_0^{q_0} \text{MC}(q)\, dq = \text{TVC}(q_0) - \text{TVC}(0) = \text{TVC}(q_0)$$

and so PS is simply total revenue less total-variable cost. That is, PS equals profit without accounting for fixed costs. The qualification that PS is nonnegative is included because if PS were negative, then the firm could choose to shut down ($q_0 = 0$) and receive zero producer surplus. Diagrammatically PS is illustrated in figure 16.12, where we see that PS equals the area below price and above the marginal-cost curve on the interval $[0, q_0]$.

If price were to increase to $p = \hat{p}$, then the PS would also increase to

$$\hat{p}\hat{q} - \int_0^{\hat{q}} \text{MC}(q)\, dq$$

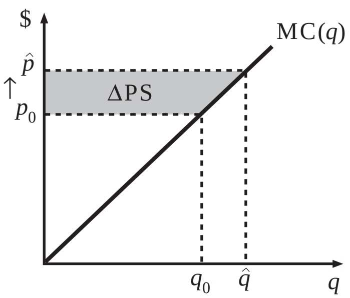


Figure 16.13 Change in producer surplus resulting from the price change p_0 to $\hat{p}$

where $\hat{q}$ is determined by the relationship $\hat{p} = \text{MC}(\hat{q})$. The change in producer surplus resulting from the price increasing from p_0 to $\hat{p}$ is illustrated in figure 16.13 as $\Delta\text{PS}\ (p : p_0 \to \hat{p})$.

Rather than working through the expressions for PS at each price p_0 and $\hat{p}$ to compute ΔPS, it is simpler to note from figure 16.13 that ΔPS is the area to the left of the $\text{MC}(q)$ curve from $p = p_0$ to $p = \hat{p}$. If we were to find the inverse of the marginal-cost function

$$q = \text{MC}^{-1}(p)$$

we could write this as the following definite integral:

$$\Delta\text{PS}(p : p_0 \to \hat{p}) = \int_{p_0}^{\hat{p}} \text{MC}^{-1}(p)\, dp$$

These relationships are illustrated by the next example:

Example 16.4 For a profit-maximizing firm with marginal-cost function $\text{MC}(q) = 2q + 1$, find

(i) PS at price $p_0 = 10$
(ii) PS at price $\hat{p} = 15$
(iii) ΔPS from the price change ($p : 10 \to 15$)

Solution

(i) At $p_0 = 10$, $\text{MC}(q_0) = p_0$ implies that $q_0 = 4.5$, and so

$$\begin{aligned}\text{PS}(p_0 = 10) &= (10)(4.5) - \int_0^{4.5} (2q + 1)\, dq \\ &= 45 - [q^2 + q]_0^{4.5} \\ &= 45 - 24.75 = 20.25\end{aligned}$$

This corresponds to the area indicated in figure 16.14 (a).

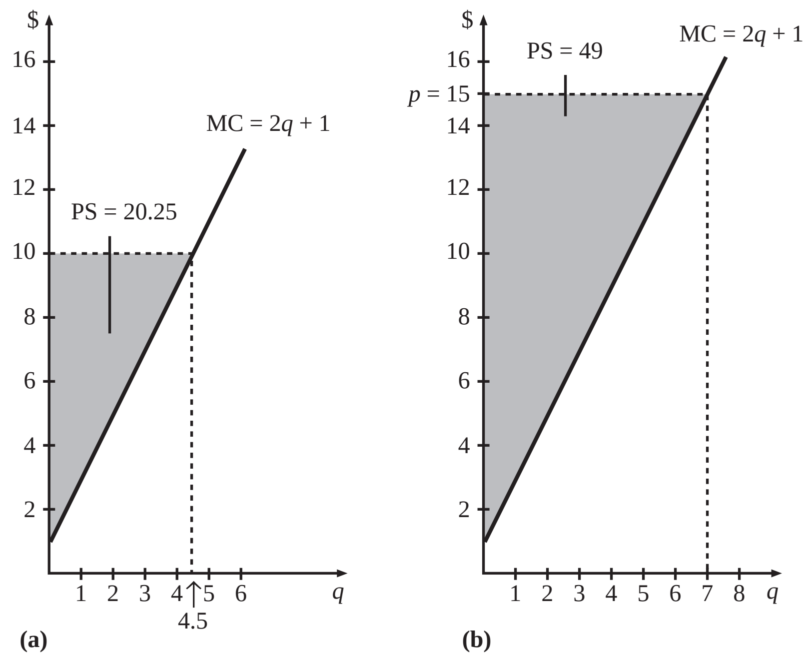


Figure 16.14 Computation of producer surplus at prices $p = 10$ and $p = 15$ (example 16.4)

(ii) At $\hat{p} = 15$ equation $\text{MC}(\hat{q}) = \hat{p}$ implies that $\hat{q} = 7$, and so

$$\begin{aligned}\text{PS}(\hat{p} = 15) &= (15)(7) - \int_0^7 (2q+1)\,dq \\ &= 105 - [q^2 + q]_0^7 \\ &= 105 - 56 = 49\end{aligned}$$

This corresponds to the area indicated in figure 16.14 (b).

(iii) From (i) and (ii) we see that the impact of the price changing from 10 to 15 is an increase in PS of amount $49 - 20.25 = 28.75$. Alternatively, noting that the inverse function of $\text{MC}(q) = 2q + 1$ is

$$\text{MC}^{-1}(p) = \frac{p}{2} - \frac{1}{2}$$

we can compute the change in PS according to the formula

$$\begin{aligned}\Delta\text{PS} &= \int_{p_0}^{\hat{p}} \text{MC}^{-1}(p)\,dp \\ &= \int_{10}^{15} \left(\frac{p}{2} - \frac{1}{2}\right) dp \\ &= \left[\frac{p^2}{4} - \frac{p}{2}\right]_{10}^{15} \\ &= \left(\frac{15^2}{4} - \frac{15}{2}\right) - \left(\frac{10^2}{4} - \frac{10}{2}\right) = 28.75\end{aligned}$$

We can see from figure 16.15 (a) that area $ABCD$ measures the ΔPS for this example. Since this is the area to the left of the MC curve between vertical points $p = 10$ and $p = 15$, it is equivalent to the area below the inverse of the MC curve between these two prices, which is the area $abcd$ in figure 16.15 (b). ■

Suppose in example 16.4 the price had changed from an initial value of $\hat{p} = 15$ to a new, lower price of $p_0 = 10$. We would then write (see property 3 of integrals)

$$\Delta\text{PS} = \int_{15}^{10} \text{MC}^{-1}(p)\,dp = -\int_{10}^{15} \text{MC}^{-1}(p)\,dp$$

and so obtain a negative value (−28.75) for the change in producer surplus. This reflects the outcome that a lower price reduces the surplus of the firm. As a final

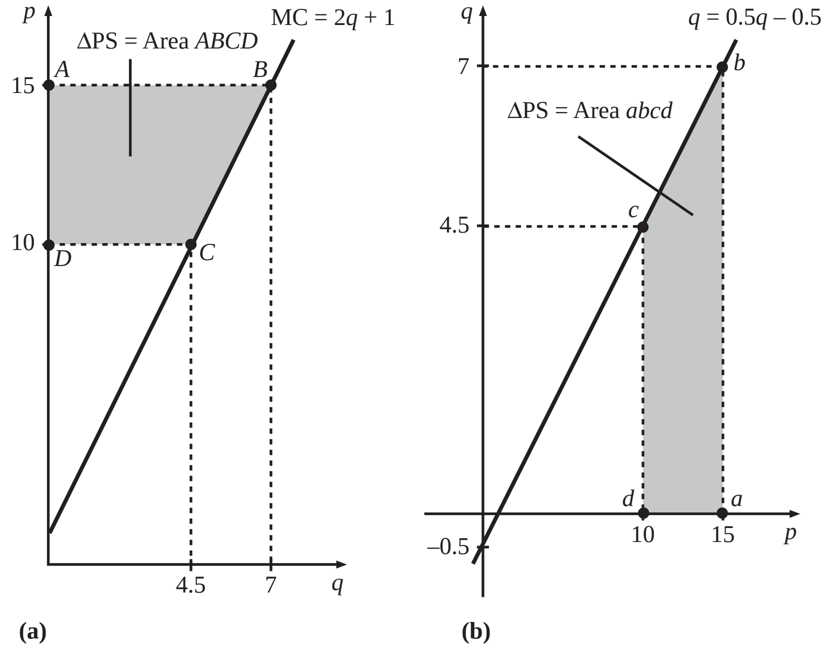


Figure 16.15 Two methods of computing the change in producer surplus. In (b) the inverse marginal-cost function is used (example 16.4 (c))

note, if a shutdown or start-up decision is not involved when price changes, then the change in producer surplus is equivalent to a change in the firm's profit.

We turn now to consumer surplus. Since a consumer's objective is to purchase that bundle of affordable goods that she most desires, it is not so obvious as in the case of the firm how one ought to measure the benefit of a given price change. Of the various possible measures, consumer surplus is the simplest one used by economists. We give the definition below and provide detailed examples in the supplementary material of the Web page http://mitpress.mit.edu/math_econ3.

Definition 16.5 Let $p = D^{-1}(q)$, $q \in R_+$, be a continuous, inverse-demand function for some consumer, and let $p = p_0$ be the price of the good purchased. If $q = q_0$ is the corresponding amount of this good purchased (i.e., q_0 satisfies $p_0 = D^{-1}(q_0)$), then the **consumer's surplus,** CS, from the availability of this good (at price p_0) is

$$\text{CS} = \int_0^{q_0} D^{-1}(q)\, dq - p_0 q_0$$

provided this value is nonnegative.

EXERCISES

1. For a profit-maximizing, perfectly competitive, firm with marginal-cost function $MC(q) = q^2 + 3$, find the following. Illustrate your results on a graph.

(a) PS at price $p_0 = 7$

(b) PS at price $\hat{p} = 12$

(c) ΔPS resulting from the price change $p_0 = 7$ to $\hat{p} = 12$

2. For a profit-maximizing, perfectly competitive, firm with a marginal-cost function $MC(q) = 3q^2 + 4q + 2$, find:

(a) PS at price $p_0 = 9$

(b) PS at price $\hat{p} = 41$

(c) ΔPS resulting from the price change $p_0 = 9$ to $\hat{p} = 41$

Illustrate your results on a graph.

3. For a consumer with demand function $q = 10 - 2p^{1/2}$, find the following. Illustrate your results on a graph.

(a) CS at price $p_0 = 1$

(b) CS at price $\hat{p} = 4$

(c) ΔCS resulting from the price change $p_0 = 1$ to $\hat{p} = 4$

4. For a consumer with demand function $q = 5 - p^{1/3}$, find:

(a) CS at price $p_0 = 1$

(b) CS at price $\hat{p} = 27$

(c) ΔCS resulting from the price change $p_0 = 1$ to $\hat{p} = 27$

Illustrate your results on a graph.

16.4 Improper Integrals

Improper integrals are of two types. The first type we consider involves definite integrals that are computed over an infinite interval of integration, namely for which the lower limit approaches $-\infty$, or the upper limit approaches $+\infty$, or both. The other type involves integrals of functions that are discontinuous at some point(s) of the interval.

The following definitions apply to improper integrals of the first type where one or other of the limits of integration is $+\infty$ or $-\infty$ (or both).

Definition 16.6

$$\int_a^{+\infty} f(x)\,dx = \lim_{c\to+\infty}\int_a^c f(x)\,dx$$

if this limit exists.

Definition 16.7

$$\int_{-\infty}^{c} f(x)\,dx = \lim_{a\to-\infty}\int_a^c f(x)\,dx$$

if this limit exists.

Definition 16.8

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{c\to+\infty}\int_0^c f(x)\,dx$$

if this limit exists.

There are many instances in economics where we must compute improper integrals. Consider, for example, the problem of computing consumer surplus measures for the case of constant elasticity of demand. Recall that this demand curve can be written as

$$q = ap^{-\epsilon}, \qquad a, \epsilon > 0$$

or, in terms of the inverse demand function

$$p = \left(\frac{q}{a}\right)^{-1/\epsilon}$$

where ϵ is the own price elasticity of demand. The graph of this demand curve illustrates that it never cuts the p or q axis (see figure 16.16). This means that the consumer surplus received at price $p = p_0$ is

$$\text{CS} = \int_{p_0}^{\infty} (ap^{-\epsilon})\,dp$$

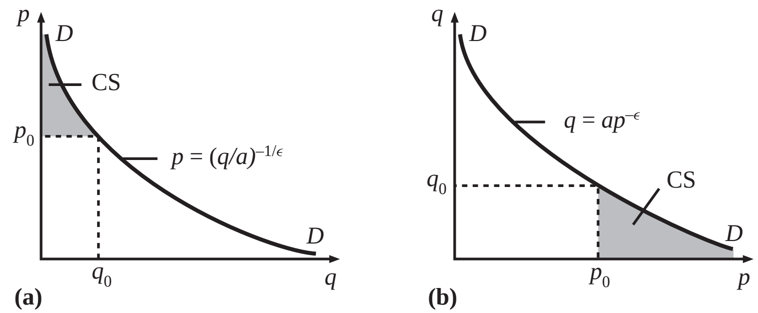


Figure 16.16 Consumer surplus for the demand function $q = ap^{-\epsilon}$

which is an improper integral of the type identified in definition 16.6. We can use this to obtain

$$\begin{aligned}
\text{CS} &= \int_{p_0}^{\infty} (ap^{-\epsilon})\, dp \\
&= \lim_{\hat{p}\to\infty} \int_{p_0}^{\hat{p}} (ap^{-\epsilon})\, dp \\
&= \lim_{\hat{p}\to\infty} \left[\frac{a}{1-\epsilon} p^{1-\epsilon}\right]_{p_0}^{\hat{p}} \\
&= \frac{a}{1-\epsilon}\left[\lim_{\hat{p}\to\infty} \hat{p}^{1-\epsilon} - p_0^{1-\epsilon}\right] \qquad (16.1)
\end{aligned}$$

The limit in equation (16.1) is well defined and converges if and only if $\epsilon > 1$. That is, the consumer surplus has a finite value for the constant elasticity demand function only if the size of the elasticity of demand is greater than 1.

Example 16.5 For the demand function $q = 50p^{-2}$, find the consumer surplus if $p = 10$.

Solution

$$\begin{aligned}
\text{CS} &= \int_{10}^{\infty} 50p^{-2}\, dp \\
&= \lim_{\hat{p}\to\infty} \int_{10}^{\hat{p}} 50p^{-2}\, dp \\
&= \lim_{\hat{p}\to\infty} -50[\hat{p}^{-1} - 10^{-1}]
\end{aligned}$$

$$= -50\left[\lim_{\hat{p}\to\infty} \frac{1}{\hat{p}} - \frac{1}{10}\right]$$

$$= -50\left[0 - \frac{1}{10}\right] = 5$$

(see figure 16.17). ■

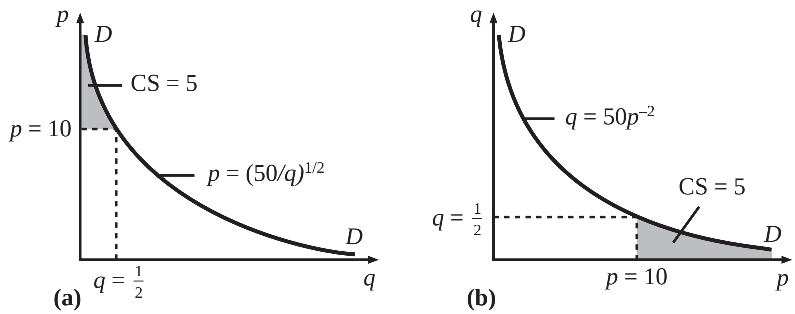


Figure 16.17 Consumer surplus for the demand function $q = 50p^{-2}$ at price $p = 10$ (example 16.5)

Example 16.6 For the demand function $q = 20p^{-1}$, find the expression for computing the consumer surplus at $p = 2$. What difficulty arises in trying to compute this value?

Solution

$$\text{CS} = \int_2^{\infty} \frac{20}{p}\, dp$$

$$= \lim_{\hat{p}\to\infty} \int_2^{\hat{p}} \frac{20}{p}\, dp$$

$$= \lim_{\hat{p}\to\infty} 20[\ln \hat{p} - \ln 2]$$

The difficulty in trying to compute this value is that $\lim_{\hat{p}\to\infty} \ln \hat{p}$ is not a finite value (see figure 16.18).

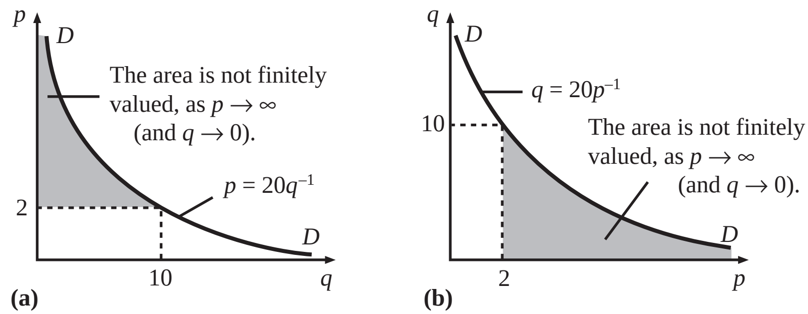


Figure 16.18 The consumer surplus for the demand function $q = 20p^{-1}$ at price $p = 2$ cannot be computed as it is not finitely valued (example 16.6) ■

Examples 16.5 and 16.6 illustrate that the value of the integral of a positively valued function over an interval of infinite length may or may not be finitely valued. It may seem impossible that the area of a figure with a positive (nonzero) distance in one dimension and an "infinite" distance in the other dimension could have a finite area. However, the intuition underlying this possible result can be understood by reconsidering some results from chapter 3 on series, where we saw that an infinite sum of positive numbers may or may not be finite in value. For example, we discovered that the infinite harmonic series is definitely divergent. That is,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty \tag{16.2}$$

while the infinite geometric series is convergent (provided $|r| < 1$); that is,

$$\lim_{N\to\infty} \sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + \cdots = \frac{a}{1-r}, \qquad |r| < 1 \tag{16.3}$$

In each of these examples the terms of the sequence that make up the series, written $a_n = f(n)$, represent a function with domain being the positive natural numbers. We can generate analogous results for functions defined on $\mathbb{R}$ by using the integral. For the two examples above we use functions $f(x) = 1/x$, $x \geq 1$, $x \in \mathbb{R}_{++}$ and $f(x) = ar^{x-1}$, $x \geq 1$, $x \in \mathbb{R}_{++}$, which at the values $x = 1, 2, 3, \ldots$ take on the same values as the functions representing the terms of the series in equations (16.2) and (16.3) respectively. The integrals in equations (16.4) and (16.5) are analogues to the sums for these two series. Related to the harmonic series we have

$$\lim_{c\to+\infty} \int_1^c \frac{1}{x}\, dx = \lim_{c\to+\infty} [\ln(x)]_1^c = \lim_{c\to+\infty} [\ln(c) - \ln(1)] = +\infty \tag{16.4}$$

and related to the geometric series, for $|r| < 1$, we have

$$\lim_{c\to+\infty}\int_1^c (ar^{x-1})\,dx = \lim_{c\to+\infty}\left[\frac{ar^{x-1}}{\ln(r)}\right]_1^c$$
$$= \lim_{c\to+\infty}\frac{1}{\ln(r)}[ar^{c-1}-a]$$
$$= -\frac{a}{\ln(r)} \tag{16.5}$$

So, just as for an infinite series, a definite integral defined over an infinite interval may either converge or diverge.

The other sort of improper integral involves computing the integral over an interval $[a, c]$ when the integrand, $f(x)$, is not continuous at some point(s) in this interval. For example, consider the function $f(x) = 1/x$ on the interval $[0, 1]$. This function is not continuous at $x = 0$ as $1/0$ is not defined, and moreover the right-hand limit $x \to 0^+$ of the function does not exist; that is, $\lim_{x\to 0^+} 1/x = +\infty$. It is natural to ask whether the area indicated in figure 16.19 (b) is finite. Suppose that we evaluate the integral of $f(x)$ over the interval $[0+\epsilon, 1]$, $\epsilon > 0$ and compute $\int_{0+\epsilon}^1 (1/x)\,dx$ in the limit as $\epsilon \to 0$. If the value of this integral is not finite, we say the integral, or area, does not exist. This turns out to be the case for this example, as illustrated below:

$$\lim_{\epsilon\to 0^+}\int_{0+\epsilon}^1\left(\frac{1}{x}\right)dx = \lim_{\epsilon\to 0^+}[\ln(x)]_\epsilon^1$$
$$= \lim_{\epsilon\to 0^+}[\ln(1)-\ln(\epsilon)] = 0-(-\infty) = +\infty$$

If we perform the same exercise with the function $f(x) = 1/\sqrt{x} = 1/x^{1/2}$, we find that the integral over the same interval does converge as $x \to 0^+$ even though $1/\sqrt{x} \to +\infty$. This is illustrated by the result

$$\lim_{\epsilon\to 0^+}\int_{0+\epsilon}^1\left(\frac{1}{x^{1/2}}\right)dx = \lim_{\epsilon\to 0^+}[2x^{1/2}]_\epsilon^1$$
$$= \lim_{\epsilon\to 0^+}[2-2\epsilon^{1/2}] = 2-0 = 2$$

In this case we say that the area beneath the function $f(x) = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ is finite, as is indicated in figure 16.19.

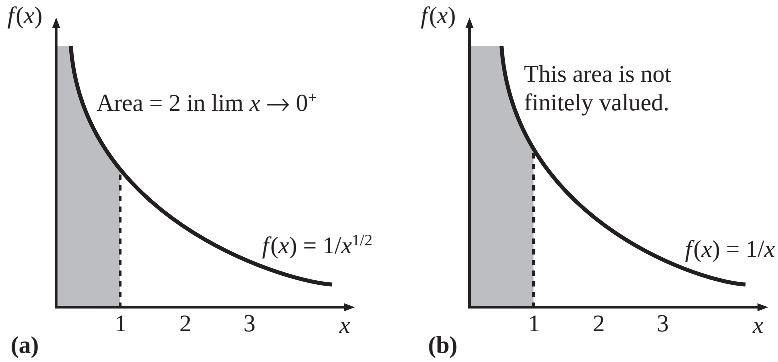


Figure 16.19 Illustration that $\int_0^1 1/x\, dx$ is not finitely valued while $\int_0^1 1/\sqrt{x}\, dx$ is finitely valued

Present Value of an Infinite Stream of Payments under Continuous Discounting

In section 3.3 we derived the result that, if interest is compounded continuously over time, then the present value of \$$b$ received at the end of t years' time is

$$PV_t = be^{-rt} \tag{16.6}$$

where r is the annual rate of interest. If a series of payments of \$$b$ per year are made at the end of each year indefinitely, then the present value of this stream is

$$\lim_{T\to\infty} \sum_{t=1}^{T} be^{-rt} \tag{16.7}$$

Now, if the annual payments are spread out uniformly over the year, rather than being made as a single annual lump sum, then the present value of the stream of payments is

$$\begin{aligned} PVB &= \lim_{T\to\infty} \int_0^T be^{-rt}\, dt \\ &= \lim_{T\to\infty} b\left[\frac{e^{-rt}}{-r}\right]_0^T \\ &= b \lim_{T\to\infty}\left[\frac{-e^{-rT}}{-r} - \frac{e^{-r0}}{-r}\right] \end{aligned}$$

$$= b\left[0 + \frac{1}{r}\right]$$

$$= \frac{b}{r} \tag{16.8}$$

Notice that the area of the rectangles below the curve be^{-rt} in figure 16.20 illustrates the sum of values for the case of lump-sum payments made at the end of each year, equation (16.7), while the area under the curve represents the value of the stream of payments when payments are made continuously throughout the year, equation (16.8). The latter value is higher because the recipient does not have to wait until the end of the year to obtain the funds.

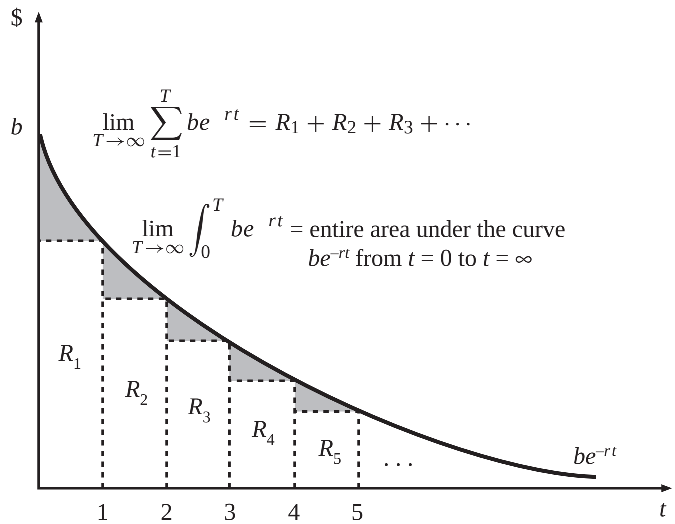


Figure 16.20 Comparison of present value of an infinite stream of payments if received at the end of each year or spread uniformly over each year

Example 16.7 Find the present value of the infinite stream of payments of amount \$1,000 per year if the annual interest rate is $r = 0.04$ (4%) and the yearly payments are spread evenly throughout the year.

Solution

To compute this value, use equation (16.8), with $b = \$1{,}000$ and $r = 0.04$, to give

$$\text{PVB} = \frac{b}{r} = \frac{\$1{,}000}{0.04} = \$25{,}000$$

■

The following definitions handle all possible cases for defining an improper integral when there is a single point of discontinuity within the interval of integration $[a, c]$. The first two cases handle discontinuities at endpoints of $[a, c]$, while the third covers the possibility of a discontinuity at an interior point of $[a, c]$. If there were more than one point of discontinuity over $[a, c]$, then one could use property 1 of integrals (see section 16.3) and apply these definitions iteratively to cover any finite number of discontinuities.

Definition 16.9

If $f(x)$ is continuous at every point in $[a, c]$ except the endpoint $x = a$, then we can say that $f(x)$ is continuous on $(a, c]$ and that

$$\int_a^c f(x)\,dx = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^c f(x)\,dx$$

if the limit exists.

Definition 16.10

If $f(x)$ is continuous at every point in $[a, c]$ except the endpoint $x = c$, then we can say that $f(x)$ is continuous on $[a, c)$ and that

$$\int_a^c f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{c-\epsilon} f(x)\,dx$$

if the limit exists.

Definition 16.11

If $f(x)$ is continuous at every point in $[a, c]$ except an interior point $x = b$, $a < b < c$, then we can say that $f(x)$ is continuous on $[a, b)$ and $(b, c]$ and that

$$\int_a^c f(x)\,dx = \lim_{\epsilon \to 0^+} \int_a^{b-\epsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{b+\delta}^c f(x)\,dx$$

if the limit exists.

Example 16.8

For the demand function $q = 5p^{-1/2} - 1$, find the increase in consumer surplus that would arise if the price fell from $p = 4$ to $p = 0$, that is, if it becomes a free good.

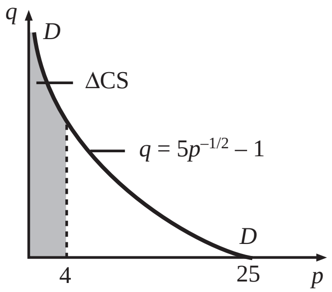


Figure 16.21 Change in consumer surplus resulting from a change in price from $p = 4$ to $p = 0$ (free good) (example 16.8)

Solution

Since $q \to \infty$ as $p \to 0$, we must write the change in consumer surplus as the improper integral

$$\begin{aligned}\Delta CS &= \lim_{\hat{p}\to 0}\int_{\hat{p}}^{4}[5p^{-1/2}-1]\,dp\\ &= \lim_{\hat{p}\to 0}[10p^{1/2}-p]_{\hat{p}}^{4}\\ &= [10(2)-4]-\left[\lim_{\hat{p}\to 0}(10p^{1/2})-\lim_{\hat{p}\to 0}\hat{p}\right]=16\end{aligned}$$

See figure 16.21. ■

EXERCISES

1. For the demand function $q = 30p^{-2}$, find the consumer surplus at price $p = 2$.

2. For the demand function $q = 10p^{-1/2}$, find the expression for the value of consumer surplus at price $p = 4$. What difficulty arises in trying to compute this value?

3. For the following demand functions, find the expression for the value of the increase in consumer surplus that would arise if the price fell from $p = 1$ to $p = 0$ (i.e., the good becomes available free of charge). If the value cannot be computed, explain why that is so.

(a) $q = 10p^{-1/3}$

(b) $q = 25p^{-2}$

4. Consider the demand function $q = Ap^{-\epsilon}$, where $A > 0$ and $\epsilon > 0$ are parameters.

(a) For some price $p_0 > 0$, find the expression for consumer surplus. Under what condition on ϵ will this value exist (i.e., have a finite value)?

(b) Suppose that the price falls from $p = p_0$ to $p = 0$. Find the expression for the change in consumer surplus. Under what condition on ϵ will this value exist?

(c) How do your answers relate to exercises 1 through 3?

5. Find the present value of the infinite stream of payments of amount \$500 per year if the annual interest rate is $r = 0.02$ (2%) and the yearly payments are spread evenly over the year.

16.5 Techniques of Integration

In this section we present two common techniques of integration that greatly expand the set of integrals that can be evaluated. The first technique follows directly from the chain rule of differentiation.

Consider a function $F(u)$, where u is in turn a function of variable x, and so we write $u = g(x)$. Let $f(u) \equiv dF(u)/du$, so that $F(u)$ is the antiderivative of $f(u)$. The chain rule of differentiation implies that

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du}\frac{du}{dx} = f(u)\frac{du}{dx} = f(u)g'(x)$$

Integrating over this expression gives us the chain rule of antidifferentiation or, as it is more commonly called, the substitution rule of integration.

The Substitution Rule of Integration

If $F(u)$ is the antiderivative of $f(u)$, namely $dF(u)/du = f(u)$ and $u = g(x)$, then

$$\int f(u)\frac{du}{dx}\,dx \quad \text{or} \quad \int f(g(x))g'(x)\,dx = F(u) + C$$

This rule is useful when the integrand can be decomposed into two (multiplicative) parts where one part is the derivative of the other. For example, the integrand in the expression

$$\int (x^3 + e^x)(3x^2 + e^x)\,dx$$

can be decomposed into two multiplicative parts with the second being the derivative of the first:

$$\frac{d(x^3 + e^x)}{dx} = 3x^2 + e^x$$

Therefore, letting $u = x^3 + e^x$ allows us to write the integral in the form

$$\int f(u)\frac{du}{dx}\,dx = F(u) + C$$

where $f(u) = u$, and so $F(u)$ is simply $u^2/2$. Substituting back for u gives us the result that

$$\int (x^3 + e^x)(3x^2 + e^x)\,dx = \frac{(x^3 + e^x)^2}{2} + C$$

A more general type of application of this technique is illustrated by changing the example slightly as follows:

Example 16.9 Find the following integral:

$$\int (x^3 + e^x)^n (3x^2 + e^x)\, dx$$

Solution

In this case, the variable u [or function $g(x)$] appears in a format in which it is the argument of a function that is easy to differentiate. Here we make the same substitution $u = x^3 + e^x$ and note that the expression for $\int f(u)(du/dx)\, dx$ has $f(u) = u^n$, and so $F(u) = u^{n+1}/n + 1$. It follows that

$$\int (x^3 + e^x)^n (3x^2 + e^x)\, dx = \frac{(x^3 + e^x)^{n+1}}{n+1} + C$$ ■

As seen by the examples above, the *trick* to successfully applying the substitution rule is to pick out one part of the integrand that is the derivative of another part. However, to be useful it must also be the case that it is easy to find the antiderivative of that part of the integrand which is a function of $u = g(x)$. In the examples presented above $f(u)$ is a polynomial, a function for which it is easy to find the antiderivative.

The second technique we present in this section is that of **integration by parts.** This method can be seen to follow from the product rule of differentiation or, equivalently, the product rule of total differentials. Consider the functions $u = f(x)$ and $v = g(x)$. It follows that

$$d(uv) = u\, dv + v\, du$$

Thus integrating this expression on both sides gives

$$\int d(uv) = \int u\, dv + \int v\, du$$

which implies that

$$uv = \int u\, dv + \int v\, du$$

or

$$\int v\,du = uv - \int u\,dv$$

Noting that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$, we can also write this expression as

$$\int g(x)f'(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

Integration by Parts

Suppose that we have continuous functions $u = f(x)$ and $v = g(x)$. It follows that

$$\int v\,du = uv - \int u\,dv$$

or

$$\int g(x)f'(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

This technique is useful when faced with an expression that is difficult to integrate but that can be broken up into two (multiplicative) parts, one part for which it is easy to find the integral $[f'(x)$ to $f(x)]$ and a second part $[g(x)]$ that, when differentiated and then multiplied by $f(x)$, generates an expression $[f(x)g'(x)]$ that turns out to be easy to integrate. At first glance this may seem a rather convoluted process. However, with some practice it becomes a reasonably straightforward exercise and one that turns out to have many useful applications.

Consider, for example, the problem of finding

$$\int xe^x\,dx$$

Since $de^x/dx = e^x$, the derivative or antiderivative of e^x is simply e^x. Thus the complication in this problem arises from the presence of the variable x. However, since $dx/dx = 1$, if we choose $v = x$ ($\Leftrightarrow dv = dx$) and $du = e^x\,dx$ (which implies that $du/dx = e^x$ and hence $u = e^x$), we can use the formula for integration by parts to rewrite the problem in a way that avoids having x and e^x multiplied together. That is, by making these substitutions we get

$$\int (x)(e^x)\,dx = e^x x - \int (e^x)\,dx = e^x(x-1) + C$$

Below we check to make sure the derivative of the result, $d[e^x(x-1)+C]/dx$, is indeed the original integrand, xe^x:

$$\begin{aligned}\frac{d[e^x(x-1)+C]}{dx} &= \frac{de^x}{dx}(x-1)+\frac{d(x-1)}{dx}e^x+\frac{dC}{dx}\\ &= e^x(x-1)+e^x+0\\ &= xe^x\end{aligned}$$

It is generally a good idea to make such a check whenever doing *any* integration exercise.

Example 16.10 Find the integral

$$\int 10xe^{2x}\,dx$$

Solution

Letting $v=10x$ and $du=e^{2x}\,dx$ gives $dv=10\,dx$ and $u=e^{2x}/2$. Using the formula for integration by parts gives

$$\begin{aligned}\int 10xe^{2x}\,dx &= \left(\frac{e^{2x}}{2}\right)10x-\int\frac{e^{2x}}{2}10\,dx\\ &= 5xe^{2x}-5\int e^{2x}\,dx\\ &= 5xe^{2x}-\frac{5}{2}e^{2x}+C\\ &= e^{2x}\left(5x-\frac{5}{2}\right)+C\end{aligned}$$

■

Integrals Depending on a Parameter

It is also useful to know how to differentiate with respect to some parameter that affects either the limits of an integral or the integrand. The economic applications of these techniques are primarily in the field of dynamic analysis (see chapter 25), and it is traditional to use the variable t, which represents time, as the variable of integration. We will use x as the parameter which may affect either the integrand or the limits of integration.

First, consider the case in which only the upper limit of an integral, U, depends on the parameter, x, which we will write as $F(x)=\int_a^{U(x)} f(t)\,dt$. By using the result of theorem 16.2 and the chain rule for differentiation, we obtain the result that

$$F'(x) = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial x} = f(U) \cdot \frac{\partial U}{\partial x}$$

Recall the intuition from theorem 16.2. That part of the above result giving $\partial F/\partial U = f(U)$ implies that a marginal increase in the value of the upper limit increases the area under the curve described by $f(t)$ at a rate that equals the height of the curve at that point, which is simply $f(U)$. Similarly, if the lower limit depends on the parameter x, writing this as $L(x)$, then a marginal increase in L of one unit *decreases* the area under the curve between the two limits of integration by the amount $f(L)$. Therefore it follows that if $F(x) = \int_{L(x)}^{b} f(t)\,dt$, then

$$F'(x) = \frac{\partial F}{\partial L} \cdot \frac{\partial L}{\partial x} = -f(L) \cdot \frac{\partial L}{\partial x}$$

where the minus sign indicates that the area is reduced by an increase in L (or is increased by a decrease in L).

Now suppose that it is the integrand that depends on the parameter x, which we write as $F(x) = \int_a^b f(t, x)\,dt$. The following result is often referred to as Leibniz's rule:

$$F'(x) = \int_a^b \frac{\partial f(t, x)}{\partial x}\,dt$$

We do not provide a formal proof for this result, but the intuition is clear. The expression on the right-hand side of the equality essentially measures the change in the area under the curve $f(t, x)$ created by changing the value of the parameter x by some *small* marginal unit. The rate of change, $F'(x)$, is found by integrating over the rate of change in $f(t, x)$ with respect to a change in x between the limits of integration.

We can summarize these three results for the case where $F(x)$ depends on the parameter x as a result of both limits of integration depending on x and the integrand depending on x; namely $F(x) = \int_{L(x)}^{U(x)} f(t, x)\,dt$. This gives us the result

$$F'(x) = -f(L, x) \cdot \frac{\partial L}{\partial x} + f(U, x) \cdot \frac{\partial U}{\partial x} + \int_L^U \frac{\partial f(t, x)}{\partial x}\,dt$$

A further generalization of this result is that if the integrand depends on a parameter z, which in turn depends on the parameter x, then we can write $F(x) = \int_{L(x)}^{U(x)} f(t, z(x))\,dt$, and applying the chain rule of differentiation, we obtain

$$F'(x) = -f(L, z) \cdot \frac{\partial L}{\partial x} + f(U, z) \cdot \frac{\partial U}{\partial x} + \int_L^U \frac{\partial f(t, z)}{\partial z} \frac{\partial z}{\partial x}\,dt$$

EXERCISES

1. Use the substitution rule to evaluate the following integrals:

(a) $\int (x^3 + 5x)^{10}(3x^2 + 5)\, dx$

(b) $\int \left(e^{x^2} + 4x\right)\left(2xe^{x^2} + 4\right)dx$

(c) $\int 2(e^x + 3x^2)(e^x + 6x)\, dx$

(d) $\int \frac{2x}{(x^2 + 2)^{10}}\, dx$

(e) $\int \frac{6x^2 + 8}{(x^3 + 4x)^2}\, dx$

2. Use the substitution rule to evaluate the following integrals:

(a) $\int (6x^3 + 3x^2 + 8x + 2)^3(18x^2 + 6x + 8)\, dx$

(b) $\int \left(e^{4x^3} + x^2\right)\left(12x^2e^{4x^3} + 2x\right)dx$

(c) $\int (4e^x + 2x^2)(e^x + x)\, dx$

(d) $\int \frac{12x + 2}{(6x^2 + 2x)^3}\, dx$

(e) $\int \frac{15x^2 + 6x + 2}{(10x^3 + 6x^2 + 4x + 11)^2}\, dx$

3. Use the technique of integration by parts to evaluate the following integrals:

(a) $\int x^2 e^x\, dx$

[Hint: There are two stages to this problem. For the first stage set $v = x^2$, and $du = e^x\, dx$. Use integration by parts again on the result obtained from the first stage.]

(b) $\int \frac{x^3}{\sqrt{1 + x^2}}\, dx$

(c) $\int x \ln x\, dx$

4. Use the technique of integration by parts to evaluate the following integrals:

(a) $\int x^3 e^x \, dx$

(b) $\int \frac{x}{\sqrt{1+x}} \, dx$

(c) $\int \ln x \, dx$

[Hint: Set $u = x$ and $v = \ln x$ ($\Rightarrow dv = (1/x)\, dx$)]

CHAPTER REVIEW

Key Concepts

antidifferentiation
constant of integration
definite integral
flow variable
fundamental theorem of integral calculus
improper integrals
indefinite integral
integrable
integrand
integration
integration by parts
lower sum
partition
producer surplus
Riemann sum
stock variable
substitution rule of integration
upper sum

Review Questions

1. Explain, using functional notation, the statement "antidifferentiation is the inverse operation of differentiation" and provide a specific example.

2. What is the difference between an indefinite integral and a definite integral?

3. What is the constant of integration?

4. Why do we ignore the constant of integration when computing a definite integral?

5. How do the processes of differentiation and antidifferentiation (or integration) relate to the distinction between a stock and flow variable in an economic model?

6. What is a partition of an interval?

7. Describe the relationship among the concepts Reimann sum, lower sum, upper sum, and the definite integral of a function $f(x)$ on the interval $[a, b]$.

8. What does it mean for a function to be integrable on a closed interval?

9. Explain, using notation and a graph, the usefulness of the fundamental theorem of integral calculus.

10. What are the various types of improper integrals?

11. Explain how the substitution rule of integration relates to the chain rule of differentiation.

12. Explain how the method of integration by parts relates to the product rule of differentiation.

Review Exercises

1. Evaluate the following integrals:

   (a) $\int x^2\,dx$

   (b) $\int (2x^3 + 5x^2 + x + 5)\,dx$

   (c) $\int \left(\sum_{i=0}^{n} a_i x^i\right) dx = \int \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\right) dx$

2. Evaluate the following integrals:

   (a) $F(x) = \int e^{2x}\,dx,\ F(0) = 1/2$

   (b) $F(x) = \int \frac{3x^2 + 2}{x^3 + 2x + 1}\,dx,\ F(0) = 0$

   (c) $F(x) = \int x^2\,dx,\ F(0) = 3$

3. If $\mathrm{MP}_L = 5L^{1/3}$ is a firm's marginal-product function, where L is the single input labor and output is zero when $L = 0$, find the production function for the firm.

4. Suppose that a firm begins at time $t = 1$ with a capital stock of $K(1) = 200{,}000$ and, in addition to replacing any depreciated capital, is planning to invest in new capital at the rate $I(t) = 50{,}000t^{-3/2}$ for the forseeable future. Find the planned level of capital stock T years from now. Will this firm's capital stock grow without bound as $T \to \infty$? Explain using a graph.

5. For a profit-maximizing firm with a marginal-cost function $\mathrm{MC}(q) = q^{3/2} + 6$, find:

   (a) PS (producer surplus) at price $p_0 = 7$

(b) PS at price $\hat{p} = 70$

(c) ΔPS resulting from the price change $p_0 = 7$ to $\hat{p} = 70$.

Illustrate your results with a graph.

6. For a consumer with demand function $q = 100 - 5p^{1/2}$, find:

(a) CS (consumer surplus) at price $p_0 = 9$

(b) CS at price $\hat{p} = 4$

(c) ΔCS resulting from the price change $p_0 = 9$ to $\hat{p} = 4$

Illustrate your results with a graph.

7. For the demand function $q = 12p^{-3}$, find the consumer surplus if $p = 1$. Illustrate your result with a graph.

8. Use the technique of integration by parts to find the integral

$$\int (2 + x)e^x \, dx$$

9. Use the substitution rule to find the integral

$$\int (x^3 + 4x^2 + 3)^4 (3x^2 + 8x) \, dx$$

10. For each scenario below find the present value of an infinite stream of payments of amount \$2,000 per year if the annual interest rate is $r = 0.05$ (i.e., 5%). For each scenario, assume that interest is compounded continuously.

Scenario A: The yearly payments are spread evenly throughout the year.
Scenario B: The yearly payments are made at the end of each year.
Scenario C: The yearly payments are made at the beginning of each year.

Use a graph (similar to that in figure 16.20) to compare these values and show that, relative to scenario A, the answer for scenario B represents a lower Riemann sum, while that of scenario C represents an upper Riemann sum.

Chapter 17 An Introduction to Mathematics for Economic Dynamics

Economic dynamics is a study of how economic variables evolve over time. Unlike economic statics, which is a study of economic systems at rest, the focus of attention in economic dynamics is on how economic systems change as they move from one position of rest (i.e., equilibrium) to another. In this sense, economic dynamics, in adding the dimension of time to economic models, goes a step beyond economic statics. Often, however, this added realism and complexity can be managed only by reducing the complexity of the economic model in some other direction.

Once we introduce time to economic models, we expand the range of questions that we can ask and issues that we can study. One of the most studied topics in economic dynamics is economic growth. What determines how quickly an economy grows? Why are growth rates different in different economies? Where does the path of growth lead an economy? Another interesting issue that has received attention is the dynamics of national debt. The concern is about whether a policy of persistent budgetary deficits leads inevitably to national insolvency or whether economic growth, if sufficiently rapid, can allow a nation to outrun insolvency forever. A third issue in economic dynamics that has been the subject of much research in recent years arises from the fact that most energy and mineral resources are nonrenewable. Should we be using so much of these resources now or should we be saving more for future use? What is the optimal rate of depletion of nonrenewable resources? These are just three examples of issues in economic dynamics. These three, and others, are explored as applications of the mathematics covered in the next few chapters on dynamics.

Economic dynamics relies on most of the mathematical tools already developed in chapters 1 through 16 of this book. In addition, however, it relies extensively on some mathematics not yet covered, primarily differential equations and difference equations. We provide an introductory, but thorough, coverage of differential and difference equations in chapters 18 through 24. Optimization in economic dynamics relies on mathematical techniques that are extensions of the techniques covered in chapters 6 and 13: optimal control theory, calculus of variations, and dynamic programming. Chapter 25 provides an introductory coverage of the main elements of optimal control theory. We feel it unnecessary to cover the calculus of variations as it can be used only for a subset of the problems that can be solved by optimal control theory. We do not cover dynamic programming, since to give it a fair treatment would take us too far afield into the realm of economic dynamics under uncertainty.

17.1 Modeling Time

Time: A Continuous or Discrete Variable?

Variables must be dated in a dynamic model. We have the option of dating variables at discrete intervals of time (e.g., once per month) or continuously (at every instant of time). Time is continuous in reality. If we model it this way and date variables at every instant, we can write

$$y(t)$$

for the value of the variable y at date t, where t is a continuous variable that represents time, or the date. Here t is a real number with larger numbers representing dates further in the future. The variable $y(t)$ is allowed to change continuously over time, like the price of copper on the London Metal Exchange.

It is sometimes more convenient to model variables as changing discretely only once per fixed period of time (like a month or day). If so, we would model time as a discrete variable and date variables at discrete intervals of time. For this we write

$$y_t$$

for the value of the variable y during period t, where t is a discrete variable that takes on the integer values $0, 1, 2, 3, 4, \ldots$. The value of y is constant for the duration of a period and can change only as t changes discretely from one period to the next. For example, even though consumer prices can change continuously, the consumer price index in many countries is calculated only once per month. A data series for the consumer price index then would consist of one price observation per month, a discrete-time data series.

What Is a Difference Equation?

As the name suggests, a **difference equation** specifies the determinants of the difference between successive values of a variable. In other words, it is an equation for the change in a variable. The *difference* or *change* in a variable between two periods is

$$\Delta y_t = y_{t+1} - y_t, \qquad t = 0, 1, 2, \ldots$$

A difference equation is any equation that contains Δy_t. In the following example, p_t stands for the consumer price index:

$$p_{t+1} - p_t = \theta p_t$$

where $0 < \theta < 1$. This difference equation says that the change in the consumer price index from period t to the next period $(t + 1)$ is equal to a fraction, θ, of the consumer price index in period t. In the next example, m_t stands for the money supply in month t, controlled by the monetary authority:

$$m_{t+1} - m_t = b + \alpha m_t$$

$b > 0$, $\alpha > 0$. This difference equation says that the monetary authorities have a policy of increasing the money supply every month by an amount b (a constant) plus a fraction α of the previous period's money supply.

If you wish to use the information in a difference equation to figure out what the consumer price index or the money supply must actually be for any month, then you are faced with the problem of solving the difference equation.

What Is a Differential Equation?

A **differential equation** is like a difference equation in that it expresses how a variable changes over time except that *time* is considered to be a continuous variable. Hence the *differential* of y can be expressed formally as the difference between successive values of y when the length of a period becomes extremely small:

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y_{t+h} - y_t}{h}$$

The differential of y is just the derivative of y with respect to the continuous variable time. We reserve a special notation for the **time derivative** of a variable:

$$\dot{y} \equiv \frac{dy}{dt}$$

That is, we put a dot over a variable to indicate its time derivative. A differential equation, then, is any equation that contains $\dot{y}$.

In the following example, $K(t)$ is the capital stock in an economy at time t:

$$\dot{K} = I(t) - \delta K(t)$$

where $0 < \delta < 1$ is the depreciation rate. This differential equation says that the change in the capital stock is equal to new investment, $I(t)$, less depreciation of existing capital, $\delta K(t)$. Given this relationship and knowledge of the path of investment, $I(t)$, you might wish to determine the size of the capital stock at some point in time. If so, the problem you face is to solve the differential equation.

There are different types of difference and differential equations. In the remainder of this chapter, we provide a brief classification of the different types and show what types will be studied in the next few chapters.

Classification of Difference Equations

A difference equation is any equation that contains a difference of a variable. A difference equation can be classified according to its **order** (whether it contains a first difference, second difference or higher difference), whether it is **linear** or **nonlinear,** and whether it is **autonomous** or **nonautonomous.**

1. **Order** The order of a difference equation is determined by the highest order of difference contained in the equation. For example, a first-order difference equation contains only the first difference of a variable: the difference in the variable between two consecutive time periods $(y_{t+1} - y_t)$. A second-order difference equation also contains the second difference of a variable: the difference in the variable between every two successive time periods $(y_{t+2} - y_t)$.

In practice, this means that a first-order difference equation contains variables at most one period apart, such as

$$y_{t+1} = 3y_t + 2$$

whereas a second-order difference equation contains variables at most two periods apart, such as

$$y_{t+2} = 2y_{t+1} + 3y_t + 2$$

or, equivalently,

$$y_t = 2y_{t-1} + 3y_{t-2} + 2$$

An nth order difference equation, then, contains variables at most n periods apart.

In this book, we will be concerned only with first- and second-order difference equations.

2. **Autonomous** A difference equation is said to be autonomous if it does not depend on time explicitly; otherwise, it is nonautonomous. For example,

$$y_{t+1} = 2y_t + 3t$$

is nonautonomous because it depends explicitly on the variable t. On the other hand,

$$y_{t+1} = 2y_t + 3$$

is an autonomous difference equation because it does not depend explicitly on the variable t.

In this book autonomous difference equations are emphasized, since these are more common in economics. However, we also show how to solve nonautonomous, linear difference equations.

3. **Linear** or **nonlinear** A difference equation is nonlinear if it involves any nonlinear terms in y_t, y_{t+1}, y_{t+2}, and so on. It is linear if all of the y terms are raised to no power other than 1. For example,

$$y_{t+1} = 2y_t^2 + 3$$

is a nonlinear, autonomous, first-order difference equation, and

$$y_{t+1} = 2 \log y_t + 3$$

is a nonlinear, autonomous, first-order difference equation. But

$$y_{t+1} = 2y_t + 3t^2$$

is a linear, nonautonomous difference equation. Note that the word linear applies only to whether the equation is linear in y terms. It can be nonlinear in t and still be a linear (in y) difference equation although, of course, it is nonautonomous in that case. An example of a linear, autonomous, second-order difference equation is

$$y_{t+2} = 5y_{t+1} + 2y_t + 3$$

and an example of a nonlinear, autonomous, second-order difference equation is

$$y_{t+2} = 5y_{t+1} + \frac{2}{y_t} + 3$$

We concentrate on linear difference equations in this book, but include a chapter on nonlinear first-order difference equations that leads naturally to a discussion of the fascinating subject of *chaos*.

4. **Solutions** The concept of a *solution* to a difference equation is different from other solution concepts discussed so far in this book. We are quite familiar with

the concept of a solution to an algebraic equation: the solution is a variable. However, a solution to a difference equation is itself a *function* that makes the difference equation true.

There are usually many solutions to a difference equation. For example, consider the linear, first-order difference equation

$$y_{t+1} = 2y_t, \qquad t = 0, 1, 2, \ldots$$

A solution is the function

$$y_t = 2^t$$

To verify that this is a solution, check that it makes the difference equation true. To do this, first note that the solution implies that $y_{t+1} = 2^{t+1}$. But 2^{t+1} is equal to $2^t(2)$; but since $y_t = 2^t$, our solution says that $y_{t+1} = 2y_t$, which is the same as the difference equation.

Another solution is the function

$$y_t = C2^t$$

where C is any arbitrary constant. To check that this, too, is a solution, note that it implies that $y_{t+1} = C2^{t+1}$. But writing 2^{t+1} as $2^t(2)$ makes this $y_{t+1} = C2^t(2)$. Using $y_t = 2^t$ makes this $y_{t+1} = 2y_t$, which again is the difference equation.

Since C can take on an infinity of values, there is an infinite number of solutions to this difference equation. It is only when we are given more information, such as the actual initial value of y, that we are able to find a unique solution. For example, if we are given that $y_0 = 1$, then we know that C must equal 1 to make the solution satisfy this equality.

In the vast majority of cases, it is usually not possible to find a solution to a difference equation. In particular, most nonlinear difference equations cannot be solved explicitly for the underlying function y_t. Instead, we would have to resort to numerical techniques with the aid of a computer to obtain a solution. Or, we could do a qualitative analysis to determine some of the properties of the solution without actually obtaining an explicit solution. Linear difference equations of any order, however, can always be solved explicitly. We study linear, first-order difference equations in chapter 18, nonlinear, first-order difference equations in chapter 19, and linear, second-order difference equations in chapter 20.

Classification of Differential Equations

A differential equation is any equation that contains a differential, or derivative. In this book, we will study only **ordinary differential equations.** Ordinary differential equations contain only ordinary derivatives as opposed to partial derivatives. An example of a partial differential equation is given later.

1. **Order** The order of a differential equation is determined by the highest order of derivative contained in the equation. For example, a first-order differential equation contains only the first derivative of a function, whereas a second-order differential equation contains the second derivative (and possibly the first derivative). An example of a second-order differential equation is

$$3\ddot{y} + 2\dot{y} + y = 2$$

Note that two dots over a variable indicate its second derivative with respect to t. An example of a third-order differential equation is

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2$$

We have not used the “dot” notation for a third-order equation because it is too cumbersome to place more than two dots over a variable.

In this book we will be concerned with only first- and second-order ordinary differential equations.

2. **Autonomous** A differential equation is said to be autonomous if it does not depend on time explicitly; otherwise, it is nonautonomous. For example,

$$\dot{y} + 5y = t$$

is a nonautonomous first-order differential equation. But

$$\dot{y} + 5y = 3$$

is an autonomous first-order differential equation. As with difference equations, we place more emphasis on autonomous differential equations, but we also show solution techniques for nonautonomous first- and second-order differential equations.

3. **Linear** or **nonlinear** A differential equation is nonlinear if it involves any nonlinear terms in y, $\dot{y}$, $\ddot{y}$, and so on. It is linear if all of the y terms are raised to no power other than 1. For example,

$$\dot{y} + t^2y = \cos t$$

is a linear, but nonautonomous first-order differential equation, whereas

$$\dot{y} + y^2 = 2$$

is a nonlinear, autonomous, first-order differential equation.

We cover linear differential equations of the first- and second-order, and include a chapter on nonlinear first-order differential equations.

4. **Solutions** A solution to a differential equation is a *function* that makes the differential equation true. As with difference equations, there are usually many solutions to a differential equation. For example, consider the linear, first-order differential equation

$$\dot{y} = b$$

One solution is

$$y(t) = bt$$

Another is

$$y(t) = bt + 1$$

The general solution is

$$y(t) = bt + C$$

where C is an arbitrary constant that can take any value. To verify that this is indeed a solution, differentiate it and see that the derivative of y does equal b.

Since C can take any value, there is an infinity of solutions to this differential equation. However, if we are given more information, such as the initial value of y, we can determine the value of C. For example, if we are given that $y(0) = 3$, which means the value of y at $t = 0$ was 3, then the value of C must be 3.

As with difference equations, we cannot solve nonlinear differential equations in general. However, it is possible to obtain explicit solutions for linear differential equations of any order. We study linear first-order differential equations in chapter 21, nonlinear first-order differential equations in chapter 22, and linear second-order differential equations in chapter 23.

Differential Equations in Economic Statics and Partial Differential Equations

Differential equations need not apply only to equations that are functions of time. Any equation that contains a derivative is a differential equation. For example, suppose that we know that the marginal-cost function for a firm is given by

$$\frac{dc(x)}{dx} = b$$

where $c(x)$ is the total-cost function defined on non-negative real values of x, where x is the output of the firm, and $dc(x)/dx$ is the marginal-cost function, which is equal to a constant b. This is a differential equation and its solution is found by integration to recover the primitive function, which in this case is the total-cost function. The solution is

$$c(x) = bx + C$$

where, as before, C is an arbitrary constant of integration. If we also know, for example, that $c(0) = F$, which means that even when the output of the firm is zero, costs are equal to F (fixed costs), then the solution becomes

$$c(x) = bx + F$$

Any equation containing an ordinary derivative is an ordinary differential equation. Thus differential equations can arise in economic statics as well as in economic dynamics. The techniques for solving and analyzing ordinary differential equations are the same in both cases. However, because our concern is with developing the tools required for economic dynamics, we adopt the convention of making all differentials refer to derivatives with respect to the variable time, as opposed to allowing differentials to refer to derivatives with respect to an arbitrary variable, such as x in the preceding example.

An equation containing partial derivatives is called a **partial differential equation.** As an example of the latter, suppose a household-utility function depends on the consumption of two goods, x and y, and suppose the marginal utility from consuming x depends on how much x and y are being consumed. Then the marginal-utility function could be

$$\frac{\partial u(x, y)}{\partial x} = \alpha x^{\alpha-1} y^{\beta}$$

where u is the utility function and α and β are each between 0 and 1. This is a partial differential equation because it contains the partial derivative of the function u. In this book, we will deal only with ordinary differential equations.

CHAPTER REVIEW

Key Concepts

autonomous equation
difference equation
differential equation
linear equation
nonautonomous equation
nonlinear equation
order of an equation
ordinary differential equation
partial differential equation
time derivative

Review Questions

1. Provide an example of a variable that varies continuously over time and an example of a variable that varies discretely over time.
2. Explain the difference between autonomous and nonautonomous difference equations and differential equations.
3. How would you determine the order of a difference or differential equation?
4. Explain how the concept of a solution to a difference or differential equation differs from the concept of the solution to an algebraic equation.
5. Explain the difference between ordinary and partial differential equations.

Review Exercises

Classify each of the following according to: (**a**) order, (**b**) linear or nonlinear, (**c**) autonomous or nonautonomous, and (**d**) difference or differential equation.

1. $y_{t+1} - y_t = 0$
2. $y_{t+2} + 3y_{t+1} + 2y_t = 1$
3. $y_t + 2y_{t-1} = t^3$
4. $y_{t+2} + 2y_t = 0$
5. $y_{t+1} = 2y_t^2 + 3t$
6. $y_{t+4} + 6y_{t+3} + y_{t+2} = 3y_t$
7. $10y_{t+2} - y_t = e^{2t}$
8. $2y_{t+1} - 5/y_t = 2$
9. $y_{t+2} + ty_{t+1} - 3y_t = 10$
10. $y_{t+1} = (\log t)y_t + 1$
11. $\dot{y} = (t+1)/y$
12. $\dot{y} + 2y = 5$
13. $\ddot{y} + 2t\dot{y} + 3y = 1$
14. $\ddot{y} + 2y^2 = t$
15. $\dot{y} = y/2 + e^t$
16. $\ddot{y} + 2\dot{y} + 2 = \log y$
17. $d^3y/dt^3 + 10\,d^2y/dy^2 = 1$
18. $5\dot{y} = 2y + 10^t$
19. $\ddot{y} = 2\dot{y} - y/2 + e^t$
20. $\dot{y} = 2y^3 + t$

Chapter 18 Linear, First-Order Difference Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Modified Cobweb Model
- A Partial Adjustment Model of Energy Demand
- Practice Exercise

In the next three chapters we introduce some elementary techniques for solving and analyzing the kinds of difference equations that are common in economics. We begin in this chapter with linear, first-order difference equations. In the next chapter we introduce nonlinear, first-order difference equations, including the famous logistic equation used extensively in the study of *chaos*. In chapter 20 we examine linear, second-order difference equations.

18.1 Linear, First-Order, Autonomous Difference Equations

In this section we explain how to solve linear, first-order difference equations that are autonomous.

Definition 18.1 The general form of the **linear, first-order, autonomous difference equation** is given by

$$y_{t+1} = ay_t + b, \qquad t = 0, 1, 2, \ldots \tag{18.1}$$

where a and b are known constants.

This difference equation is of the *first-order* because the largest difference to appear is a first difference (a difference of one period); it is *linear* because the y_{t+1}

and y_t terms are not raised to any power other than 1. It is *autonomous* because a and b are constants. If a or b vary with t, the difference equation is nonautonomous. This case is the subject of section 18.2.

Notice that an equation of the form $gy_{t+1} = hy_t + k$ can always be put in the form of equation (18.1) by dividing through by g, provided that g is nonzero.

Solving a difference equation means finding the underlying *function* of time, y_t, that gives rise to it. We begin by doing this directly for the case in which the initial condition, y_0, is known.

If y_0 is known, then at $t = 0$, equation (18.1) implies that

$$y_1 = ay_0 + b$$

At $t = 1$,

$$y_2 = ay_1 + b = a(ay_0 + b) + b$$

After simplifying, we have

$$y_2 = a^2 y_0 + b(a + 1)$$

At $t = 2$,

$$y_3 = ay_2 + b = a[a^2 y_0 + b(a + 1)] + b$$

which becomes

$$y_3 = a^3 y_0 + b(a^2 + a + 1)$$

The expressions for y_1, y_2, and y_3 reveal a pattern developing. Let's conjecture that the solution for y_t is

$$y_t = a^t y_0 + b(a^{t-1} + a^{t-2} + \cdots + a + 1) \tag{18.2}$$

It turns out that this conjecture is correct, but rather than take it on faith, we really should prove it. Before doing so, simplify the expression by noting that inside the brackets is a sum of t terms in a geometric progression. We can reduce the sum, using what we learned in chapter 3, to

$$1 + a + a^2 + \cdots + a^{t-1} = \begin{cases} \dfrac{1 - a^t}{1 - a} & \text{if } a \neq 1 \\ t & \text{if } a = 1 \end{cases}$$

Therefore the conjectured solution to the difference equation can be expressed as

$$y_t = \begin{cases} a^t y_0 + b\left(\dfrac{1-a^t}{1-a}\right) & \text{if } a \neq 1;\ t = 0, 1, 2, \ldots \\ y_0 + bt & \text{if } a = 1 \end{cases} \tag{18.3}$$

We now prove that our conjecture leading to equation (18.3) is correct.

Theorem 18.1 The function y_t given by equation (18.3) is the unique solution to the linear, autonomous, first-order difference equation (18.1), where y_0 is the given initial condition.

Proof

We prove this for the case $a \neq 1$ in two steps (and leave the proof for the case $a = 1$ to the reader). In the first step, we prove that equation (18.3) is a solution. In the second, we prove that there is only one solution.

Step 1 If equation (18.3) is a solution for y_t, then it satisfies equation (18.1); i.e., y_{t+1} is equal to $ay_t + b$. Let's try this. From equation (18.3) we see that

$$y_{t+1} = a^{t+1} y_0 + b\left(\frac{1-a^{t+1}}{1-a}\right)$$

Now add and subtract the term $ab(1-a^t)/(1-a)$ to the right-hand side to get

$$y_{t+1} = a^{t+1} y_0 + \frac{ab(1-a^t)}{1-a} - \frac{ab(1-a^t)}{1-a} + b\left(\frac{1-a^{t+1}}{1-a}\right)$$

This becomes

$$y_{t+1} = a\left[a^t y_0 + b\left(\frac{1-a^t}{1-a}\right)\right] + \frac{b}{1-a}[1 - a^{t+1} - a(1-a^t)]$$

Substituting the solution for y_t into the above and simplifying gives

$$y_{t+1} = ay_t + \frac{b}{1-a}(1-a)$$

and therefore

$$y_{t+1} = ay_t + b$$

Thus equation (18.3) is a solution to equation (18.1).

Step 2 We now prove that there is only one solution. To do this, we use the method of proof by induction. Since y_0 is given as part of the problem, the value for y_1 is uniquely determined by the straightforward calculation $y_1 = ay_0 + b$. In addition, if any value y_t is known (as it is given that y_2 can be calculated from y_1, then y_3 can be calculated from y_2, etc.), then y_{t+1} is also uniquely determined by the calculation $y_{t+1} = ay_t + b$. Thus the solution to the difference equation with y_0 given, is unique. Putting the two parts of this proof together then allows us to conclude that the solution exists and is unique. ■

Example 18.1 Suppose that you deposit \$100 at $t = 0$ in a bank account. Assume that interest, at the annual rate of 10%, is deposited in the account at the end of each year. How much money is in the bank account after seven years?

Solution

The amount of money in the account in year $t + 1$ is 10% larger than in year t. The difference equation, then, is

$$y_{t+1} = 1.1y_t$$

In this case, we have that $a = 1.1$, $b = 0$, and $y_0 = 100$. The solution is

$$y_t = 100(1.1)^t$$

After 7 years, $y_7 = 100(1.1)^7 = \$194.87$. ■

Example 18.2 Suppose you deposit \$100 at the beginning of every year, starting at $t = 0$, in a bank account that earns 10% interest per year. Derive an expression showing the amount of money in the account at the beginning of year t.

Solution

In year $t + 1$ the account is \$100 larger than in the previous year, and has earned 10% interest on the previous year's balance. The difference equation then is

$$y_{t+1} = 1.1y_t + 100$$

We have $a = 1.1$ and $b = 100$. The solution is

$$y_t = 100(1.1)^t + 100\left(\frac{1 - (1.1)^t}{1 - 1.1}\right)$$

We could use this expression to calculate the amount of money in the account after t years (assuming the interest rate remains constant). For example, in year 25, there would be \$10,918.17 in the account. ■

The General Solution

So far we have found the solution to the linear, autonomous, first-order difference equation directly when the initial value of y is known. Here we step back for a moment and show that although there is only *one* solution that satisfies both the difference equation *and* the initial condition, there is, in general, an infinite number of solutions to the linear, first-order difference equation itself.

As an example, consider the difference equation

$$y_{t+1} = 5y_t, \qquad t = 0, 1, 2, \ldots \tag{18.4}$$

A solution to this difference equation is a function defined over $t = 0, 1, 2, \ldots$ that makes the difference equation a true statement for all possible values of t in the domain. A solution is

$$y_t = 5^t$$

We can verify that this equation is a solution by showing that it satisfies the original difference equation. At $t + 1$ the equation above becomes

$$y_{t+1} = 5^{t+1}$$

But this equation can be simplified to get

$$y_{t+1} = 5(5^t) = 5y_t$$

which shows that it does satisfy the difference equation and is therefore a solution. However, another solution is

$$y_t = 2 \cdot 5^t$$

which can be verified as follows:

$$y_{t+1} = 2 \cdot 5^{t+1} = 5 \cdot (2 \cdot 5^t) = 5y_t$$

There are many more solutions to the difference equation (18.4). We can express the solution, in general, by writing

$$y_t = C5^t \tag{18.5}$$

where C is an arbitrary constant. To verify that this function is a solution, ensure that it satisfies the difference equation (18.4):

$$y_{t+1} = C5^{t+1} = 5(C5^t) = 5y_t$$

The general solution to equation (18.4) is equation (18.5). Because C can be any value, there is, in general, an infinite number of solutions to the difference equation itself. This leads us to a formal statement of the general solution to the linear, first-order, autonomous difference equation:

Theorem 18.2 There exists a constant C such that any solution to the linear, first-order, autonomous difference equation can be expressed as

$$y_t = \begin{cases} Ca^t + b\left(\dfrac{1-a^t}{1-a}\right) & \text{if } a \neq 1;\ t = 0, 1, 2, \ldots \\ y_t = C + bt & \text{if } a = 1 \end{cases} \tag{18.6}$$

Proof

The theorem can be proved either by direct substitution of equation (18.6) into the difference equation, as was done in theorem 18.1, or by simply noting that we could set C equal to the initial value of y and then apply theorem 18.1. ■

Theorem 18.2 expresses the general solution to the difference equation (18.1). As such, we know that it satisfies that difference equation. If we also require the solution to satisfy an initial condition, then we will have to choose a particular value for the constant C.

Example 18.3 Solve the difference equation

$$y_{t+1} = 0.5y_t + 10$$

Solution

Applying theorem 18.2 gives

$$y_t = C(0.5)^t + 10\left(\frac{1-(0.5)^t}{1-0.5}\right)$$

as the general solution. ■

Example 18.4 Solve the difference equation in example 18.3 and ensure that it also satisfies the initial condition: $y_0 = 1$.

Solution

Setting $t = 0$ in the general solution in example 18.3 gives

$$y_0 = C + 0$$

Therefore we must set $C = 1$ to satisfy the given initial condition. The solution becomes

$$y_t = (0.5)^t + 10\left(\frac{1 - (0.5)^t}{1 - 0.5}\right)$$ ■

Although it is important to remember that the general solution to a linear, first-order difference equation involves an infinite number of particular solutions, we will be working with initial value problems exclusively for the remainder of this chapter.

The Steady State and Convergence

A difference equation determines the value of y_{t+1}, given the value of y_t. Generally, the value of y changes over time, tracing out the dynamic path of the variable. However, a property of autonomous difference equations that is important in economics, is that they often have a steady state. A steady state is the value of y at which the dynamic system becomes stationary (so it is sometimes called the stationary value of y). That is to say, y_{t+1} takes the same value as y_t for all values of t. In a linear, autonomous, first-order difference equation, there always exists a steady state, as long as $a \neq 1$. We show how to find it below.

Definition 18.2 The **steady-state** or **stationary value** in a linear, first-order, autonomous difference equation is defined as the value of y at which the system comes to rest. This implies that $y_{t+1} = y_t$.

To find the steady-state value of y, which we will call $\bar{y}$, set $y_{t+1} = y_t \equiv \bar{y}$ in the difference equation. This gives

$$\bar{y} = a\bar{y} + b$$

Solving for $\bar{y}$ gives

$$\bar{y} = \frac{b}{1-a}, \qquad a \neq 1$$

If $a = 1$, there is no steady-state solution.

If y ever becomes equal to its steady-state value, it will remain at that value for all successive time periods. The important question then is: If y starts off at any arbitrary value, will it always tend to converge towards its steady-state value?

To answer this question, rearrange the solution stated in equation (18.3) to get

$$y_t = a^t\left(y_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} \qquad \text{if } a \neq 1;\ t = 0, 1, 2, \ldots \tag{18.7}$$

Inspection of this expression makes it apparent that the question of **convergence** and **divergence** is determined entirely by the term a^t, since this is the only term in the solution that depends on t. If this term converges to zero as t goes to infinity, then y_t converges to $b/(1-a)$. On the other hand, if this term diverges to infinity as t goes to infinity, then y_t will diverge also. It is therefore imperative that we understand the behavior of a^t as $t \to \infty$.

We can think of the term a^t, with $t = 0, 1, 2, \ldots$, as a sequence of numbers

$$\{a^t\} = 1, a, a^2, a^3, \ldots, a^t, \ldots$$

In chapter 3 we learned that a sequence like this converges to zero as t goes to infinity if $|a| < 1$ and diverges if $|a| > 1$. This gives us our main convergence result:

Theorem 18.3 In the case of a linear, autonomous, first-order difference equation, y_t converges to its steady-state value, $b/(1-a)$, if and only if $|a| < 1$.

While convergence is guaranteed if $|a| < 1$, the *path* that y_t takes over time is very different, depending on the sign of a. If $0 < a < 1$, then y_t will converge monotonically to $b/(1-a)$. We know this because each term in the sequence $\{a^t\}$ is smaller than the previous one. For example, if $a = 1/2$, the sequence is

$$\left\{\left(\frac{1}{2}\right)^t\right\} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

However, if $-1 < a < 0$, y_t will converge to $b/(1-a)$ on an **oscillating path.** We know this because each term in the sequence $\{a^t\}$ will have the opposite sign to the previous one. For example, if $a = -1/2$, the sequence is

$$\left\{\left(-\frac{1}{2}\right)^t\right\} = 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}, \ldots$$

So far we have found that if a is positive, the path of y_t is **monotonic** and if a is negative, y_t oscillates between positive and negative values. In either case, the path of y_t converges to the steady-state value only if the absolute value of a is less than 1. There are three additional cases that warrant separate consideration:

(i) If $a = 0$, we see from equation (18.7) that y_t is constant over time and equal to b.
(ii) If $a = 1$, we see from equation (18.3) that y_t diverges to infinity if $b > 0$ and minus infinity if $b < 0$.
(iii) If $a = -1$, then y_t oscillates between the two values y_0 and $b - y_0$.

Example 18.5

Let y_t denote the number of individuals in a population of fish. Let the dynamic behavior of the fish population be governed by the difference equation

$$y_{t+1} = ay_t + 10$$

Find the steady-state number of fish and sketch a graph of y_t, first for the case $a = 0.5$ and second for the case $a = -0.5$.

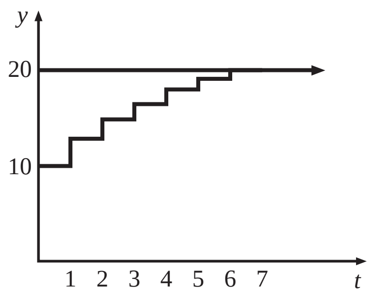


Figure 18.1 Approach path in example 18.5 for $a = 0.5$

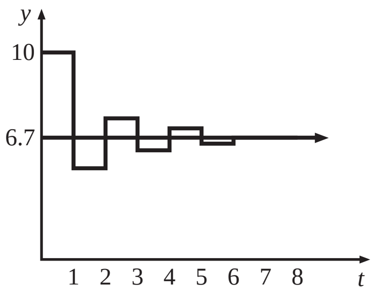


Figure 18.2 Approach path in example 18.5 for $a = -0.5$

Solution

The steady-state value of y is found by setting $y_{t+1} = y_t = \bar{y}$. This gives

$$\bar{y} = \frac{10}{1-a}$$

The solution to the difference equation can be expressed as

$$y_t = a^t\left(y_0 - \frac{10}{1-a}\right) + \frac{10}{1-a}$$

Clearly, if $|a| < 1$, then y_t converges to $10/(1-a)$ as t goes to infinity. Thus, if $a = 0.5$, then y_t approaches the steady-state value $\bar{y} = 20$ smoothly. In figure 18.1 we show an approach path starting from an initial value of 10.

If $a = -0.5$, y_t converges to $\bar{y} = 20/3 = 6.7$ on an oscillating path. In figure 18.2 we show an approach path again starting from 10. In both cases y_t is very close to its respective steady-state value after 7 or 8 time periods. ■

The Cobweb Model of Price Adjustment

In most markets, suppliers must commit to a supply decision before they know the price at which their product will sell. For example, in some agricultural markets, farmers plant their crops in the spring and harvest them in the fall when prices may be quite different than they were in the spring. In some labor markets (e.g., those for lawyers, teachers, nurses) individuals commit to an investment in human capital by entering a specialized university training program but do not learn their employment and salary prospects until they graduate some years later. What are the implications of this type of lag in the supply process for the behavior of market price over time?

We use the following model of price determination to investigate this question. We will learn that price oscillations are inevitable if suppliers make their supply decision as if the current price will prevail when their supply reaches the market.

Let the market-demand function be given by

$$q_t^D = A + Bp_t$$

where q_t^D is the quantity demanded in period t and p_t is the market price that prevails in period t.

We assume that supply decisions are made one period before the product reaches the market. Thus the supply reaching the market in period t is decided upon in period $t-1$ on the basis of what suppliers expect price to be in the next period. Let $E_{t-1}(p_t)$ represent this expected price. Then the quantity supplied in period t is assumed to be given by

$$q_t^S = F + GE_{t-1}(p_t)$$

To close this model, we need to specify the way in which price expectations are formed. In the basic cobweb model, the assumption is

$$E_{t-1}(p_t) = p_{t-1}$$

which means that suppliers expect the next period price to equal the current price.

Assuming that the price adjusts to clear the market each period, then supply and demand will be equal in each period. This means that

$$A + Bp_t = F + Gp_{t-1}$$

Rearranging and solving for p_t gives

$$p_t = \frac{G}{B}p_{t-1} + \frac{F-A}{B} \tag{18.8}$$

which shows that the time path of price is governed by a linear, autonomous, first-order difference equation (expressed in terms of t and $t-1$ instead of $t+1$ and t). The steady-state price, which we shall call $\bar{p}$, is found by setting $p_t = p_{t-1} = \bar{p}$. Doing this and rearranging gives

$$\bar{p} = \frac{A - F}{G - B}$$

Note that the steady-state price is also the price at which supply equals demand.

Comparing equation (18.8) to the form in equation (18.1), we see that

$$a = \frac{G}{B} \quad \text{and} \quad b = \frac{F - A}{B}$$

Applying theorem 18.1 gives the solution

$$p_t = p_0 \left(\frac{G}{B}\right)^t + \frac{F - A}{B}\left(\frac{1 - (G/B)^t}{1 - G/B}\right)$$

Rearranging this result and using the expression for $\bar{p}$ gives

$$p_t = (p_0 - \bar{p})\left(\frac{G}{B}\right)^t + \bar{p} \tag{18.9}$$

Price converges to $\bar{p}$ if and only if $-1 < G/B < 1$, for only then will the first term in equation (18.9) go to zero as t goes to infinity. Unfortunately, there is no particular reason for the ratio G/B to satisfy this condition, since it is just the ratio of the slopes of the supply and demand functions. Thus price may or may not converge, depending on the relative slopes.

Usually $B < 0$ (demand slopes negatively) and $G > 0$ (supply slopes positively). As a result B/G is usually negative. The implication is that price oscillates in this model because $(G/B)^t$ will be alternately positive and negative as t is an even- or odd-numbered period. If $|G/B| < 1$, then price follows an oscillating but converging path to its steady-state level. If $|G/B| > 1$, then price follows an oscillating path but the oscillations become larger and larger over time. In this case, price never converges to its steady-state equilibrium.

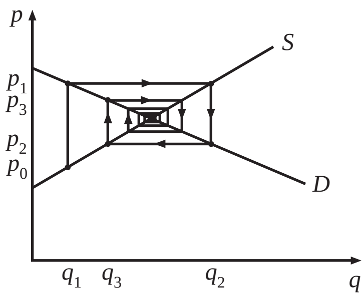


Figure 18.3 Price adjustment in the basic cobweb model

In figure 18.3 we depict the case of a *stable* market in which price converges to the steady-state equilibrium price. The market begins out of equilibrium (for some unspecified reason such as a demand shift) at price p_0. In period 0, suppliers plan to produce the quantity q_1 for period 1. When this quantity reaches the market in period 1, price rises to p_1 to clear the market. In period 1, suppliers plan to produce the amount q_2 for period 2. When this quantity reaches the market in period 2, price falls to p_2 to clear the market. At this price, suppliers plan q_3 for period 3. In period 3, price rises to p_3 to clear the market. Notice that price went from p_0 up to p_1, down to p_2, and then up to p_3. This process of price oscillations continues as price gradually converges to its steady-state equilibrium value where the supply and demand curves intersect. The diagram gives one the impression of a cobweb; hence, we call this the **cobweb model.**

Although the oscillations of the cobweb model may be its most interesting feature, its most disturbing feature is that it may be highly unstable. Since the absolute value of the ratio G/B is just as likely to exceed 1 as it is to be less than 1, divergence is just as likely as convergence. This undesirable feature of the cobweb model is a direct result of the assumption that suppliers form extremely naive price expectations. Even as price oscillates and price expectations are never realized (i.e., price in period t is never equal to price in period $t-1$), the model assumes that suppliers continue to forecast that the current price will prevail in the next period. In the next example, price expectations are formed in a slightly more sophisticated way. As we will see, this change makes the model more stable.

Summary of Convergence Analysis

For the difference equation

$$y_{t+1} = ay_t + b$$

the solution is

$$y_t = \begin{cases} a^t(y_0 - \bar{y}) + \bar{y} & \text{if } a \neq 1, t = 0, 1, 2, \ldots \\ y_0 + bt & \text{if } a = 1 \end{cases}$$

where

$$\bar{y} = \frac{b}{1-a} \qquad \text{if } a \neq 1$$

is the steady-state (stationary) equilibrium that exists when $a \neq 1$.

The steady-state equilibrium is *stable* (y_t converges to $\bar{y}$) if and only if

$$-1 < a < 1$$

The path of y_t as it approaches $\bar{y}$ (called the **approach path**) is

- *monotonic* if a is positive (and less than 1)
- *oscillatory* if a is negative (and greater than -1)

Furthermore, if $a \geq 1$, then y_t diverges from $\bar{y}$ monotonically. If $a < -1$, then y_t diverges from $\bar{y}$ with ever-increasing oscillations. If $a = -1$, then y_t never approaches $\bar{y}$ but instead alternates in value between y_0 and $b - y_0$. If $a = 0$, then y_t is constant and equal to b.

EXERCISES

1. For each of the following difference equations:

(a) Obtain the general solution (i.e., when y_0 is unspecified).

(b) Obtain the unique solution for the case when y_0 is known.

(c) Solve for the steady state if it exists and indicate whether or not y_t converges to the steady state.

(i) $y_{t+1} = 2y_t - 10$

(ii) $y_{t+1} = y_t$

(iii) $y_t = 0.5y_{t-1} + 1$

2. Repeat exercise 1 for the following difference equations:

(i) $y_{t+1} = 0.1y_t + 9$

(ii) $y_t = y_{t-1} - 1$

(iii) $y_{t+1} = 5y_t - 2$

For exercises 3 and 4, you will need the following information: Interest is paid into most bank accounts more frequently than once per year. For example, if the interest is paid twice per year, we have semiannual compounding; if it is paid once per month, we have monthly compounding. Let r be the annual interest rate and let n be the number of periods of compounding per year. If there are T years, then the interest rate per period is r/n and the total number of periods is nT.

3. If you invest $1,000 today in a savings account that pays interest at the rate of 0.5% per month, how much money will be in the savings account after 120 months?

4. Suppose you deposit $50 at the beginning of every month in a bank account that earns 1% interest per month. How many months will it take before you have $4175.00 in your account?

5. A firm has a capital stock in period 0 of K_0, and invests a constant amount, I, that augments the capital stock at the end of each period. However, a proportion δ of the capital stock depreciates during each period. Write out the difference equation for the capital stock and solve it.

6. Solve the difference equation for price in the basic cobweb model for the following values of the parameters of the supply and demand equations. Assume that $p_0 = 2$.

(a) $B = -10$, $G = 5$, $A = 100$, $F = 25$.

(b) $B = -10$, $G = 20$, $A = 100$, $F = 10$.

(c) $B = -10$, $G = 10$, $A = 100$, $F = 20$.

In each case, sketch a diagram showing the supply and demand curves, the equilibrium price and quantity, and the cobweb of price movements. Be particularly careful to indicate whether or not price converges to the equilibrium price.

7. Suppose that aggregate consumption in period t, C_t, is a linear function of aggregate income in the previous period, Y_{t-1}

$$C_t = A + BY_{t-1}$$

where $A, B > 0$ are constant. If aggregate investment is a constant amount, I, and aggregate income is equal to consumption plus investment

$$Y_t = C_t + I$$

write out the difference equation for aggregate income and solve it. What restriction must be placed on B to ensure that income converges monotonically to the steady-state equilibrium? What is the short-run (one period) and the long-run (steady state) impact of an increase in I on aggregate income?

18.2 The General, Linear, First-Order Difference Equation

If a and b in the linear difference equation vary over time, the difference equation is nonautonomous. However, rather than refer to this version as the *nonautonomous* linear, first-order difference equation, we will just call it the *general* version of the linear, first-order difference equation because everything else is a special case, including the autonomous case (a and b constant), and three possible nonautonomous cases (a and b both vary over time, a constant but b varies over time, a varies over time but b constant).

Definition 18.3

The general form of the linear, nonautonomous first-order difference equation is given by

$$y_{t+1} = a_t y_t + b_t, \qquad t = 0, 1, 2, \ldots \tag{18.10}$$

where a_t and b_t are known functions defined over $t = 0, 1, 2, \ldots$.

Like its autonomous counterpart, the difference equation (18.10) can be solved directly if y_0 is known. Since a_0 and b_0 are also known, we have

$$y_1 = a_0 y_0 + b_0$$

With y_1 now determined, y_2 is given by

$$y_2 = a_1 y_1 + b_1 = a_1 a_0 y_0 + a_1 b_0 + b_1$$

since a_1 and b_1 are known. Next, y_3 is given by

$$\begin{aligned} y_3 &= a_2 y_2 + b_2 \\ &= a_2 a_1 a_0 y_0 + a_2 a_1 b_0 + a_2 b_1 + b_2 \end{aligned}$$

since a_2 and b_2 are known. Looking at the successive solutions for y_1, y_2, and y_3 reveals an emerging pattern. We conjecture that the solution for y_t is

$$\begin{aligned} y_t = {} & (a_{t-1}a_{t-2}\cdots a_0)y_0 + (a_{t-1}a_{t-2}\cdots a_1)b_0 \\ & + (a_{t-1}a_{t-2}\cdots a_2)b_1 + \cdots + a_{t-1}b_{t-2} + b_{t-1} \end{aligned}$$

A more compact way of expressing this solution is to use the multiplication symbol Π. We then have

$$\begin{aligned} y_t = {} & \prod_{i=0}^{t-1} a_i y_0 + b_0 \prod_{i=0}^{t-1} \frac{a_i}{a_0} + b_1 \prod_{i=1}^{t-1} \frac{a_i}{a_1} + \cdots \\ & + b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k} + \cdots + b_{t-1} \prod_{i=t-1}^{t-1} \frac{a_i}{a_{t-1}} \end{aligned}$$

Simplifying this expression one step further gives the following:

Theorem 18.4 The unique solution to the general, linear, first-order difference equation that also satisfies initial condition y_0 is

$$y_t = \prod_{i=0}^{t-1} a_i y_0 + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i}{a_k}, \qquad t = 0, 1, 2 \ldots \tag{18.11}$$

Proof

To prove this is a solution, we show it satisfies the difference equation. We do this by calculating $y_{t+1} - a_t y_t$ and showing it is indeed equal to b_t. First, calculate y_{t+1}:

$$y_{t+1} = \prod_{i=0}^{t} a_i y_0 + \sum_{k=0}^{t} b_k \prod_{i=k}^{t} \frac{a_i}{a_k}$$

Next, calculate $a_t y_t$. This gives

$$a_t y_t = a_t \prod_{i=0}^{t-1} a_i y_0 + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t-1} \frac{a_i a_t}{a_k}$$

which simplifies to

$$a_t y_t = \prod_{i=0}^{t} a_i y_0 + \sum_{k=0}^{t-1} b_k \prod_{i=k}^{t} \frac{a_i}{a_k}$$

Subtract $a_t y_t$ from y_{t+1}. All terms cancel except the tth term in the summation in the expression for y_{t+1}. But that term is simply equal to b_t. Therefore equation (18.11) is a solution. That it is the only solution follows directly from the fact that with y_0 known, y_1 can be directly calculated from y_0, and hence y_{t+1} can be directly calculated from y_t. ■

Three special cases can arise. First, suppose that $a_t = a$, a constant, but b_t varies. In this case equation (18.11) becomes

$$y_t = a^t y_0 + \sum_{k=0}^{t-1} b_k a^{t-k-1} \tag{18.12}$$

On the other hand, if $b_t = b$, a constant, but a_t varies, then equation (18.11) becomes

$$y_t = \prod_{i=0}^{t-1} a_i y_0 + b \sum_{k=0}^{t-1} \prod_{i=k}^{t-1} \frac{a_i}{a_k} \tag{18.13}$$

Finally, if both a and b are constant, we have

$$y_t = a^t y_0 + b \sum_{k=0}^{t-1} a^{t-k-1} \tag{18.14}$$

We can simplify this further by writing out the terms in the summation

$$\sum_{k=0}^{t-1} a^{t-k-1} = a^{t-1} + a^{t-2} + \cdots + a^0 \qquad \text{for } k = 0, 1, 2, \ldots, t-1$$

and noticing that we can re-express this summation as

$$\sum_{k=0}^{t-1} a^{t-k-1} = \sum_{k=0}^{t-1} a^k$$

Substituting into equation (18.14) gives

$$y_t = a^t y_0 + b \sum_{k=0}^{t-1} a^k$$

This reduces to

$$y_t = \begin{cases} a^t y_0 + b\left(\dfrac{1-a^t}{1-a}\right) & \text{if } a \neq 1 \\ y_0 + bt & \text{if } a = 1 \end{cases}$$

which we have already shown, in section 18.1, to be the solution to the special case of the *autonomous* difference equation.

Example 18.6 Solve $y_{t+1} = (t+1)y_t + 3^t$.

Solution

Here $a_t = (t+1)$ and $b_t = 3^t$. Therefore the solution, using equation (18.11), is

$$y_t = \prod_{i=0}^{t-1}(i+1)\left[y_0 + \sum_{k=0}^{t-1} \frac{3^k}{(0+1)(1+1)(2+1)\cdots(t-1+1)}\right]$$

which can be simplified to

$$y_t = t!\left[y_0 + \sum_{k=0}^{t-1} \frac{3^k}{k!}\right]$$

■

A Multiplier Model with Exogenous Growth

Suppose that aggregate consumption in period t is given by

$$C_t = A + BY_{t-1}$$

where $0 < B < 1$ is the marginal propensity to consume out of the previous year's income. Assume that the aggregate national income is equal to investment plus consumption

$$Y_t = C_t + I_t$$

and assume that investment is given by

$$I_t = (1+g)^t$$

where $g > 0$ is the exogenous growth rate in investment spending. After substituting the consumption and investment functions into the national income identity, we obtain the following first-order linear difference equation:

$$Y_t = BY_{t-1} + A + (1+g)^t$$

Comparing this to the general form in definition 18.3, $a_t = B$, a constant, and $b_t = A + (1+g)^t$. By equation (18.12) then, the solution is

$$Y_t = B^t Y_0 + \sum_{k=0}^{t-1} [A + (1+g)^k] B^{t-1-k}$$

Because $0 < B < 1$, the solution shows that the first term on the right-hand side goes to zero as t goes to infinity. As a result Y_t converges to the second term as t goes to infinity. The second term can therefore be thought of as the long-run growth path of national income.

EXERCISES

1. For given y_0, solve

$$y_{t+1} = \frac{\alpha}{t+1} y_t$$

2. For given y_0, solve

$$y_{t+1} = ay_t + t$$

3. For given y_0, solve

$$y_{t+1} = (t + 1)y_t + b$$

4. For given y_0, solve

$$y_{t+1} = ay_t + \beta^t$$

5. For given y_0, solve

$$y_{t+1} = \alpha^t y_t + b$$

6. A firm invests I_t in its capital stock in period t where

$$I_t = I_0(1 + g)^{-t}$$

and where I_0 and g are positive constants. If a portion, δ, of the capital stock depreciates each period, write out the difference equation for the capital stock and solve it, given that the initial value of the capital stock is K_0.

7. If you invest $100 per year (at the beginning of the year) in a bank account that pays the going annual interest rate, r_t, (interest compounded annually), write out the difference equation and solve it as far as possible as a function of the sequence for r_t.

CHAPTER REVIEW

Key Concepts

approach path
cobweb model
convergence
divergence
monotonic path
oscillating path
stationary value
steady state

Review Questions

1. Explain the difference between the general form and the autonomous form of the linear, first-order difference equation.

2. Explain the difference between the solution in equation (18.3) and the solution in equation (18.6).

3. Explain what is meant by the steady state of a linear, first-order difference equation.
4. Explain why not all linear, first-order difference equations have a steady state.
5. Explain the difference between monotonic convergence and oscillatory convergence.
6. State and explain the necessary and sufficient conditions for convergence in a linear, autonomous, first-order difference equation.

Review Exercises

1. For each of the following difference equations:
   - (a) Obtain the general solution (i.e., when y_0 is unspecified).
   - (b) Obtain the unique solution for the case when y_0 is known.
   - (c) Solve for the steady state if it exists and indicate whether or not y_t converges to the steady state.
     - (i) $y_{t+1} = 0.8y_t + 1$
     - (ii) $y_{t+1} = y_t + 10$
     - (iii) $y_{t+1} + 0.1y_t = 9.9$
2. Repeat exercise 1 above for the following difference equations:
   - (i) $y_{t+1} + y_t = 0$
   - (ii) $2y_{t+1} = y_t - 6$
   - (iii) $y_{t+1} = 0.5y_t + 50$
3. For each of the following difference equations:
   - (a) Obtain the unique solution that satisfies the given value for y_0.
   - (b) Calculate the values for y_1, y_2, y_3, y_4, and y_5 directly from the difference equation, and observe the speed of convergence to the steady state, if it exists.
   - (c) Calculate the value for y_5 from the solution obtained in (a) and check that it matches the value obtained in (b).
     - (i) $y_{t+1} + y_t = 2$, $y_0 = 2$
     - (ii) $y_{t+1} = 3y_t - 1$, $y_0 = 1$
     - (iii) $y_{t+1} = 0.5y_t + 50$, $y_0 = 50$
     - (iv) $3y_{t+1} + 2y_t = 1$, $y_0 = 2$
     - (v) $y_{t+1} = y_t + (-1)^t$, $y_0 = 2$

(vi) $y_{t+1} = -y_t + (-1)^t,\ y_0 = 2$

(vii) $y_{t+1} = (-1)^t y_t + 1,\ y_0 = 2$

4. Repeat exercise 3 above for each of the following difference equations:

(i) $y_{t+1} = -0.1y_t + 9.9,\ y_0 = 15$

(ii) $y_{t+1} = -2y_t + 12,\ y_0 = 10$

(iii) $y_{t+1} = 10y_t + 90,\ y_0 = -5$

(iv) $y_{t+1} = 0.5y_t + (-1)^t,\ y_0 = 2$

(v) $y_{t+1} = (-0.5)^t y_t + (-1)^t,\ y_0 = 2$

(vi) $y_{t+1} = (0.5)^t y_t + 1,\ y_0 = 2$

5. A perfectly competitive industry has the following supply function:

$$Q_t = F + Gp_t, \qquad F, G > 0;\ t = 0, 1, 2, \ldots$$

The demand for this product is a function of price and the lagged value of quantity

$$Q_t = A + Bp_t + \theta Q_{t-1}, \qquad A > 0,\ B < 0;\ 0 < \theta < 1;\ t = 1, 2, 3, \ldots$$

This says that current quantity demanded is equal to a fraction θ of last period's value of quantity plus a constant A, plus an amount that depends negatively on price, Bp_t. A demand function such as this might apply to a commodity characterized by habit formation (like cigarettes). Alternatively, it might apply to a raw material input in production, like energy, which is used in conjunction with a capital stock that is fixed in the short run, leading to slow adjustments of energy demand to price changes.

Assuming that the market clears each period, derive a first-order difference equation for quantity, Q. Solve the difference equation, find the steady state, and determine whether quantity converges monotonically, in oscillations, or not at all.

6. Suppose now that a perfectly competitive industry has supply function

$$Q_t = F + Gp_t + \alpha Q_{t-1}, \qquad F, G > 0,\ 0 < \alpha < 1;\ t = 1, 2, 3, \ldots$$

which says that the current quantity supplied is equal to a fraction α of lagged supply as well as a function of current price. (Can you think of an economic rationale for such a supply function?)

Assuming that the demand function is given by

$$Q_t = A + Bp_t, \qquad A > 0,\ B < 0;\ t = 0, 1, 2, \ldots$$

and that the market clears each period, derive a first-order difference equation for quantity, Q_t. Solve it, find the steady-state equilibrium, and determine whether quantity converges monotonically, in oscillations, or not at all.

7. Assume that we can model the unemployment rate as

$$U_t = \alpha + \beta U_{t-1}, \qquad \alpha, \beta > 0; t = 1, 2, 3, \ldots$$

where U_t is the unemployment rate in period t. Labor economists refer to the steady-state level of unemployment in this model as the *natural rate of unemployment*. Solve this difference equation (assume that U_0 is known), solve for the natural rate of unemployment, and determine the necessary and sufficient condition for U_t to converge to the natural rate of unemployment.

(a) Suppose that there are occasional shifts of (shocks to) the demand for labor causing it to sometimes rise and sometimes fall. These shifts translate into occasional decreases and increases in the unemployment rate. The modified model of unemployment that captures these shifts is

$$U_t = \alpha + \beta U_{t-1} + e_t, \qquad \alpha, \beta > 0; t = 1, 2, 3, \ldots$$

where e_t is a term that is typically assumed to be equal to zero on average, but which actually varies over time (representing random shifts or shocks), being sometimes positive and sometimes negative. Obtain the solution to this difference equation.

(b) Prove that the solution obtained in (a) can be expressed as

$$U_t = \beta^t U_0 + \frac{\alpha}{1-\beta}(1-\beta^t) + e_0\beta^{t-1} + e_1\beta^{t-2} + e_2\beta^{t-3} + \cdots + e_{t-2}\beta + e_{t-1}$$

Discuss the following statement: “If $0 < \beta < 1$, the effect of past shocks wears off over time and the unemployment rate tends to converge toward the natural rate; however, recent shocks prevent it from actually getting there."

(c) Suppose that $U_0 = 6$, $\alpha = 3$, $\beta = 0.5$, $e_0 = 3$, $e_1 = e_2 = e_3 = 0$, $e_4 = -1$, and $e_5 = 0$. Find the natural rate of unemployment, and calculate the values of U_1 to U_6.

(d) Suppose that $U_0 = 6$, $\alpha = 1.2$, $\beta = 0.8$, and e_0 to e_5 are as given above. Find the natural rate of unemployment, calculate the values of U_1 to U_6, and compare the results to those obtained in (c).

Chapter 19 Nonlinear, First-Order Difference Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- An Economic Growth Model
- A Malthusian Growth Model
- Practice Exercises

In the previous chapter we saw that linear, first-order difference equations can be solved explicitly. We will see in the next chapter that this is also true for linear, second-order difference equations. *Nonlinear* difference equations, on the other hand, cannot be solved explicitly in general. However, it is still possible to obtain qualitative information about the solution by analyzing the nonlinear difference equation with the aid of a phase diagram. This technique can be very useful in economics because we are often mainly concerned with the qualitative properties of dynamic models. In this chapter we do this analysis for first-order difference equations, and we focus on the problem of determining the stability properties of the solution. Our analysis will lead us to a brief consideration of the study of *chaos*.

19.1 The Phase Diagram and Qualitative Analysis

The general expression for the nonlinear, first-order difference equation is

$$y_{t+1} = g(y_t, t), \qquad t = 0, 1, 2, \dots$$

However, we will consider only *autonomous*, nonlinear difference equations; that is, nonlinear difference equations that do not depend on time explicitly.

Definition 19.1 The **nonlinear, first-order, autonomous difference equation** is expressed as

$$y_{t+1} = f(y_t), \qquad t = 0, 1, 2, \dots \tag{19.1}$$

If there exists a **steady-state equilibrium** (or equilibria if there are more than one), it is found as usual by setting $y_{t+1} = y_t = \bar{y}$, where $\bar{y}$ is a steady-state value of y. In general, this gives

$$\bar{y} = f(\bar{y})$$

The main concern in doing a qualitative analysis of a nonlinear difference equation is to determine whether or not y_t converges to a steady-state equilibrium. If it does, then no matter what the starting value, y_0, the path of y_t will eventually lead to the value $\bar{y}$. Then, even though we cannot solve for y_t explicitly as a function of t, we can say where its path always leads. If it does not converge, then we might try to determine whether y_t diverges endlessly, or whether it cycles back and forth between particular values, or whether it displays chaotic behavior.

To make things concrete, consider a specific example for a function f that gives the following nonlinear difference equation:

$$y_{t+1} = y_t^{\alpha}, \qquad \alpha > 0;\ t = 0, 1, 2, \ldots \tag{19.2}$$

The steady-state (or stationary) values of y are found by setting $y_{t+1} = y_t = \bar{y}$. Doing this and rearranging yields

$$\bar{y}(\bar{y}^{\alpha-1} - 1) = 0$$

Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the steady-state values. If y_t ever becomes equal to 0 or 1, it will remain at that value forever. How can we determine whether y_t converges to one of these values?

It is helpful to construct a **phase diagram** to see whether y_t tends to move toward or away from the steady-state values. A phase diagram for a difference equation is a graph showing y_{t+1} against y_t. As a result it is just a graph of $f(y_t)$. The steady-state points will be located at intersections of $f(y_t)$ with the 45° line, for it is along this line that $y_{t+1} = y_t$.

We have drawn the phase diagram for equation (19.2) in figure 19.1 for the case $\alpha = 1/2$. We confine our diagrams to the positive quadrant because economic variables are constrained typically to be nonnegative.

To determine the dynamic behavior of y_t, consider an arbitrary initial value such as $y_0 = 0.5$. Given y_0, the value for y_1 is found by following a vertical line up from y_0 to the function, and then tracing across horizontally to the vertical axis. This procedure gives us $y_1 = f(y_0)$. To find y_2, we first make y_1 the current value of y by transposing it onto the horizontal axis. We do this by extending a horizontal line from y_1 to the 45° line and then down to the horizontal axis. Now use the diagram to find $y_2 = f(y_1)$. This gives the y_2 shown on the vertical axis. To find y_3, repeat the same steps: first make y_2 the current value of y by transposing it onto the horizontal axis. Then use the diagram to find $y_3 = f(y_2)$. As we do this,

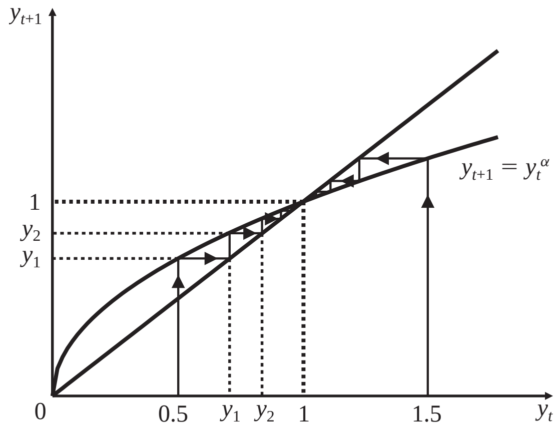


Figure 19.1 Phase diagram for equation (19.2) when $\alpha = 1/2$

notice that each successive value of y is getting closer to the stationary point $\bar{y} = 1$. We conclude that y_t appears to be diverging from the point $\bar{y} = 0$ but converging to $\bar{y} = 1$ from an arbitrary starting point $0 < y_0 < 1$.

We next investigate the motion of the system to the right of the steady-state point $\bar{y} = 1$. Consider a value for y_0 such as 1.5. Finding the value of y_1, then y_2, and so on, in the same way as we did above, we see that y_t again appears to be converging to the point $\bar{y} = 1$ from a starting point $y_0 > 1$.

We conclude that from any starting point $y_0 > 0$, the path of y_t appears to converge to $\bar{y} = 1$ making $\bar{y} = 1$ a stable equilibrium. On the other hand, the point $\bar{y} = 0$ is an unstable equilibrium because for $y > 0$, y_t diverges from 0. We will confirm these conclusions with a more rigorous test presently.

Example 19.1 Construct the phase diagram and conduct a qualitative analysis of the difference equation

$$y_{t+1} = y_t^2$$

Solution

We have drawn the phase diagram in figure 19.2. Starting at $y_0 = 0.5$, for example, we find the value y_1 as before, and then transpose this value onto the horizontal axis by using the 45° line. The value for y_2 is then found by tracing a line up to $f(y)$ to obtain $y_2 = f(y_1)$. Already we see that the system is moving away from $\bar{y} = 1$ but toward $\bar{y} = 0$. Indeed, continuing with this procedure, we see that y_t converges to $\bar{y} = 0$.

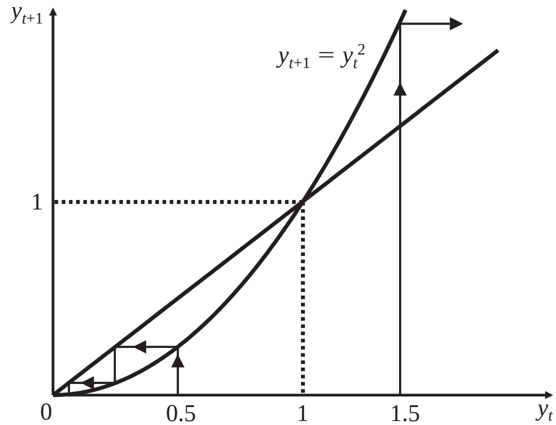


Figure 19.2 Phase diagram for example 19.1

Next, starting to the right of the point $\bar{y} = 1$, we consider a value such as $y_0 = 1.5$. Using the diagram to find the successive values of y_t, we see rather quickly that y_t increases monotonically. We conclude that the point $\bar{y} = 1$ is an **unstable equilibrium** and the point $\bar{y} = 0$ is a **locally stable equilibrium.**

Why is the point $\bar{y} = 1$ stable when $\alpha = 1/2$ but not when $\alpha = 2$? The crucial difference between these two cases, and the factor that determines whether or not a steady-state point is stable, is the *slope* of $f(\bar{y})$; this is the slope of the curve in the phase diagram where it intersects the 45° line. However, we postpone consideration of this important point until we have examined the case of a negatively sloping phase diagram. ■

Example 19.2 Draw the phase diagrams and conduct a qualitative analysis of the difference equation (19.2) when $\alpha = -1/2$ and $\alpha = -2$.

Solution

In figures 19.3 and 19.4 we have drawn phase diagrams for the difference equation (19.2) for the cases of $\alpha = -1/2$ and $\alpha = -2$ respectively. There is only one steady-state point now, namely $\bar{y} = 1$.

Starting at $y_0 = 0.5$ in figure 19.3, we find the value of y_1 as usual, and use the 45° line to transpose y_1 onto the horizontal axis. This time the system *overshoots* the stationary point in the sense that y_1 is larger than 1. However, we will see that the system converges nevertheless, albeit in an oscillatory fashion. Find $y_2 = f(y_1)$, and transpose this value onto the horizontal axis. Note that again the system overshoots the steady-state point, this time in the opposite direction, but is closer to $\bar{y} = 1$ than was y_0. It appears that the system is indeed converging. Find $y_3 = f(y_2)$, and transpose its value onto the horizontal axis; its value is closer

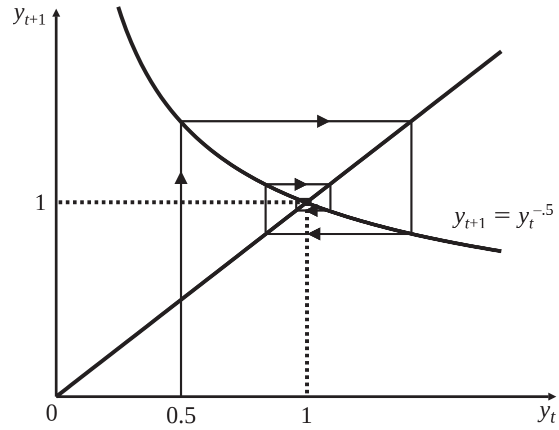


Figure 19.3 Phase diagram for example 19.2 when $\alpha = -1/2$

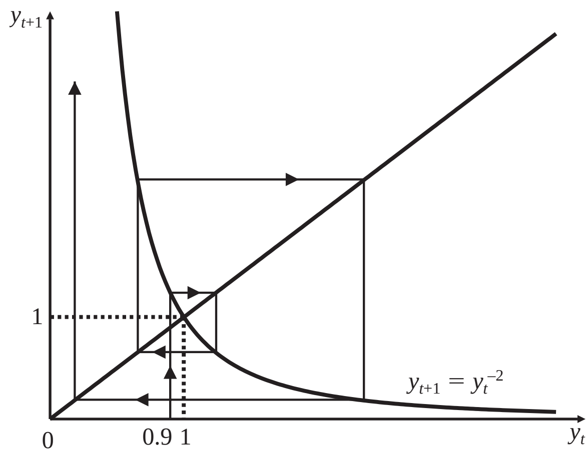


Figure 19.4 Phase diagram for example 19.2 when $\alpha = -2$

to $\bar{y} = 1$ than was y_1. Thus, although the system is oscillating around $\bar{y} = 1$, the oscillations are becoming smaller over time. If we continued this procedure, we would observe a cobweb approach path to $\bar{y} = 1$. In addition, if we chose $y_0 > 1$, such as $y_0 = 1.5$, we would find a qualitatively similar, oscillating but convergent, approach path. We conclude that $\bar{y} = 1$ appears to be a stable equilibrium, although we wait for a more rigorous test to confirm this.

In figure 19.4, the phase diagram is drawn for $\alpha = -2$. Starting at $y_0 = 0.9$, we find $y_1 = f(y_0)$, and we transpose this value onto the horizontal axis. Then we find $y_2 = f(y_1)$, and we transpose this value onto the horizontal axis. As above, the system is oscillating. But unlike the stable case, we now find that y_2 is farther away from $\bar{y} = 1$ than was y_0. In other words, the system appears to be diverging. This is confirmed by continuing this procedure. We find that y_3 is further from y_0 than was y_1 and so on. The system is oscillating, and the oscillations are becoming larger over time. We conclude that $\bar{y} = 1$ is an unstable equilibrium. ■

So far we have discovered that for the difference equation (19.2), the point $y = 1$ is a steady-state point no matter what the value of α. We have also found that it appears to be a stable steady-state point (i.e., y_t converges to it from any $y_0 > 0$) when $\alpha = -1/2$ or $1/2$ but is unstable when $\alpha = -2$ or 2. It would be useful if we could draw conclusions about the stability of the point $\bar{y} = 1$ for any value of α without having to resort to a phase diagram each time. Fortunately there is a way to do this, not just for this example but for any autonomous, nonlinear difference equation. The key, as hinted at above, is the slope of the graph in the phase diagram as it intersects the 45° line. A steady-state point is stable if and only if the absolute value of this slope is less than 1 at $\bar{y}$. Because the slope of the function as it cuts the 45° line is just the derivative of f evaluated at $\bar{y}$, we can state this result as follows:

Theorem 19.1 A steady-state equilibrium point of any first-order, autonomous, nonlinear difference equation is locally stable if the absolute value of the derivative, $f'(\bar{y})$, is less than 1 and is unstable if the absolute value of the derivative is greater than 1 at that point.

The implications of this theorem are powerful. It says that for any autonomous, first-order difference equation, we can determine whether a steady-state point is locally stable by taking the derivative of f and evaluating that derivative at the point $\bar{y}$. If its absolute value is less than 1, we know the point is locally stable. If not, we know it is unstable. And we can do all of this without any knowledge of the solution to the difference equation!

Although powerful, this result does have one limitation: it can be used only to determine local stability. If $\bar{y}$ is locally stable, the system converges to $\bar{y}$ from any point in the neighborhood of $\bar{y}$. However, it does not necessarily converge to $\bar{y}$ from all points (this would be called **global stability**), such as ones far away from the steady state. Although there is no general test for global stability in nonlinear dynamics, the phase diagram analysis, combined with the test for local stability, can usually tell us what we need to know about global stability. This is demonstrated in the example below.

Example 19.3 Use theorem 19.1 to determine the local stability properties of

$$y_{t+1} = y_t^{\alpha}$$

for different values of α.

Solution

The derivative of f is

$$f'(y_t) = \alpha y_t^{\alpha-1}$$

Evaluating the derivative at the stationary point $\bar{y} = 1$ gives

$$f'(1) = \alpha$$

Applying theorem 19.1, we conclude that the point $\bar{y} = 1$ is locally stable only when $-1 < \alpha < 1$. For all other values, $\bar{y} = 1$ is unstable. This result explains and confirms our phase diagram analyses.

For $\alpha > 0$, we found another steady-state point to be $\bar{y} = 0$. Evaluating the derivative at this point gives

$$f'(0) = 0 \quad \text{if} \quad \alpha > 1 \tag{19.3}$$

$$f'(0) = \text{undefined if} \quad 0 < \alpha < 1. \tag{19.4}$$

We conclude that if $\alpha > 1$, the point $\bar{y} = 0$ is locally stable. However, the phase diagram analysis in figure 19.2 proved that it is not globally stable. That is, we found that y_t converges to 0 from any $y_t < 1$ but does not converge to 0 for any $y_t \geq 1$.

When $0 < \alpha < 1$, equation (19.4) shows that the point $\bar{y}$ is unstable because the derivative becomes infinitely large (α divided by 0). ■

We found that the difference equation (19.2) led to oscillations in y_t when $\alpha = -1/2$ and $\alpha = -2$ but that y_t moved monotonically when $\alpha = 1/2$ or $\alpha = 2$. It turns out that the sign of the slope of the graph in the phase diagram determines whether y_t oscillates or moves in one direction. We state this important point as follows:

Theorem 19.2 A first-order difference equation will lead to oscillations in y_t if the derivative f' is negative for all $y_t > 0$, but y_t will move monotonically if the derivative is positive for all $y_t > 0$.

Before proceeding, we make two comments about theorems 19.1 and 19.2. First, both apply to *linear* as well as *nonlinear* first-order difference equations. Second, theorem 19.2 does not apply if the derivative f' changes sign as a function of y_t as it does in section 19.2. In that case, y_t can display very complex behavior involving both monotonic and oscillatory segments.

EXERCISES

1. For the difference equation

$$y_{t+1} = \frac{3}{16} + y_t^2$$

find the steady-state points, determine whether they are locally stable using theorem 19.1, and sketch a phase diagram to investigate the global stability.

2. For the difference equation

$$y_{t+1} = 2 - 3y_t^2$$

find the steady-state points, determine whether the one in the positive quadrant is locally stable using theorem 19.1, and sketch a phase diagram to investigate the global stability.

3. For the difference equation

$$y_{t+1} = 4 + \frac{9}{4y_t}$$

find the steady-state points, determine whether the one in the positive quadrant is locally stable using theorem 19.1, and sketch a phase diagram to investigate the global stability.

4. For the difference equation

$$y_{t+1} = 1 + \frac{3}{4y_t}$$

find the steady-state points, determine whether the one in the positive quadrant is locally stable using theorem 19.1, and sketch a phase diagram to investigate the global stability.

19.2 Cycles and Chaos

Up to this point we have considered only nonlinear difference equations for which the slope, f', does not change sign. The implication is that the graph of y_{t+1} against y_t in phase diagrams is either monotonically increasing or decreasing, but never is hill-shaped or valley-shaped. In the remainder of the chapter, we consider nonlinear difference equations that produce hill-shaped curves in the phase diagram. This kind of difference equation can lead to very interesting dynamic behavior, such as cycles that repeat themselves every two or more periods, and even *chaos*, in which there is no apparent regularity in the behavior of y_t. Providing a formal analysis of these nonlinear difference equations is beyond the scope of this book; however, we hope to give students a rudimentary exposure to the subject.

Consider the first-order, nonlinear difference equation

$$y_{t+1} = ry_t(1 - y_t), \qquad t = 0, 1, 2, \ldots \tag{19.5}$$

The steady-state points, $\bar{y}$, are obtained by solving

$$\bar{y} - r\bar{y}(1 - \bar{y}) = 0$$

Simplifying gives

$$\bar{y}\left(\frac{1-r}{r} + \bar{y}\right) = 0$$

The two steady-state points are

$$\bar{y} = 0 \tag{19.6}$$

and

$$\bar{y} = \frac{r-1}{r} \tag{19.7}$$

A strictly positive steady-state equilibrium exists only if $r > 1$. We will therefore assume that $r > 1$. If $r \le 1$, then the steady states are 0 and negative respectively, which are of no interest in economics.

To determine the stability properties of these two equilibria, we apply theorem 19.1, which requires that we evaluate the derivative of equation (19.5) at the points $\bar{y}$. We get

$$\frac{dy_{t+1}}{dy_t} = r - 2ry_t = \begin{cases} r & \text{at } \bar{y} = 0 \\ 2 - r & \text{at } \bar{y} = (r-1)/r \end{cases}$$

This result tells us that the point $\bar{y} = 0$ is unstable given our assumption that $r > 1$; the point $\bar{y} = (r-1)/r$ is locally stable only if $|2-r| < 1$. Expressed differently, the positive steady-state point is locally stable if and only if

$$1 < r < 3$$

Let's draw the phase diagram for this difference equation. We know that the graph cuts through the 45° line at the points 0 and $(r-1)/r$. From our calculation of the slope, we know that the graph peaks at $y = 1/2$ (where the slope is 0). This information, plus a quick calculation that shows that the second derivative of the function is negative ($= -2r$), assures us that the graph is hill-shaped. Whether the peak of the graph occurs to the left or right of the point $\bar{y} = (r-1)/r$ depends on whether r is larger or smaller than 2. If smaller than 2 (but larger than 1), the graph intersects the 45° line to the left of the peak. As a result, the slope of the graph is positive at the stable steady-state point. The more interesting cases arise when $r > 2$, which makes the slope of the graph negative at the steady-state point. Figure 19.5 shows the phase diagram for the case in which $2 < r < 3$. This satisfies the condition for local stability and also makes the slope of the graph negative at the stable steady-state point.

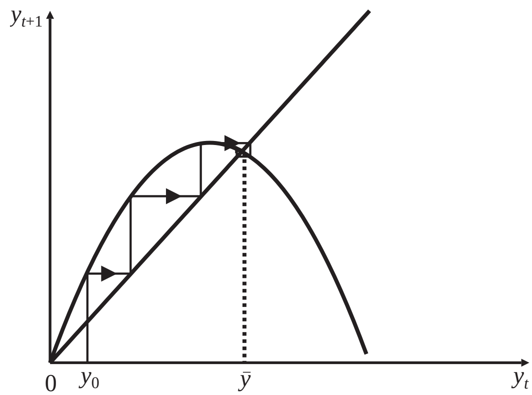


Figure 19.5 Path of y_t that converges monotonically at first, but eventually oscillates as it converges to the steady state

Because the slope is negative but less than 1 in absolute value, y_t converges to $\bar{y}$ from either direction (within a neighborhood) but the approach paths will oscillate locally. The phase diagram helps us to see what goes on globally. For example, starting at y_0 in figure 19.5, which is definitely outside the neighborhood of $\bar{y}$, the slope is positive. This condition causes y_t to increase monotonically for a few time periods as shown. However, as y_t approaches the neighborhood of $\bar{y}$, the slope becomes negative, causing y_t to begin oscillating as it converges to the steady state. This is interesting dynamic behavior, but even more fascinating dynamics occur when $r \geq 3$.

What happens when $r \geq 3$? First, the point $\bar{y} = (r-1)/r$ is no longer a stable steady state. However, an important characteristic of a hill-shaped phase diagram, which does not apply to a monotonic phase diagram, is that when the steady-state point is unstable, the path of y_t does not diverge endlessly to infinity or zero. Instead, although it never converges to $\bar{y}$, y_t oscillates within a bounded range and could even converge to a regular periodic behavior.

Figure 19.6 shows the phase diagram for the case of $r = 3.5$. We know that $f'(\bar{y})$ is negative in this case so that paths oscillate in the region of the stationary point. We also know that $f'(\bar{y})$, being smaller than -1, is outside the stable range. As a result, paths do not converge to the steady-state point, $\bar{y}$. However, they also do not diverge to zero or infinity. The reason is that $f(y)$ is nonmonotonic, causing the phase curve to be hill-shaped. Consider what happens in figure 19.6 as a path diverges from $\bar{y}$. Beginning at y_0, the path diverges for two time periods. That is, y_1 is further away from $\bar{y}$ than was y_0; y_2 is even further away. However, the path next runs into the positively sloped region of the phase curve, causing it to bounce back towards $\bar{y}$. This returns the path immediately to the negatively sloped region of the phase curve, which leads once again to diverging oscillations. However, the path will eventually run into the positively sloped region again, which pushes it back toward $\bar{y}$ again and thereby keeps the path within finite bounds. Although it never reaches $\bar{y}$, it also never gets too far away.

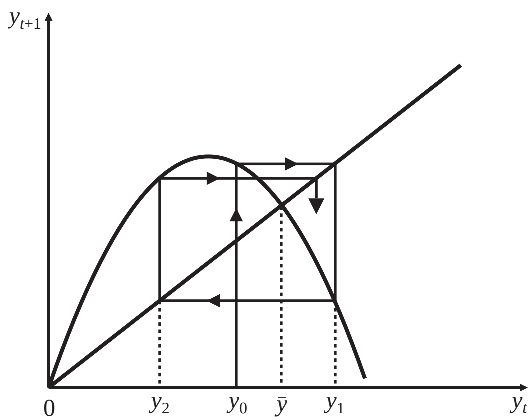


Figure 19.6 Phase diagram when $r = 3.5$

So far we have found that for $1 < r < 3$, y_t converges to the steady-state point and for $r \geq 3$, y_t will not converge to the steady-state point but neither will it diverge endlessly. What is interesting is that for values of r slightly larger than 3, y_t converges to a **stable limit cycle,** two periods long. That is, regardless of the starting value, y_t converges to a path that cycles back and forth in successive periods between two values. In the language of nonlinear dynamics, it is said that the system *bifurcates* at $r = 3$, which means that it changes from having one steady-state value, $\bar{y}$, to having an equilibrium in which there are two values between which the path of y oscillates. If r is as large as about 3.5, the two-period limit cycle itself becomes unstable and the system bifurcates again, producing a different stable limit cycle, this one being four periods long.

Figure 19.7 shows the phase diagram for $r = 3.2$. At this value, we get a limit cycle of two periods. Starting at any value, such as at y_0 shown in the diagram, y_t eventually converges to the path shown, which alternates back and forth between $\bar{y}_H$ and $\bar{y}_L$ (H and L stand for high and low respectively).

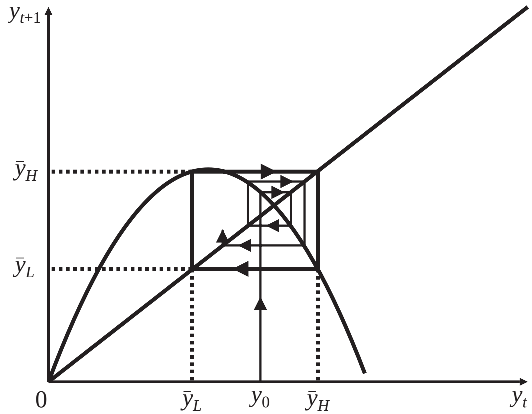


Figure 19.7 Stable limit cycle of two periods when $r = 3.2$

Higher values of r produce repeated bifurcations and hence, repeated doublings of the period of the limit cycles. Eventually, however, at some values of r, the time paths do not have cycles of any length, although they continue to be bounded. This is said to be **chaos.** Roughly speaking, this means that the path of y_t fluctuates in an apparently random fashion over time, not settling down into any regular pattern whatsoever.

One of the important findings in the study of chaotic systems is the *extreme* sensitivity of the path of y_t to the initial value, y_0. This finding is sometimes metaphorically called the **butterfly effect** after the notion that a butterfly disturbing the air in Beijing today could affect weather systems in New York next month. Seemingly insignificant changes in the initial value, y_0, produce such large differences in the path of y_t, that the new path seems to bear no resemblance whatsoever to the original one. This is very different from stable dynamic systems in which all paths, regardless of their initial values, converge to the same steady-state point. Figure 19.8 shows two paths of y_t, both generated from equation (19.5) when $r = 3.7$ but one starts at $y_0 = 0.20$ and the other starts at $y_0 = 0.21$. In the diagram the two

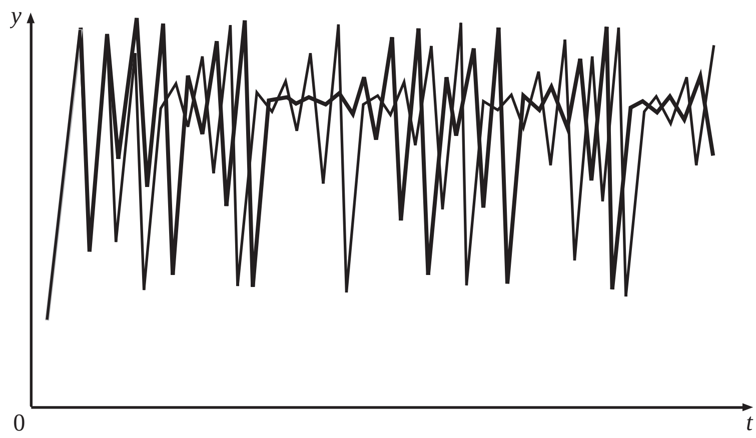


Figure 19.8 Small differences in initial conditions that can lead to very different chaotic paths

paths look identical for the first four or five time periods. However, any similarity between the two paths disappears very rapidly.

The paths of y_t in figure 19.8 look like the plots of random variables, but they are of course completely determined by the difference equation. This finding has led some economists to argue that some of the apparently random fluctuations of the business cycle could indeed be due, in some part at least, to the inherently nonlinear dynamic relationships in the economy. Most of these models are too complex to consider in this book; however, we provide on the Web page, http://mitpress.mit.edu/math_econ3, a relatively simple example of an economic model that is capable of generating stable equilibria, cycles, or chaos depending on parameter values. One final point of information before proceeding is that *chaos* is not produced only by nonlinear *difference* equations; nonlinear differential equations can also produce chaotic behavior, provided that they are of the third-order or higher.

EXERCISES

1. For the difference equation

$$y_{t+1} = \frac{3}{2} y_t (1 - y_t)$$

find the steady-state points; determine which of them are stable; and in a phase diagram, sketch the path of y_t starting from $y_0 = 2/3$.

2. For the difference equation

$$y_{t+1} = \frac{5}{2} y_t(1 - y_t)$$

find the steady-state points; determine which of them are stable; and in a phase diagram, sketch the path of y_t starting from $y_0 = 1/3$.

3. Suppose that a firm's advertising expenditures affect profits as follows:

$$\Pi_t = aE_t(1 - E_t)$$

where Π is profit and E is advertising expenditure. Assume further that the firm devotes a constant share, b, of its profits to the next year's advertising campaign. In particular, assume that

$$E_{t+1} = b\Pi_t$$

Derive the difference equation for advertising expenditure. Find the steady-state values of E and derive the conditions the parameters must satisfy for E_t to converge to a steady state.

4. For the model in exercise 3, what happens if the parameter values are such that $ab = 3$? Given $E_0 = 0.690$, calculate E_t for $t = 1, 2, 3, 4$ to four decimal places.

5. Suppose that the world's Minke whale population is governed by the following difference equation:

$$y_{t+1} = 2y_t(1 - y_t) - H, \qquad t = 0, 1, 2, \ldots$$

where H is the world harvest of whales. Now suppose that the International Convention on Whaling has been signed by all nations and limits the annual harvest of Minke whales to $H = 0.08$. Find the steady-state value of the population and draw the phase diagram. Calculate y_t for $t = 1, 2, 3, 4$ if $y_0 = 0.75$ to 4 decimal places.

CHAPTER REVIEW

Key Concepts

- butterfly effect
- chaos
- global stability
- locally stable equilibrium
- phase diagram
- stable limit cycle
- steady-state equilibrium
- unstable equilibrium

Review Questions

1. Explain the construction of a phase diagram for a nonlinear, first-order autonomous difference equation and explain the significance of the 45° line.
2. Explain the value and the limitations of relying on a phase diagram to analyze the dynamic properties of a nonlinear difference equation.
3. Explain the difference between a locally stable and a globally stable steady state.
4. Describe the necessary and sufficient condition for a steady state to be locally stable.
5. Explain the conditions under which a convergent approach path is likely to be (i) monotonic and (ii) oscillatory.
6. Explain the meaning of a stable limit cycle of two periods.

Review Exercises

1. For the difference equation

$$y_{t+1} = 2y_t - 10y_t^2$$

find the steady-state points and analyze the dynamic behavior of y_t in the phase diagram.

2. For the difference equation

$$y_{t+1} = \frac{1}{2}y_t^2 - y_t$$

find the steady-state points and analyze the dynamic behavior of y_t in the phase diagram.

3. For the difference equation

$$y_{t+1} = ay_t - by_t^2$$

find the steady-state points. How is the stability affected by b?

4. Let the aggregate consumption function be

$$C_{t+1} = a + bY_t^{\alpha}, \qquad 0 < \alpha < 1; \quad 0 < b < 1$$

where C is aggregate consumption and Y is aggregate income. Assuming that

$$Y_t = C_t + I$$

where I is a constant level of aggregate investment, derive the difference equation for aggregate income. Draw a phase diagram and determine the behavior of Y_t. In particular, does it converge to a steady-state value or not?

5. Consider the following nonlinear cobweb model. Let the market-demand curve for corn be

$$Q_t^D = 2 - P_t$$

Let the market-supply curve for corn be

$$Q_t^S = P_{t-1}^{1/2}$$

In equilibrium, supply equals demand. Derive the first-order difference equation for price implied by this model. Find the steady-state price level, determine whether it is stable, and conduct a qualitative analysis of the dynamic behavior of price in this model.

6. Suppose that the Atlantic codfish population is governed by the following difference equation:

$$y_{t+1} = 6y_t - 6y_t^2 - H, \qquad t = 0, 1, 2, \ldots$$

where H is the constant annual harvest of codfish. If $H = 2/3$, find the steady-state value of the population and sketch the phase diagram. Determine whether the steady state is stable.

7. Suppose that the Peruvian shrimp population is governed by the following difference equation:

$$y_{t+1} = 3y_t - 3y_t^2 - H, \qquad t = 0, 1, 2, \ldots$$

where H is the constant annual harvest of shrimp. If $H = 7/48$, find the steady-state values of the population and determine which is stable.

8. For review exercise 7, if the annual harvest is increased to $H = 1/3$, how are the steady state and the dynamics of the population affected?

Chapter 20 Linear, Second-Order Difference Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Multiplier-Accelerator Model
- Practice Exercises

In this chapter we turn to linear difference equations of the second order. We focus our attention on the *autonomous* case in section 20.1 and consider a special nonautonomous case in section 20.2. In addition we introduce a new solution technique in this chapter. The technique involves breaking up the relatively difficult problem of finding the general solution to the difference equation into two parts, each of which is easier to solve than the whole. Not only does this simplify matters in this chapter, but it proves to be indispensable in later chapters in solving differential equations, and systems of difference and differential equations.

20.1 The Linear, Autonomous, Second-Order Difference Equation

We begin with the case of a second-order, linear difference equation that is autonomous; that is, one that does not depend explicitly on time.

Definition 20.1 The general form of the **linear, autonomous, second-order difference equation** is

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b, \qquad t = 0, 1, 2, \ldots \tag{20.1}$$

This equation is linear because y_t, y_{t+1}, and y_{t+2} are not raised to any power other than 1. It is of the second-order because the largest difference to appear in the equation is a two-period difference. It is autonomous because it has constant coefficients, a_1 and a_2, and a constant term, b. If the coefficients or term vary with t, then the equation in nonautonomous. In section 20.2 we consider the case of a varying term.

We wish to find the *general* solution to equation (20.1), by which we mean an expression that includes all possible solutions as special cases. It is easier to break this problem up into two parts. In the first part, instead of finding all the solutions to equation (20.1), find just one. One that will usually suffice is the steady-state solution, $\bar{y}$. In the second part, find the general solution of the *homogeneous form* of equation (20.1), by which we mean the form that occurs when the term b is set equal to zero. The reason for the two-part attack is that adding these two solutions together gives us the general solution to equation (20.1)! We restate this important result more formally.

Theorem 20.1 If y_p is any **particular solution** to equation (20.1), such as the steady-state solution, and y_h is the general solution to the **homogeneous form** of equation (20.1) then the general solution of the *complete* difference equation in equation (20.1) is given by

$$y_t = y_h + y_p$$

where we have dropped the t subscripts on y_h and y_p for convenience.

We do not prove the theorem at this point. Stating it, however, gives a good indication of the organization of our approach in this section. We begin with a thorough analysis of the problem of finding the general solution to the homogeneous form of equation (20.1). Following this, we solve the relatively easy problem of finding the steady-state solution, for this will serve as the particular solution we need to apply theorem 20.1. We then go on to study the convergence properties of the solution.

The General Solution of the Homogeneous Equation

Our first problem is to solve the homogeneous form of equation (20.1). This is given by:

Definition 20.2 The **homogeneous form** of the linear, autonomous, second-order difference equation is

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0, \qquad t = 0, 1, 2, \ldots \tag{20.2}$$

Note that the only difference between this and the complete equation is the absence of the b term in the homogeneous form. We proceed by first stating the general solution; we then go on to provide a justification for it.

Theorem 20.2 The general solution to the homogeneous form of the linear, autonomous, second-order difference equation is as follows:

- If $a_1^2 - 4a_2 > 0$ (**real and distinct roots**),

$$y_h = C_1 r_1^t + C_2 r_2^t \tag{20.3}$$

- If $a_1^2 - 4a_2 = 0$ (**real and equal roots**),

$$y_h = C_1 r^t + C_2 t r^t \tag{20.4}$$

- If $a_1^2 - 4a_2 < 0$ (**complex roots**),

$$y_h = a_2^{t/2}(C_1 \cos \theta t + C_2 \sin \theta t) \tag{20.5}$$

where C_1 and C_2 are arbitrary constants (the values for which would be determined by initial conditions if given) and r_1 and r_2 are given by

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{20.6}$$

and, in the case of complex roots, where θ can be calculated using either of the following relationships:

$$\cos \theta = \frac{-a_1}{2a_2^{1/2}}, \quad \sin \theta = \frac{\sqrt{4a_2 - a_1^2}}{2a_2^{1/2}}$$

Theorem 20.2 uses terminology that we have not yet encountered in the study of difference equations: roots. As we will see, every linear, second-order difference equation has associated with it two roots, sometimes called eigenvalues or characteristic roots. Like the roots of any quadratic equation, they can be real valued (distinct or equal) or complex valued. As theorem 20.2 indicates, the solution takes slightly different forms depending on which of these cases arises. We will analyze the three cases individually after providing a justification (rather than a formal proof) for theorem 20.2.

If we knew nothing about the form of the solution to the homogeneous difference equation (20.2), how could we go about finding it? We know that the solution to the homogeneous version of the *first-order* difference equation is of the form $y_t = Ar^t$ (where $r = a$) and A is an arbitrary constant. Why not try the same form for the *second-order* difference equation? Thus suppose that we try

$$y_t = Ar^t$$

as a solution to equation (20.2). If this is a solution, then it has to satisfy equation (20.2). To see if it does, first note that the proposed solution implies that $y_{t+1} = Ar^{t+1}$ and $y_{t+2} = Ar^{t+2}$. Substituting these and our proposed expression for y_t into equation (20.2) gives

$$Ar^{t+2} + a_1Ar^{t+1} + a_2Ar^t = 0$$

Simplifying gives

$$(r^2 + a_1r + a_2)Ar^t = 0$$

Our proposed solution will therefore work provided that we choose values for r that satisfy the quadratic equation $(r^2 + a_1r + a_2) = 0$ (since we rule out the trivial solutions $r = 0$ and $A = 0$). This quadratic equation is known as the characteristic equation of difference equation (20.2).

Definition 20.3

The **characteristic equation** of the linear, second-order difference equation with constant coefficients is

$$r^2 + a_1r + a_2 = 0$$

The values of r that solve the characteristic equation are known as the **roots** (or **eigenvalues** or **characteristic roots**) of the characteristic equation. There are two roots that solve this equation; we will call them r_1 and r_2. Their values are given in equation (20.6).

Suppose the two roots that solve the characteristic equation are real valued and different. Then we have actually found *two* solutions that satisfy equation (20.2). They are

$$y_t^{(1)} = A_1r_1^t \quad \text{and} \quad y_t^{(2)} = A_2r_2^t \tag{20.7}$$

Let's confirm that $y_t^{(1)}$ is a solution to equation (20.2) (and we leave it to the reader to do the same for $y_t^{(2)}$). Given $y_t^{(1)}$ in equation (20.7), then

$$y_{t+1}^{(1)} = A_1 r_1^{t+1} \quad \text{and} \quad y_{t+2}^{(1)} = A_1 r_1^{t+2}$$

Substituting these values into equation (20.2) gives

$$\begin{aligned} y_{t+2}^{(1)} + a_1 y_{t+1}^{(1)} + a_2 y_t^{(1)} &= A_1 r_1^{t+2} + a_1 A_1 r_1^{t+1} + a_2 A_1 r_1^t \\ &= A_1 r_1^t \left(r_1^2 + a_1 r_1 + a_2\right) \\ &= 0 \end{aligned}$$

The final equality follows because we know that r_1 solves the characteristic equation. Therefore $y_t^{(1)}$ satisfies equation (20.2) and is a solution.

Although having two solutions may appear to present a problem, since we are, after all, looking for one general solution, in fact the general solution is a linear combination of two linearly independent solutions. As a result we actually require two solutions. Intuitively the reason for this is that two linearly independent solutions are required in order to recover the two constants that are lost in taking the first and then the second difference of the underlying equation for y_t. The real problem arises therefore in the case of real-valued but equal roots. That is, when $a_1^2 - 4a_2 = 0$, so that $r_1 = r_2$, we appear to have just one solution. However, it is possible even in this case to find a second distinct solution. Rather than derive it, we state it and then verify that it is correct.

If $r_1 = r_2 = r$, the two distinct solutions are

$$y_t^{(1)} = A_1 r^t \quad \text{and} \quad y_t^{(2)} = A_2 t r^t \tag{20.8}$$

These are linearly independent of one another (and are therefore distinct) because one cannot be made equal to the other by multiplying it by any constant. It is possible to verify that both of these are solutions to equation (20.2) by substitution as was done above. We do this for the second solution.

Given $y_t^{(2)} = A_2 t r^t$, then

$$y_{t+1}^{(2)} = A_2(t+1) r^{t+1} \quad \text{and} \quad y_{t+2}^{(2)} = A_2(t+2) r^{t+2}$$

Substituting these values into equation (20.2) gives

$$\begin{aligned} y_{t+2}^{(2)} + a_1 y_{t+1}^{(2)} + a_2 y_t^{(2)} &= A_2(t+2)r^{t+2} + a_1 A_2(t+1)r^{t+1} + a_2 A_2 t r^t \\ &= A_2 r^t[(t+2)r^2 + a_1(t+1)r + a_2 t] \\ &= A_2 r^t[t(r^2 + a_1 r + a_2) + r(2r + a_1)] \\ &= A_2 r^t[0 + r(-a_1 + a_1)] \\ &= 0 \end{aligned}$$

The second-to-last equality follows from the previous one because r solves the characteristic equation and because the case of equal roots arises only when $a_1^2 - 4a_2 = 0$, which means that $r = -a_1/2$.

We now have derived two solutions to equation (20.2) for the case of real-valued, different roots and for the case of real-valued but equal roots. As we mentioned above, the *general* solution is obtained by taking a linear combination of the two distinct solutions. We state this formally:

Theorem 20.3 Let $y^{(1)}$ and $y^{(2)}$ be the two solutions to equation (20.2) given by equation (20.7) in the case of different roots or by equation (20.8) in the case of equal roots. The general solution to the homogeneous difference equation (20.2) is

$$y_t = C_1 y_t^{(1)} + C_2 y_t^{(2)}$$

where C_1 and C_2 are arbitrary constants.

The implication of the result above is that the *general* solution is the one given in equation (20.3) or equation (20.4), depending on whether the roots are equal or different. Rather than provide a formal proof, we demonstrate the validity of theorem 20.3 by the following example:

Example 20.1 Solve the following difference equation and prove that the solution is the general solution:

$$y_{t+2} - 6y_{t+1} + 8y_t = 0, \qquad t = 0, 1, 2, \ldots \tag{20.9}$$

Solution

The characteristic equation is $r^2 - 6r + 8 = 0$, for which the roots are 2 and 4. Thus the two distinct solutions are

$$y_t^{(1)} = 2^t \quad \text{and} \quad y_t^{(2)} = 4^t$$

According to theorem 20.3, the general solution is

$$y_t = C_1 2^t + C_2 4^t \tag{20.10}$$

where C_1 and C_2 are arbitrary constants, the values for which would be determined by initial conditions.

To prove that we have found the general solution, we need to prove (i) that equation (20.10) is a solution to equation (20.9) and (ii) that *any* solution to equation (20.9) can be expressed as equation (20.10). Only then have we verified that we have found an expression that includes *all* solutions as special cases.

The first part is relatively easy to prove by direct substitution. Using equation (20.10) to calculate y_{t+2} and y_{t+1}, substitute into equation (20.9) to get

$$\begin{aligned} y_{t+2} - 6y_{t+1} + 8y_t &= C_1[2^{t+2} - 6(2^{t+1}) + 8(2^t)] + C_2[4^{t+2} - 6(4^{t+1}) + 8(4^t)] \\ &= C_1 2^t[2^2 - 6(2) + 8] + C_2 4^t[4^2 - 6(4) + 8] \\ &= 0 \end{aligned}$$

Therefore equation (20.10) is a solution to equation (20.9).

The second part is a bit more difficult to prove. Let y' represent *any* arbitrary solution. Can we find unique values of C_1 and C_2 that make this solution a special case of our general solution? In other words, do there exist C_1 and C_2 that make the following true?

$$y_t' = C_1 y_t^{(1)} + C_2 y_t^{(2)}$$

The answer is yes. Given initial values for y', say y_0' and y_1', we can determine the values of the constants by making the solution satisfy these initial conditions as follows:

$$y_0' = C_1 y_0^{(1)} + C_2 y_0^{(2)}$$

$$y_1' = C_1 y_1^{(1)} + C_2 y_1^{(2)}$$

Substituting the values at $t = 0$ and $t = 1$ from equation (20.10) gives

$$\begin{aligned} y_0' &= C_1 + C_2 \\ y_1' &= 2C_1 + 4C_2 \end{aligned}$$

This is a system of two equations in two unknowns. Provided these are linearly independent equations (which is why the two solutions must be linearly independent), they can be solved for unique values of C_1 and C_2. Doing this gives

$$C_1 = 2y'_0 - \frac{y'_1}{2} \quad \text{and} \quad C_2 = -y'_0 + \frac{y'_1}{2}$$

This tells us we can find unique values for C_1 and C_2 that make the first two values (at $t = 0$ and $t = 1$) of any arbitrary solution just a special case of our general solution. But if we have any two consecutive values of y'_t, such as y'_0 and y'_1, then the next value, y'_2, is uniquely determined by the difference equation; and then y'_3 is uniquely determined, and so on. Thus, if we have found unique values of C_1 and C_2 that make the first two values of y' a special case of our general solution, then all subsequent values (from $t = 2$ onwards) must also be a special case of our general solution. This verifies that (20.10) is the general solution. ■

Example 20.2 Solve the following homogeneous difference equation:

$$y_{t+2} + 4y_{t+1} + 3y_t = 0$$

Solution

The roots are

$$r_1, r_2 = \frac{-4}{2} \pm \frac{1}{2}\sqrt{16 - 12}$$

Therefore the roots are -1 and -3. The solution to the homogeneous difference equation then is

$$y_t = C_1(-1)^t + C_2(-3)^t$$ ■

Example 20.3 Solve the following difference equation:

$$y_{t+2} - 4y_{t+1} + 4y_t = 0$$

Solution

The roots of this homogeneous difference equation are

$$r_1, r_2 = \frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 16}$$

Therefore both roots are 2. Using theorem 20.2, we find the solution to be

$$y_t = (C_1 + C_2 t)2^t$$ ■

Complex Roots

If the roots of the characteristic equation of the second-order difference equation turn out to be complex-valued numbers, (this occurs if $a_1^2 - 4a_2 < 0$), we can still obtain a solution, as we shall now show. However, some students will have to review the section on complex numbers and trigonometric functions, which appears at http://mitpress.mit.edu/math_econ3, before they can understand the details of what follows.

When $a_1^2 - 4a_2 < 0$, we can write the algebraic solution to the characteristic equation as

$$\begin{aligned} r_1, r_2 &= \frac{-a_1 \pm \sqrt{(-1)\big(4a_2 - a_1^2\big)}}{2} \\ &= \frac{-a_1 \pm \sqrt{-1}\sqrt{4a_2 - a_1^2}}{2} \end{aligned}$$

Using the concept of the *imaginary number,* $i = \sqrt{-1}$, the roots of the characteristic equation can then be written as the conjugate *complex numbers*

$$r_1, r_2 = h \pm vi$$

where

$$h = \frac{-a_1}{2}, \quad v = \frac{\sqrt{4a_2 - a_1^2}}{2}$$

The solution to the homogeneous difference equation becomes

$$y_h = c_1(h + vi)^t + c_2(h - vi)^t \tag{20.11}$$

To make equation (20.11) easier to interpret, we use the fact that a complex number can be expressed in polar or trigonometric form as

$$h \pm vi = R(\cos\theta \pm i \sin\theta)$$

where $R = \sqrt{h^2 + v^2}$ is said to be the *absolute value* of the complex roots and $\cos\theta = h/R$ and $\sin\theta = v/R$. Next we make use of *de Moivre's theorem* (see the appendix) to reduce equation (20.11) to a more easily interpreted expression. This theorem states that

$$[R(\cos\theta + i\sin\theta)]^n = R^n[\cos\theta n + i\sin\theta n]$$

Thus equation (20.11) becomes

$$y_h = c_1 R^t(\cos\theta t + i\sin\theta t) + c_2 R^t(\cos\theta t - i\sin\theta t) \tag{20.12}$$

This can be simplified further by noting that

$$R = (h^2 + v^2)^{1/2} = \left(\frac{a_1^2}{4} + \frac{4a_2 - a_1^2}{4}\right)^{1/2} = (a_2)^{1/2}$$

By collecting like terms and defining new constants that subsume those in equation (20.12), we obtain the solution for the homogeneous form of the linear, first-order difference equation in the case of complex roots

$$y_h = a_2^{t/2}(C_1\cos\theta t + C_2\sin\theta t) \tag{20.13}$$

In deriving this, we have set $C_1 = c_1 + c_2$ and $C_2 = (c_1 - c_2)i$. C_1 and C_2 are real-valued numbers even though they are functions of the imaginary number. As shown in the appendix, this is because c_1 and c_2 are themselves conjugate complex numbers. Another way to verify that C_1 and C_2 are real valued is to use the initial conditions (y_0 and y_1 given) to evaluate C_1 and C_2. Thus we obtain a real-valued solution to the difference equation. Also note that $a_2 > 0$ is guaranteed in the case of complex roots because $a_1^2 - 4a_2 < 0$ only if $a_2 > 0$.

Example 20.4 Solve the following homogeneous difference equation:

$$y_{t+2} - 2y_{t+1} + 2y_t = 0$$

Solution

The roots are

$$r_1, r_2 = \frac{2}{2} \pm \frac{1}{2}\sqrt{4-8}$$

which are complex valued since they involve the square root of a negative number. We write them as

$$r_1, r_2 = 1 \pm i$$

From equation (20.5) the solution to the homogeneous form is

$$y_t = 2^{t/2}(C_1 \cos\theta t + C_2 \sin\theta t)$$

where the value of θ can be found by solving

$$\cos\theta = \frac{h}{R} = \frac{1}{\sqrt{2}}$$

From trigonometric tables (some parts of which are included in the appendix), we find that $\cos(\pi/4) = 1/\sqrt{2}$. Therefore $\theta = \pi/4$ and the solution becomes

$$y_t = 2^{t/2}\left[C_1 \cos\left(\frac{\pi}{4}t\right) + C_2 \sin\left(\frac{\pi}{4}t\right)\right]$$ ■

The Complete Solution

The complete solution to the second-order difference equation (20.1) is obtained by applying theorem 20.1. This tells us that by adding together the general solution to the *homogeneous* form and a *particular* solution to the complete form in equation (20.1), we obtain the general solution to the complete equation. We now turn to the problem of finding a particular solution.

For autonomous difference equations, the particular solution we look for is the steady-state value of y. As usual, this occurs when y_t is stationary, which implies that $y_{t+2} = y_{t+1} = y_t$ which, as before, we call $\bar{y}$.

Setting $y_{t+2} = y_{t+1} = y_t = \bar{y}$ gives

$$\bar{y} + a_1\bar{y} + a_2\bar{y} = b$$

Solving gives

$$\bar{y} = \frac{b}{1 + a_1 + a_2}, \qquad a_1 + a_2 \neq -1$$

If $a_1 + a_2 = -1$, a steady-state solution does not exist. In that case we would have to find an alternative particular solution to use in obtaining the general solution. The particular solution to use in this case is $y_p = At$, where A is a constant to be determined using the method outlined in section 20.2. For the remainder of this section, we will assume that $a_1 + a_2 \neq -1$, which guarantees the existence of a steady-state solution.

Adding $\bar{y}$ to the solution to the homogeneous form gives the general solution to the complete difference equation (20.1):

Theorem 20.4 The general solution to the complete difference equation in equation (20.1), when $a_1 + a_2 \neq 1$, is as follows:

- If $a_1^2 - 4a_2 > 0$ (*real and distinct roots*),

$$y_t = C_1 r_1^t + C_2 r_2^t + \frac{b}{1 + a_1 + a_2} \tag{20.14}$$

- If $a_1^2 - 4a_2 = 0$ (*real and equal roots*),

$$y_t = C_1 r^t + C_2 t r^t + \frac{b}{1 + a_1 + a_2} \tag{20.15}$$

- If $a_1^2 - 4a_2 < 0$ (*complex roots*),

$$y_t = a_2^{t/2}(C_1 \cos\theta t + C_2 \sin\theta t) + \frac{b}{1 + a_1 + a_2} \tag{20.16}$$

where C_1 and C_2 are arbitrary constants (the values for which would be determined by initial conditions, if given) and r_1, r_2, and θ are defined in theorem 20.2.

Example 20.5 Solve the following difference equation:

$$2y_{t+2} + 8y_{t+1} + 6y_t = 32$$

Solution

Put the difference equation in standard form by dividing through by 2. This gives

$$y_{t+2} + 4y_{t+1} + 3y_t = 16$$

The homogeneous form of this difference equation is

$$y_{t+2} + 4y_{t+1} + 3y_t = 0$$

which is identical to the one solved in example 20.2. We therefore use that solution and add to it the particular solution to the complete equation given by the steady-state solution. The steady-state solution is obtained by solving

$$\bar{y} + 4\bar{y} + 3\bar{y} = 16$$

which gives $\bar{y} = 2$. Therefore the general solution to the complete equation is

$$y_t = C_1(-1)^t + C_2(-3)^t + 2$$ ■

Example 20.6 Solve the following difference equation:

$$y_{t+2} - 4y_{t+1} + 4y_t = 5$$

Solution

The homogeneous form of this difference equation is identical to the one solved in example 20.3. We therefore use that solution and add to it the particular solution to the complete equation given by the steady-state solution. This is obtained as the solution to

$$\bar{y} - 4\bar{y} + 4\bar{y} = 5$$

which gives $\bar{y} = 5$. Therefore the general solution to the complete equation is

$$y_t = (C_1 + C_2 t)2^t + 5$$ ■

Example 20.7 Solve the following difference equation:

$$y_{t+2} - 2y_{t+1} + 2y_t = 10$$

Solution

The homogeneous form of this difference equation is identical to the one solved in example 20.4. Using that solution plus the steady-state solution gives the general solution to the complete equation. The steady-state solution is obtained as the solution to

$$\bar{y} - 2\bar{y} + 2\bar{y} = 10$$

which gives $\bar{y} = 10$. The general solution to the complete equation then is

$$y_t = 2^{t/2}\left[C_1 \cos\left(\frac{\pi}{4}t\right) + C_2 \sin\left(\frac{\pi}{4}t\right)\right] + 10$$ ■

Initial Values

If the solution to a second-order difference equation is required to satisfy two specified initial conditions, the values of the constants must be set accordingly. The following example demonstrates the procedure for doing this.

Example 20.8 Solve for the constants in examples 20.5, 20.6, and 20.7 that make the solutions satisfy the initial conditions $y_0 = 1$ and $y_1 = 2$.

Solution

At $t = 0$, the solution to example 20.5 becomes

$$y_0 = C_1 + C_2 + 2$$

At $t = 1$, the solution becomes

$$y_1 = -C_1 - 3C_2 + 2$$

Setting $y_0 = 1$ and $y_1 = 2$ and solving these two equations for C_1 and C_2 gives $C_1 = -3/2$ and $C_2 = 1/2$. The solution to example 20.5 which also satisfies the initial conditions then is

$$y_t = -\frac{3}{2}(-1)^t + \frac{1}{2}(-3)^t + 2$$

The solution to example 20.6 at $t = 0$ is

$$y_0 = C_1 + 5$$

At $t = 1$, the solution is

$$y_1 = (C_1 + C_2)2 + 5$$

Setting $y_0 = 1$ and $y_1 = 2$ and solving gives $C_1 = -4$ and $C_2 = 2.5$. The solution to example 20.6 which also satisfies the initial conditions then is

$$y_t = (-4 + 2.5t)2^t + 5$$

The solution to example 20.7 at $t = 0$ is

$$y_0 = C_1 + 10$$

At $t = 1$, the solution is

$$y_1 = \sqrt{2}\left(C_1 \cos\frac{\pi}{4} + C_2 \sin\frac{\pi}{4}\right) + 10$$

But $\cos(\pi/4) = 1/\sqrt{2}$ and $\sin(\pi/4) = 1/\sqrt{2}$. Therefore

$$y_1 = \sqrt{2}\left(\frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}}\right) + 10$$

Solving gives $C_1 = -9$ and $C_2 = 1$. The solution becomes

$$y_t = 2^{t/2}\left[-9\cos\left(\frac{\pi}{4}t\right) + \sin\left(\frac{\pi}{4}t\right)\right] + 10$$

■

The Steady State and Convergence

We have found the complete solution to the linear, autonomous, second-order difference equation. Our concern now is to determine the conditions under which y_t converges to its steady-state value, $\bar{y}$, and under what conditions it diverges. The following theorem states these conditions:

Theorem 20.5 The path of y_t in a linear, autonomous, second-order difference equation converges to the steady-state value $\bar{y}$ from any starting value, where

$$\bar{y} = \frac{b}{1 + a_1 + a_2}$$

if $a_1 + a_2 \neq -1$, if and only if the absolute values of both roots are less than 1.

Proof

We consider three cases:

Real and distinct roots The solution in this case is

$$y_t = C_1 r_1^t + C_2 r_2^t + \bar{y}$$

To determine if y_t converges, take the limit of the solution for y_t as $t \to \infty$. It is apparent that the result depends entirely on the behavior of the terms involving r_1^t

and r_2^t. From our analysis of first-order difference equations, we know that these terms converge to 0 if the absolute values of r_1 and r_2 are less than 1, and diverge otherwise. Thus, y_t converges to $\bar{y}$ if and only if

$$|r_1| < 1 \quad \text{and} \quad |r_2| < 1$$

Note that both roots must be less than 1 in absolute value. If just one of them is greater than 1 in absolute value, y_t will diverge.

Real and equal roots The solution in this case is

$$y_t = C_1 r^t + C_2 t r^t + \bar{y}$$

If the root r is greater than 1 in absolute value, y_t clearly diverges. If the absolute value of the root is less than 1, y_t converges to $\bar{y}$ but it is a bit more difficult to see this than in the previous case. It is clear that $C_1 r^t$ goes to 0 as $t \to \infty$ if $|r| < 1$, but the limit of $C_2 t r^t$ as $t \to \infty$ is not so clear. We can ignore C_2 because it is a constant. The term tr^t is of the form $(\infty \times 0)$ in the limit. To evaluate this, we convert it to t/r^{-t} so that its limit is in the form (∞/∞) when $|r| < 1$ and then l'Hôpital's rule (see section 5.4) can be applied. Using the differentiation rule that $da^x/dx = a^x \ln(x)$, take the derivative of the numerator and denominator with respect to t to get $1/[-r^t \ln(r)]$. We now take the limit of this expression as $t \to \infty$ and see that the denominator goes to infinity, so the whole term goes to 0. Thus, both terms involving t go to 0 as $t \to \infty$ if and only if $|r| < 1$.

Complex roots The solution in this case is

$$y_t = a_2^{t/2}(C_1 \cos \theta t + C_2 \sin \theta t) + \bar{y}$$

where $\cos \theta = h/R$; $\sin \theta = v/R$, where $h = -a_1/2$, $v = (4a_2 - a^2)^{1/2}/2$, and $R = a_2^{1/2}$. The appendix at the end of the book shows that the cosine and sine functions are bounded between $-C_i$ and C_i, $i = 1, 2$ respectively, as $t \to \infty$. Thus, the convergence of y_t is entirely determined by the behavior of the term $a_2^{t/2}$. If $a_2^{1/2} < 1$, this term converges to 0; otherwise, it diverges. Recalling that $\sqrt{a_2} = \sqrt{h^2 + v^2}$ is the absolute value of the complex root, $h + vi$, then we again have found that y_t converges if and only if the absolute value of the root is less than 1. ■

The Cobweb Model

In chapter 18 we examined a model of price determination in which suppliers form their supply decision for next period on the basis of this period's price. We found that the model is highly unstable because it leads to *explosive* price oscillations unless the slope of the supply function is less than the absolute value of the slope of the demand function. We criticized the model for the naive way in which suppliers are assumed to form price expectations.

We will consider an alternative specification of the price expectations process here that leads to a second-order difference equation for price instead of a first-order difference equation, as was the case in chapter 18.

As before, suppose that the supply function for period t is

$$Q_t^S = F + GE_{t-1}(p_t)$$

where Q^S is the quantity supplied and $E_{t-1}(p_t)$ is the price that suppliers in period $t-1$ expected to prevail in period t. We assume that

$$E_{t-1}(p_t) = p_{t-1} - \rho \Delta p_{t-2}$$

where

$$\Delta p_{t-2} = p_{t-1} - p_{t-2}$$

is the price change from period $t-2$ to period $t-1$, and ρ is a parameter, which we discuss below. The price expectation process modeled here is one in which suppliers in period $t-1$ use information on the (then) current price, p_{t-1}, and the price change over the previous period, Δp_{t-2}, to predict the next period price.

If $0 \leq \rho \leq 1$, suppliers expect the next price change, $E_{t-1}(p_t) - p_{t-1}$, to be in the opposite direction of the previous price change, Δp_{t-2}. This would be a reasonable expectation given our knowledge of the oscillations that occur in the cobweb model. On the other hand, if $-1 \leq \rho \leq 0$, suppliers expect the next price change to be in the same direction as the previous price change. We will investigate the stability properties of this model for both positive and negative ρ.

The demand function is given by

$$Q_t^D = A + Bp_t$$

Equilibrium occurs when $Q_t^D = Q_t^S$. After making substitutions we find that this condition gives

$$A + Bp_t = F + G[p_{t-1}(1-\rho) + \rho p_{t-2}]$$

Simplifying gives

$$p_t - \frac{G}{B}(1-\rho)p_{t-1} - \frac{G}{B}\rho p_{t-2} = \frac{F-A}{B}, \qquad t = 2, 3, 4 \ldots$$

This second-order difference equation for price holds for all $t \geq 2$. We rewrite it for convenience as

$$p_{t+2} - \frac{G}{B}(1-\rho)p_{t+1} - \frac{G}{B}\rho p_t = \frac{F-A}{B}, \qquad t = 0, 1, 2, \ldots \tag{20.17}$$

To solve the complete equation, we begin by obtaining the general solution to its homogeneous form. This is

$$p_{t+2} - \frac{G}{B}(1-\rho)p_{t+1} - \frac{G}{B}\rho p_t = 0, \quad t = 0, 1, 2, \ldots \tag{20.18}$$

The roots of this difference equation are

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

where $a_1 = -G(1-\rho)/B$ and $a_2 = -G\rho/B$.

The solution to the homogeneous form of the difference equation is

$$p_h = C_1 r_1^t + C_2 r_2^t \tag{20.19}$$

Note that if $\rho < 0$, then the roots are real valued and distinct in the normal case of downward-sloping demand, $B < 0$, because the expression under the square root symbol is then positive. On the other hand, if $\rho > 0$, the roots could turn out to be real valued (distinct or equal) or complex valued.

The steady-state solution, which we will call $\bar{p}$ (assuming that it exists), provides a particular solution to the complete difference equation. This is obtained by solving

$$\bar{p} - \frac{G}{B}(1-\rho)\bar{p} - \frac{G}{B}\rho\bar{p} = \frac{F-A}{B}$$

which gives

$$\bar{p} = \frac{F-A}{B-G}$$

the usual market-clearing price. The general solution to the complete equation then is

$$p_t = C_1 r_1^t + C_2 r_2^t + \bar{p}$$

Price converges to its steady-state value if the absolute values of both roots are less than 1. However, it is difficult to tell if this condition is satisfied in this model. We consider two numerical examples of this model and then conduct a more general stability analysis.

Example 20.9 Determine the solution to the difference equation in the cobweb model and determine whether price converges to its steady-state value when the parameter values are as follows: $B = -16$, $G = 13$, $A = 60$, $F = 2$, and $\rho = -3/13$.

Solution

Using the expressions shown for a_1 and a_2 in the cobweb model, we get

$$a_1 = \frac{-13}{-16}\left(1 + \frac{3}{13}\right) = 1$$

$$a_2 = \frac{-13}{-16}\frac{3}{13} = \frac{3}{16}$$

The roots therefore are

$$r_1, r_2 = \frac{-1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{3}{4}} = -\frac{3}{4}, -\frac{1}{4}$$

The steady-state value of price is

$$\bar{p} = \frac{2 - 60}{-16 - 13} = 2$$

The general solution then is

$$p_t = C_1\left(-\frac{3}{4}\right)^t + C_2\left(-\frac{1}{4}\right)^t + 2$$

Because the absolute values of both roots are less than 1, price converges to a value of 2 as t goes to infinity, regardless of the value of the constants C_1 and C_2 (and therefore regardless of the starting values for p_0 and p_1).

We arbitrarily chose starting values of $p_0 = 6$ and $p_1 = 1$ and obtained values for the constants of $C_1 = 0$ and $C_2 = 4$. We then calculated p_t for $t = 0, 1, 2, 3, \ldots$ and obtained the following sequence of values: 6.000, 1.000, 2.250, 1.938, 2.016, 1.996, 2.001, 2.000, 2.000, ... As we see, price converges to its steady-state value rather quickly in this example. ■

Example 20.10 Determine the solution to the difference equation in the cobweb model and determine whether price converges to its steady-state value when the parameter values are as follows: $B = -2$, $G = 3/2$, $\rho = 1/3$, $F = 10$, and $A = 27.5$.

Solution

In this case we obtain $a_1 = 0.5$, $a_2 = 0.25$, and find that the roots turn out to be complex valued. The homogeneous solution then is

$$p_h = (0.5)^t[C_1 \cos(\theta t) + C_2 \sin(\theta t)]$$

The value of θ is determined by using either the relationship that $\cos(\theta) = h/R$ or $\sin(\theta) = v/\text{R}$. We shall use the former. Recalling that $h = -a_1/2$, then

$$\cos(\theta) = \frac{h}{R} = \frac{-0.25}{0.5} = -0.5$$

Since $\cos(2\pi/3) = -0.5$, we conclude that the value of θ is $2\pi/3$. The homogeneous solution becomes

$$p_h = (0.5)^t\left[C_1 \cos\left(\frac{2\pi}{3}t\right) + C_2 \sin\left(\frac{2\pi}{3}t\right)\right]$$

After calculating the steady-state price to be 5, the general solution to the complete difference equation becomes

$$p_t = (0.5)^t\left[C_1 \cos\left(\frac{2\pi}{3}t\right) + C_2 \sin\left(\frac{2\pi}{3}t\right)\right] + 5$$

Figure 20.1 depicts the motion of price given by this equation. Because $(0.5)^t$ approaches 0 quickly as t increases, the cyclical fluctuations in price are damped

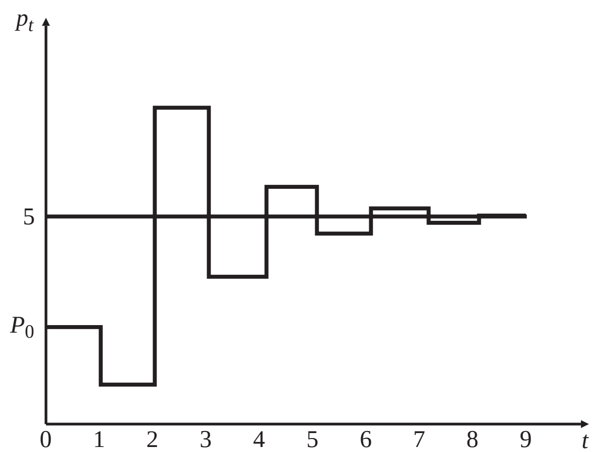


Figure 20.1 Path of price in the cobweb model for the case of complex roots

quite rapidly, leading to fairly rapid convergence. For a larger value of $\sqrt{a_2}$, such as 0.95, the damping would be more gradual leading to a slower convergence. Of course, if $a_2 > 1$, price would diverge in explosive oscillations. ■

Convergence Revisited

So far we have determined that the condition for **convergence** is that the absolute values of both roots be less than 1. Although this was indeed the case in the previous numerical examples of the cobweb model, a fair question to ask is whether the stability of these examples was just a lucky choice of parameter values or whether the modification of the price expectation formation process has added some stability to the inherently unstable cobweb model.

To answer this question, we have to determine what restrictions on the coefficients of the difference equation guarantee that the roots of the characteristic equation will have an absolute value less than 1. The following conditions do this:

Theorem 20.6 The absolute value of the roots of the characteristic equation for the linear, autonomous, second-order difference equation are less than 1 if and only if the following three conditions are satisfied:

(i) $1 + a_1 + a_2 > 0$
(ii) $1 - a_1 + a_2 > 0$
(iii) $a_2 < 1$

Before providing an explanation of these conditions, let us check that the parameters in example 20.10 satisfy these restrictions. We have $a_1 = 0.5$ and $a_2 = 0.25$. Since both a_1 and a_2 are positive, condition (i) is satisfied. Checking condition (ii), $1 - 0.5 + 0.25 = 0.75 > 0$ so this, too, is satisfied. Finally, condition (iii) is satisfied because $a_2 = 0.25$.

Panels (a), (b), (c), and (d) in figure 20.2 display graphs of the characteristic equation

$$f(r) = r^2 + a_1 r + a_2$$

for different values of a_1 and a_2.

The roots that solve the characteristic equation occur at the intersection of the function with the horizontal axis. We seek to determine the parameter restrictions that ensure both intersections occur between -1 and $+1$.

In the first three panels of figure 20.2, the roots are between -1 and $+1$. Notice that in all three cases, $f(1) > 0$ and $f(-1) > 0$; in fact these are necessary conditions for the roots to be between -1 and $+1$. Panel (d) is an example where

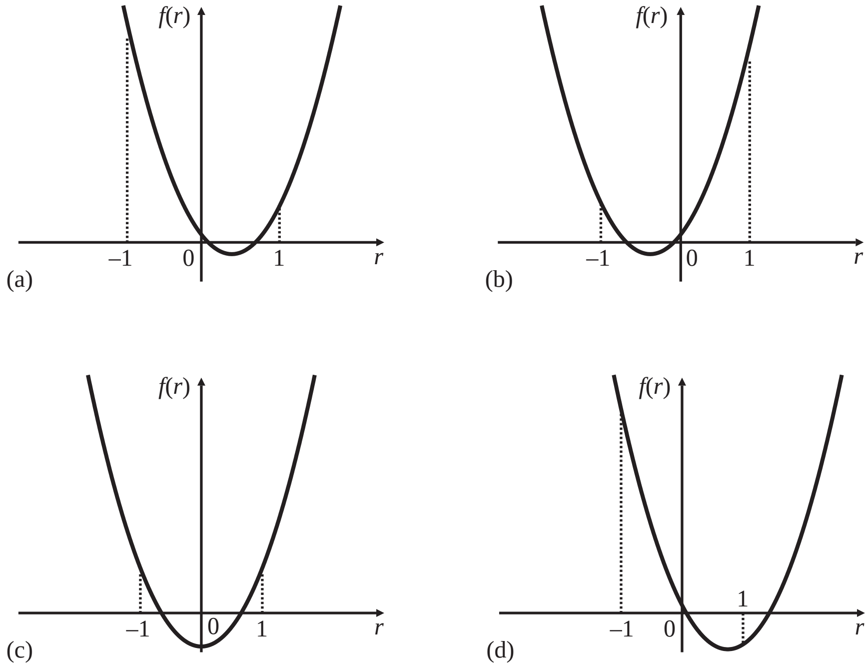


Figure 20.2 (a) Roots positive but less than 1; (b) roots negative but greater than -1; (c) absolute values of both roots less than 1, but one root positive and one root negative; (d) one root exceeding 1

one of the roots exceeds 1; notice that $f(1) < 0$ in this case. Hence conditions (i) and (ii) in theorem 20.6 are the necessary conditions $f(1) > 0$ and $f(-1) > 0$.

Although these conditions rule out most of the other possibilities, such as the roots equal to -1 and $+1$, one root less than -1 and one root larger than $+1$, one root less than -1 and one between -1 and $+1$, and one root between -1 and $+1$ and one larger than $+1$, they do not rule out two important possibilities. Namely, both roots could be less than -1 or both roots could be greater than $+1$, and the conditions $f(1) > 0$ and $f(-1) > 0$ would be satisfied. Thus we need an additional condition to rule out these two possibilities. In each of these cases, the product of the two roots exceeds $+1$. But, since the product of the characteristic roots of a quadratic function is equal to a_2, then we need only add the condition that $a_2 < 1$ to the two conditions we already have. These three together then comprise the necessary and sufficient conditions for both roots to be between -1 and $+1$.

Example 20.11 Use theorem 20.6 to determine whether the cobweb model has been made more or less likely to converge to its steady state by modifying the price expectations process in the basic cobweb model.

Solution

We do this separately for the cases $\rho > 0$ and $\rho < 0$.

Case 1 $0 < \rho < 1$

The first condition for stability in theorem 20.6 is

(i) $1 + a_1 + a_2 > 0$. In the modified cobweb model, we get

$$\begin{aligned} 1 + a_1 + a_2 &= 1 - \frac{G}{B}(1-\rho) - \frac{G}{B}\rho \\ &= 1 - \frac{G}{B} \end{aligned}$$

which is positive for the usual case of downward-sloping demand. The first condition for stability is therefore satisfied. The second condition is

(ii) $1 - a_1 + a_2 > 0$. We get

$$\begin{aligned} 1 - a_1 + a_2 &= 1 + \frac{G}{B}(1-\rho) - \frac{G}{B}\rho \\ &= 1 + \frac{G}{B}(1-2\rho) \end{aligned}$$

which is positive if $\rho > 1/2$ (when $B < 0$). The third condition is

(iii) $a_2 < 1$. We get

$$a_2 = -\frac{G}{B}\rho$$

which is less than 1 if

$$-\frac{G}{B} < \frac{1}{\rho}$$

We conclude that, compared to the basic cobweb model, the modified model is more stable when $\rho > 0$. (Recall that $\rho > 0$ means that suppliers expect the price change to be in the opposite direction to the most recent price change.) We reach this conclusion as follows. First, condition (i) is always satisfied. Second, if the absolute value of G/B is less than 1, (the requirement for convergence in the basic model), conditions (ii) and (iii) are also always satisfied. So the modified model is at least as stable as the basic model. But for $\rho > 1/2$, condition (ii) is satisfied no matter what the absolute value of G/B and condition (iii) requires the absolute value of G/B to be less than $1/\rho$ which is a larger number than 1. Thus the conditions for convergence in this modified version are less restrictive than in the simple version. In other words, the model is likely to converge for a larger range of parameter values.

Case 2 $-1 < \rho < 0$

(i) For the first stability condition, we get

$$\begin{aligned} 1 + a_1 + a_2 &= 1 - \frac{G}{B}(1-\rho) - \frac{G}{B}\rho \\ &= 1 - \frac{G}{B} \end{aligned}$$

As above, the first condition is always satisfied for the usual case of downward-sloping demand.

(ii) For the second condition, we get

$$\begin{aligned} 1 - a_1 + a_2 &= 1 + \frac{G}{B}(1-\rho) - \frac{G}{B}\rho \\ &= 1 + \frac{G}{B}(1-2\rho) \end{aligned}$$

which is positive if

$$-\frac{G}{B} < \frac{1}{1-2\rho}$$

(iii) For the third condition, we get

$$a_2 = -\frac{G}{B}\rho$$

which is less than 1 for the usual case of $B < 0$.

The first and the third condition are always satisfied (for the case of $B < 0$) but the second presents a problem. It is satisfied only if the absolute value of G/B is less than a number less than 1 (since $1/(1-2\rho) < 1$). We conclude that the modified model is *less* stable than the simple model if $\rho < 0$. (Recall that $\rho < 0$ means that suppliers expect the price to change in the same direction as the most recent price change.) ■

A Model of Cournot Duopoly

Suppose there are only two firms in a market. Let x_t be firm 1's output in period t and y_t be firm 2's output in period t. The market- (inverse-) demand function is assumed to be

$$p(x + y) = 120 - (x + y)$$

Assume costs are zero for both firms. Firm 1 must choose how much output to produce in period $t + 1$. We apply the Cournot assumption that each firm chooses its output for $t + 1$ to maximize profit in the belief that the other firm will maintain its output level at the period t value. Since cost is zero, profit in $t + 1$, π_{t+1}, is just price times quantity. For firm 1, profit is

$$\pi_{t+1} = (120 - x_{t+1} - y_t)x_{t+1}$$

The first-order necessary condition for choosing x_{t+1} to maximize π_{t+1} is

$$120 - y_t - 2x_{t+1} = 0$$

Firm 2 has a symmetrical maximization problem so we would obtain a similar-looking first-order condition with the roles of x and y reversed. Together, these two first-order conditions give the following *reaction* functions for the two firms:

$$x_{t+1} = 60 - \frac{1}{2}y_t$$

$$y_{t+1} = 60 - \frac{1}{2}x_t$$

Using the second reaction function to substitute for y_t in the first reaction function gives the following second-order difference equation for x:

$$x_{t+1} - 0.25x_{t-1} = 30, \qquad t = 1, 2, \ldots$$

We rewrite this in the more familiar form

$$x_{t+2} - 0.25x_t = 30, \qquad t = 0, 1, 2, \ldots$$

In terms of the coefficients of the general form of the second-order difference equation, this problem gives

$$a_1 = 0, \quad a_2 = -0.25, \quad b = 30$$

The stationary value of x_t, which we call $\bar{x}$, is obtained by setting $x_{t+2} = x_t$ and solving the difference equation, which gives

$$\bar{x} = 40$$

Does x_t converge to 40? Theorem 20.5 gives the answer before even solving the difference equation:

(i) $1 + a_1 + a_2 = 1 + 0 - 0.25 = 0.75 > 0$
(ii) $1 - a_1 + a_2 = 1 - 0 - 0.25 = 0.75 > 0$
(iii) $a_2 = -0.25 < 1$

All three conditions are satisfied, so we can be sure that x_t converges to 40. This is confirmed by calculating the roots, which are $1/2$ and $-1/2$. The solution to the difference equation then is

$$x_t = C_1(0.5)^t + C_2(-0.5)^t + 40$$

Example 20.12 Suppose that the initial conditions in the Cournot duopoly model are $x_0 = 50$ and $x_1 = 30$. Find the values of the constants that ensure that the solution also satisfies these initial conditions.

Solution

Setting $t = 0$ and $x_0 = 50$ and solving gives $C_1 = 10 - C_2$. Setting $t = 1$ and $x_1 = 30$ and solving gives $C_2 = 15$. The solution to this initial-value problem then is

$$x_t = -5(0.5)^t + 15(-0.5)^t + 40$$ ■

EXERCISES

1. Solve the following difference equations:

(a) $y_{t+2} - y_t = 0$

(b) $2y_{t+2} - 5y_{t+1} + 2y_t = 4$

(c) $y_{t+2} + 2y_{t+1} + y_t = 16$

(d) $y_{t+2} - 6y_{t+1} + 18y_t = 26$

2. Solve the following difference equations:

(a) $y_{t+2} + \frac{1}{2} y_{t+1} - \frac{3}{16} y_t = 6$

(b) $9y_{t+2} - 6y_{t+1} + y_t = 4$

(c) $y_{t+2} + 2y_{t+1} + 4y_t = 21$

(d) $3y_{t+2} - 9y_{t+1} - 12y_t = -48$

3. Suppose that there are only two firms supplying a market which has demand function

$$p(x+y) = A - B(x+y)$$

where x is firm 1's output level and y is firm 2's output level. Find and solve the second-order difference equation for x, assuming each firm makes a Cournot conjecture about the other firm's output level.

4. As in exercise 3, suppose that only two firms supply the market and each makes a Cournot assumption about the other's output level. Let the (inverse-) demand function be

$$p(x+y) = 126 - (x+y)$$

Now suppose the firms have cost functions given by

$$C(x) = \frac{1}{2}x^2, \quad C(y) = \frac{1}{2}y^2$$

Profit for firm 1 is now

$$\pi_{t+1} = (126 - x_{t+1} - y_t)x_{t+1} - \frac{1}{2}x_{t+1}^2$$

A similar expression can be derived for firm 2's profit.

(a) Find the second-order difference equation for x.

(b) Use theorem 20.6 to determine whether or not price converges to its steady-state level.

(c) Solve the difference equation.

5. Consider the following *multiplier-accelerator* model of an economy. Let aggregate national income, Y, be equal to

$$Y_t = C_t + I_t + G_t$$

where C, I, and G are consumption, investment, and government expenditure, respectively. Assume that government expenditure is constant at $\bar{G}$. However, assume that consumption is given by

$$C_t = mY_t$$

where $0 < m < 1$ is the marginal propensity to consume. In addition assume that investment is a fraction α of the growth of national income in the previous year

$$I_t = \alpha(Y_{t-1} - Y_{t-2})$$

Derive the second-order difference equation for national income implied by this model and solve it. Use theorem 20.6 to determine what restrictions on the parameters of the model must be made to ensure convergence.

6. For the modified cobweb model examined in this chapter, find the complete general solution for the following two sets of parameter values and determine whether price converges to the equilibrium.

(a) $\rho = -1/2$, $G = 10$, $B = -20$, $F = 20$, $A = 140$

(b) $\rho = -1/2$, $G = 10$, $B = -50$, $F = 20$, $A = 260$

20.2 The Linear, Second-Order Difference Equation with a Variable Term

When the term b_t is not constant, then the linear, second-order difference equation is nonautonomous. The method of solving a nonautonomous difference equation still involves adding together the solution to the homogeneous form and a particular solution to the complete equation. However, we can no longer rely on using the steady-state solution as a particular solution since it no longer exists necessarily. Even when b is constant, a steady-state solution does not exist when $1 + a_1 + a_2 = 0$. In this section, we explain an alternative technique for finding a particular solution.

Various techniques have been developed for finding a particular solution when b_t is not constant. We provide an introduction to one of these techniques, known as the **method of undetermined coefficients.** This method relies on one's ability to "guess" the form of the particular solution. While this might at first seem an arbitrary approach, we provide three guidelines that can be followed.

Case 1 If b_t is an nth degree polynomial in t, say $p_n(t)$, then assume that the particular solution is also a polynomial. That is, assume that

$$y_p = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$$

where the A_i are constants to be determined using the procedure outlined in example 20.13 below.

Case 2 If b_t is of the form k^t, where k is some constant, then assume that

$$y_p = Ak^t$$

where A is a constant to be determined using the procedure outlined in the economic application at http://mitpress.mit.edu/math_econ3.

Case 3 If b_t is of the form $k^t p_n(t)$, then assume that

$$y_p = Ak^t(A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n)$$

An important exception to these guidelines for guessing solutions should be noted. If any term in the assumed solution is also a term (solution) of the homogeneous solution disregarding multiplicative constants, then the assumed solution must be modified as follows. Multiply the assumed solution by t^k, where k is the smallest positive integer such that the common terms are then eliminated. Two examples below illustrate this procedure.

Example 20.13 Solve $y_{t+2} - 3y_{t+1} + 2y_t = 10$.

Solution

The characteristic equation is $r^2 - 3r + 2 = 0$, with roots 1 and 2. Therefore the solution to the homogeneous form is

$$y_h = C_1 2^t + C_2$$

We wish to find a particular solution but we note that $1 + a_1 + a_2 = 1 - 3 + 2 = 0$ for this problem. Thus, we cannot find a stationary value for y to use as a particular solution. The alternative is to use the method of undetermined coefficients to find a particular solution. Because b_t in this case is a constant (a polynomial of order 0), we would first try a particular solution of the same form, that is, $y_p = A$, where A is a constant to be determined. However, this is the same as the constant term, C_2, in the homogeneous solution, disregarding the multiplicative constant. As a result we need to modify our “guess” by trying $y_p = At$. (Note that another way of knowing that the first guess would not work is by the fact that we know a constant solution does not exist; otherwise, we would have used it.)

The particular solution must satisfy the difference equation. We use this fact to solve for A. Substitute the particular solution into the difference equation to get

$$A(t+2) - 3A(t+1) + 2At = 10$$

Solving this gives us $A[t + 2 - 3t - 3 + 2t] = 10$, for which we get $A = -10$. Therefore the particular solution is $y_p = -10t$, so the general solution to the complete equation is

$$y_t = C_1 2^t + C_2 - 10t$$ ■

Example 20.14 Solve $y_{t+2} - 3y_{t+1} + 2y_t = 1 + t$.

Solution

The homogeneous solution is the same as for the previous example. Only the particular solution will differ. Because the form of b_t now is a first-degree polynomial in t, our initial "guess" for the particular solution is

$$y_p = A_0 + A_1 t$$

However, the same problem arises here as in the previous example. That is, our "guess" has a term in common (ignoring multiplicative constants) with the homogeneous solution. Specifically, both have a constant term. Thus we multiply our first guess by t to obtain our next trial solution

$$y_p = A_0 t + A_1 t^2$$

This trial solution has no terms in common with the homogeneous solution so we may proceed. Substituting the particular solution into the complete difference equation (after obtaining the y_{t+2}, and y_{t+1} terms for the particular solution) produces

$$\{A_0(t+2) + A_1(t+2)^2\} - 3\{A_0(t+1) + A_1(t+1)^2\} + 2\{A_0 t + A_1 t^2\} = 1 + t$$

Now collect constant terms, terms in t, and terms in t^2:

$$(A_1 - A_0 - 1) - t(2A_1 + 1) + t^2(0) = 0$$

Since this must hold for all values of t, each of the terms in brackets must be identically equal to zero.

$$A_1 = -\frac{1}{2} \quad \text{and} \quad A_0 = -\frac{3}{2}$$

The complete solution to the difference equation therefore is

$$y_t = C_1 2^t + C_2 - \frac{3}{2}t - \frac{1}{2}t^2$$ ■

EXERCISES

1. Solve the following difference equations:

(a) $y_{t+2} - 3y_{t+1} + 2y_t = 12$

(b) $y_{t+2} - \frac{5}{2}y_{t+1} + y_t = 3^t$

(c) $y_{t+2} + 2y_{t+1} + y_t = t$

2. Solve the following difference equations:

(a) $9y_{t+2} - 6y_{t+1} - 3y_t = 4$

(b) $y_{t+2} - 2y_{t+1} + \frac{3}{4}y_t = 5 + 3t$

(c) $3y_{t+2} - 9y_{t+1} - 12y_t = 3^t$

CHAPTER REVIEW

Key Concepts

characteristic equation
characteristic roots
complete solution
complex roots
convergence
eigenvalues
homogeneous form
initial values
particular solution
real and distinct roots
real and equal roots
steady state
method of undetermined coefficients

Review Questions

1. Explain the usefulness of theorem 20.1.

2. What is the characteristic equation and what role does it play in finding the solution to a linear, second-order difference equation?

3. How are the two solutions to a homogeneous, linear, second-order difference equation (given by the two roots) used to obtain the general solution?

4. Under what conditions is the particular solution given by the steady-state solution?
5. If initial conditions are given, should the values of the arbitrary constants, C_1 and C_2, be determined after obtaining the general solution to the *homogeneous* form, or only after obtaining the general solution to the *complete* equation?
6. State the necessary and sufficient conditions for convergence to a steady state both in terms of the roots and in terms of the coefficients of a linear, second-order difference equation.
7. When does one use the method of undetermined coefficients to find the particular solution to a linear, second-order difference equation?

Review Exercises

1. Solve $9y_{t+2} - 6y_{t+1} + y_t = 16$.
2. Solve $y_{t+2} - 5y_{t+1} + 6y_t = 2$.
3. Solve $2y_{t+2} - 5y_{t+1} + 2y_t = 10$.
4. Solve $y_{t+2} + 2y_{t+1} + y_t = 120$.
5. Solve $y_{t+2} - 2y_{t+1} + y_t = 2$.
6. Solve $2y_{t+2} - 5y_{t+1} + 2y_t = 2^t$.
7. Solve $3y_{t+2} - 6y_{t+1} + 4y_t = 14$.
8. Solve $y_{t+2} - y_t = 1$.
9. Find the solutions of the difference equations in review exercises 1, 3, 5, and 7 that satisfy the initial conditions: $y_0 = 1$ and $y_1 = -1$.
10. Find the solutions of the difference equations in review exercises 2, 4, 6, and 8 that satisfy the initial conditions: $y_0 = 1$ and $y_1 = -1$.
11. Two firms share the market for a product. Firm 1's output is x; firm 2's output is y. The two reaction functions of the firms are

$$x_{t+1} + \beta y_t = b, \qquad \beta \neq 1$$
$$y_{t+1} + \alpha x_t = b, \qquad \alpha \neq 1$$

Derive and solve the second-order difference equation for x implied by this model.

12. Suppose that the national product, y_t, is composed of production plus investment. Part of current production is intended for current consumption, and part is intended to maintain an inventory of consumer goods. Let q_t

be production intended for consumption and x_t be production intended for inventory maintenance. Assuming no lag in the consumption function, then actual consumption in any period is

$$C_t = my_t$$

Producers must decide production for period t before C_t is actually known. Assume that producers expect consumption in period t to be the same as consumption for period $t-1$. They therefore produce an amount

$$q_t = my_{t-1}$$

for consumption. Assuming that producers wish to maintain a constant inventory, they produce an amount for inventory equal to the difference between actual and planned sales in the previous period. Planned sales in the previous period were q_{t-1}, whereas actual sales were an amount C_{t-1}. Therefore

$$x_t = C_{t-1} - q_{t-1} = my_{t-1} - my_{t-2}$$

In period t, national product is $y_t = q_t + x_t + \bar{I}$, where $\bar{I}$ is an exogenous and constant investment. Derive the second-order difference equation for y_t implied by this model and obtain the general solution.

13. In the study of econometrics, we sometimes encounter an autoregressive process, which means that a variable is regressed upon itself. For example, let

$$y_t = \beta y_{t-1} + e_t, \qquad \beta < 1$$

be the autoregressive process where y is a regressive lagged function of itself, and where e is a random error term in the relationship. Often the error term will have the following form:

$$e_t = \rho e_{t-1} + u_t$$

where u_t also is an error term. However, we assume that the *expected value* of u_t is zero. In other words, if we took the average value of u_t over many periods, it would be zero. Derive the second-order difference equation implied for y in this model by eliminating e_t and e_{t-1} from the equation.

14. For the problem in review exercise 12, let $m = 0.25$, $y_0 = 4$, $\bar{I} = 3$, and $y_1 = 1$. Solve the difference equation.

15. For review exercise 13, set u_t equal to its expected value and solve the difference equation. This solution gives the expected value of y_t. Use theorem 20.6 to analyze the convergence properties of this model.

Chapter 21 Linear, First-Order Differential Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- The Dynamics of National Debt Accumulation
- The Dynamics of the IS-LM Model
- Practice Exercises

In the next three chapters we explain techniques for solving and analyzing ordinary differential equations. We do not attempt to provide exhaustive coverage of the topic but instead focus on the types of differential equations and techniques of analysis that are most common in economics. We begin in this chapter with linear, first-order differential equations. In the next chapter we turn to an examination of nonlinear, first-order differential equations, and in the chapter after that we examine linear, second-order differential equations. In this chapter and throughout, we will solve a large number of examples and economic applications to illustrate the uses of ordinary differential equations in economics.

21.1 Autonomous Equations

In this section we explain how to solve linear, first-order differential equations that are *autonomous*, meaning ones in which the variable t does not enter the equation explicitly.

Definition 21.1 The general form of the **linear, autonomous, first-order differential equation** is

$$\dot{y} + ay = b \tag{21.1}$$

where a and b are known constants.

The differential equation is linear because $\dot{y}$ and y are not raised to any power other than 1. It is of the first-order because that is the highest-order derivative in the equation. It is autonomous because the coefficient a and the term b are constant. If a or b vary with t (i.e., are explicit functions of t), the equation is nonautonomous—this case is taken up in section 21.2.

The solution method we use in this section relies on the technique of separating the problem of finding the general solution for the complete differential equation into two simpler subproblems. If y_h denotes the general solution to the *homogeneous form* (obtained by setting $b = 0$) and y_p denotes a **particular solution** (as opposed to the general solution) to the complete equation, then we can use the result that

$$y = y_h + y_p$$

That is, the **general solution,** y, for the complete equation is the sum of the general solution to the homogeneous form and a particular solution to the complete equation, such as the steady-state solution. The two subproblems, finding y_h and y_p, are relatively easy to solve, and they provide the means by which we will solve the complete differential equation in this section. After showing how to find y_h and y_p, we prove that adding them together provides the desired general solution.

The Homogeneous Solution

We begin by obtaining the general solution to the homogeneous form of the differential equation in definition 21.1.

Definition 21.2

The **homogeneous** form of the linear, autonomous, first-order differential equation is

$$\dot{y} + ay = 0, \qquad a \neq 0$$

If $a = 0$, the solution is easy to obtain by direct integration. It is simply $y(t) = C$, where C is an arbitrary constant of integration. From now on, we focus on the more general case, in which $a \neq 0$.

We can solve the homogeneous form by direct integration after manipulating it into a suitable form. Subtract ay from both sides of the equation and then divide through by y. This gives

$$\frac{\dot{y}}{y} = -a$$

In this form we can integrate each side with respect to t without too much difficulty. The integral of the right-hand side is just $-at + c_1$, where c_1 is a constant of integration. The integral of the left-hand side is written as

$$\int \frac{\dot{y}}{y}\, dt$$

Recalling that $\dot{y}$ is actually dy/dt, this becomes

$$\int \frac{dy/dt}{y}\, dt$$

and after canceling the dt terms, this becomes

$$\int \frac{1}{y}\, dy$$

Since the integral of $1/y$ is just $\ln y + c_2$, where c_2 is a constant of integration, we now have integrated both sides, giving

$$\ln y + c_2 = -at + c_1$$

To obtain an explicit solution for y, take the antilogarithm of both sides. This gives

$$y = e^{-at+c_1-c_2} \tag{21.2}$$

$$= e^{-at}e^{c_1-c_2} \tag{21.3}$$

$$= Ce^{-at} \tag{21.4}$$

where $C = e^{c_1-c_2}$ is still an arbitrary constant of integration.

This gives the solution to the homogeneous form. To avoid confusion later on, we will use the notation y_h to refer to the solution to the homogeneous form (the h subscript stands for homogeneous), and write the solution as $y_h(t)$ to make it clear that because it is a solution, we express it as an explicit function of t.

Theorem 21.1 The general solution to the **homogeneous** form of the linear, autonomous, first-order differential equation is

$$y_h(t) = Ce^{-at}$$

Before proceeding, let's check that our solution is correct. To do this, it is sufficient to prove that it satisfies the differential equation. The solution implies that

$$\dot{y}_h = -aCe^{-at}$$

Substitute this and the solution, $y_h(t)$, into $\dot{y} + ay$, and then check that it equals 0 as in the original differential equation

$$-aCe^{-at} + aCe^{-at}$$

This expression does equal 0, so we can be certain that $y_h(t)$ is a correct solution.

Example 21.1 Solve the *homogeneous* form of the differential equation

$$\dot{y} = 3y + 2$$

Solution

The homogeneous form is

$$\dot{y} - 3y = 0$$

Rewrite this as

$$\frac{\dot{y}}{y} = 3$$

Integrating both sides gives

$$\ln y + c_2 = 3t + c_1$$

Taking the antilogarithm of both sides and simplifying gives the solution:

$$y_h(t) = Ce^{3t}$$ ■

Example 21.2 Let y represent national energy consumption and suppose it grows at a constant rate of 2%. Derive and solve the differential equation implied by this statement.

Solution

The rate of growth of something is just its growth divided by its level ($\dot{y}/y$). If the *percentage* rate of growth is a constant 2%, then the rate of growth itself is just 0.02. Thus we can express the statement that energy consumption grows at a constant 2% as follows:

$$\frac{\dot{y}}{y} = 0.02$$

The solution, using theorem 21.1, is

$$y(t) = Ce^{0.02t}$$

The solution gives the level of energy consumption at time t. ■

The Particular Solution

Having obtained the general solution to the homogeneous form of the differential equation in definition 21.1, we now look for a *particular* solution to the complete differential equation. A particular solution that is easy to find is the steady-state equilibrium value of y. The concept of a steady-state value for a differential equation is identical to that for a difference equation, although the method of finding it is slightly different.

Definition 21.3

A **steady-state value** of a differential equation is defined by the condition $\dot{y} = 0$. It is the value of y, which we call $\bar{y}$, at which y is stationary.

To find the steady-state value of y, set $\dot{y} = 0$ in the complete differential equation in definition 21.1. This gives

$$0 + a\bar{y} = b$$

Solving gives

$$\bar{y} = \frac{b}{a}$$

as the steady-state value of y. As long as $a \neq 0$, as we have assumed, there does exist a steady-state value. In general, we will use the notation y_p to stand for a *particular* solution to the complete equation. However, in the case of an autonomous, linear, first-order differential equation with $a \neq 0$, we will always use the steady-state value as the particular solution. That is, we will use

$$y_p = \bar{y}$$

To confirm that $\bar{y}$ is indeed a particular solution to the complete differential equation, it is sufficient to substitute it back in and ensure that it satisfies the complete differential equation. To do this, note that the time-derivative of the solution, $\bar{y}$, is 0. Substitute this and $\bar{y}$ into the complete differential equation to get

$$0 + a\bar{y} = b$$

Substituting the expression for $\bar{y}$ confirms that $\bar{y}$ satisfies the complete differential equation and is therefore a solution.

The General Solution

Because of its importance for linear differential equations, theorem 21.2 restates why we go to the trouble of finding y_h and y_p:

Theorem 21.2 The solution to any linear differential equation is equal to the sum of the homogeneous solution and any particular solution to the complete differential equation. That is,

$$y = y_h + y_p \tag{21.5}$$

Proof

Let y_1 and y_2 be any two solutions of the complete differential equation, and define $z = y_1 - y_2$ as the difference between these two solutions. Then we can show that z is a solution to the homogeneous form of the differential equation. We do this as follows:

$$\begin{aligned} \dot{z} &= \dot{y}_1 - \dot{y}_2 \\ &= (-ay_1 + b) - (-ay_2 + b) \\ &= -a(y_1 - y_2) \\ &= -az \end{aligned}$$

Therefore

$$\dot{z} + az = 0$$

which means that z satisfies the homogeneous form of the differential equation and is therefore a solution to its homogeneous form. This result allows us to say the following: let y be the general solution to the complete differential equation and let y_p be any particular solution. Since y and y_p are two solutions of the complete equation, then we just proved that $z = y - y_p$ is a solution of its homogeneous form. But, since y_h is what we are calling a solution to the homogeneous form, it follows that $y_h = y - y_p$, and from this it follows that $y = y_h + y_p$. ■

If we now use this result to add y_h and y_p together, the general solution to the complete equation is given by:

Theorem 21.3 The general solution to the complete autonomous, linear, first-order differential equation is

$$y(t) = Ce^{-at} + \frac{b}{a} \tag{21.6}$$

Example 21.3 Solve the differential equation

$$\dot{y} + 2y = 8$$

Solution

The homogeneous form is

$$\dot{y} + 2y = 0$$

Therefore the solution to the homogeneous form is

$$y_h(t) = Ce^{-2t}$$

The particular solution we use is the steady-state value of y, which is obtained by solving

$$0 + 2\bar{y} = 8$$

This gives

$$\bar{y} = 4$$

The general solution to the complete differential equation is therefore

$$y(t) = Ce^{-2t} + 4$$

■

Example 21.4 Let $K(t)$ represent the quantity of capital available in an industry at time t. Suppose that capital depreciates at the rate δ and that the rate of investment in the industry is a constant $\bar{I}$. Derive and solve the differential equation implied by these statements.

Solution

If $\delta > 0$ is the constant *rate* of depreciation, then $\delta K(t)$ is the total amount of depreciation at time t. The change in the stock (quantity) of capital then is just $\bar{I} - \delta K$. The differential equation for capital is therefore

$$\dot{K} = \bar{I} - \delta K$$

The homogeneous form is

$$\dot{K} + \delta K = 0$$

The solution to the homogeneous form is

$$K_h = Ce^{-\delta t}$$

The particular solution we use is the steady-state solution, which is found by setting $\dot{K} = 0$ and solving. This gives

$$\bar{K} = \frac{\bar{I}}{\delta}$$

This tells us that if the capital stock ever reaches the level $\bar{I}/\delta$, depreciation will just equal new investment, so there will be no further increases or decreases in the size of the capital stock.

The general solution to the complete differential equation therefore is

$$K(t) = Ce^{-\delta t} + \frac{\bar{I}}{\delta}$$

■

The Initial-Value Problem

When we are also given the initial value for y, i.e., the value of y at $t = t_0$, where t_0 is the initial value of t, then the solution to the **initial-value problem** is one which both solves the differential equation and satisfies the initial value of y.

Example 21.5 Solve the differential equation

$$\dot{y} = 0.1y - 1$$

and ensure that it satisfies the initial condition $y(0) = 5$ at $t = 0$.

Solution

The solution to the homogeneous form of the differential equation is

$$y_h(t) = Ce^{0.1t}$$

The particular solution we use is the steady-state solution, which is

$$\bar{y} = 10$$

The general solution then is

$$y(t) = Ce^{0.1t} + 10$$

To find the solution that also satisfies the initial condition, evaluate the general solution at $t = 0$. This gives

$$y(0) = C + 10$$

To ensure that $y(0) = 5$, we set $C = -5$. The solution to this initial-value problem then is

$$y(t) = -5e^{0.1t} + 10$$ ■

As this example demonstrates, the arbitrary constant takes on a particular value when the solution is also required to satisfy an initial condition. This turns the general solution into a unique solution that is true only for the given initial condition. In economics, we are almost always interested in the unique solution to a differential equation that also satisfies a known initial condition.

In general, if the initial time is t_0, and the initial condition is $y(t_0) = y_0$, then the general solution at time t_0 becomes

$$y_0 = Ce^{-at_0} + \frac{b}{a}$$

which means that

$$C = \left(y_0 - \frac{b}{a}\right)e^{at_0}$$

The solution then becomes

$$y(t) = \left(y_0 - \frac{b}{a}\right)e^{at_0}e^{-at} + \frac{b}{a}$$

After simplifying, this becomes

$$y(t) = \left(y_0 - \frac{b}{a}\right)e^{-a(t-t_0)} + \frac{b}{a} \tag{21.7}$$

Usually we take $t_0 = 0$, in which case the expression simplifies to

$$y(t) = \left(y_0 - \frac{b}{a}\right)e^{-at} + \frac{b}{a} \tag{21.8}$$

Example 21.6 Find the solution to the differential equation in example 21.4, which also satisfies the condition that the capital stock at time $t = 0$ is known to have been K_0.

Solution

The solution to the differential equation itself is

$$K(t) = Ce^{-\delta t} + \frac{\bar{I}}{\delta}$$

Setting $t = 0$ and $K(0) = K_0$ gives

$$K_0 = C + \frac{\bar{I}}{\delta}$$

which means that

$$C = K_0 - \frac{\bar{I}}{\delta}$$

Substituting for C in the general solution and rearranging gives

$$K(t) = \left(K_0 - \frac{\bar{I}}{\delta}\right)e^{-\delta t} + \frac{\bar{I}}{\delta}$$

as the solution to this initial-value problem. ■

The Steady State and Convergence

One of the main concerns in economic dynamics is to determine whether the dynamic system under analysis [the variable $y(t)$ here] converges to the steady-state equilibrium or not. For example, if y represents price in a model of market equilibrium, it is important to know whether it converges over time to an equilibrium value.

Let us rewrite the solution to the initial-value problem as

$$y(t) = (y_0 - \bar{y})e^{-at} + \bar{y}$$

We wish to determine whether $y(t)$ converges to $\bar{y}$ as t goes to infinity. To do this, we calculate the limit of $y(t)$

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}[(y_0 - \bar{y})e^{-at} + \bar{y}]$$

The limiting behavior of $y(t)$ is determined entirely by the sign of a. If $a > 0$, the term e^{-at} tends toward zero as t gets larger. The limit of $y(t)$ therefore, as $t \to \infty$, is $\bar{y}$. On the other hand, if $a < 0$, the term e^{-at} tends to infinity as $t \to \infty$. Therefore $y(t)$ itself goes to infinity [or negative infinity depending on the sign of $(y_0 - \bar{y})$].

We summarize this important result as

Theorem 21.4 The solution to a linear, autonomous, first-order differential equation, $y(t)$, converges to the steady-state equilibrium, $\bar{y} = b/a$, no matter what the initial value, y_0, if and only if the coefficient in the differential equation is positive: $a > 0$.

Example 21.7 Determine whether the solution for $K(t)$ in example 21.6 converges to the steady state.

Solution

We found the solution to the differential equation for the capital stock to be

$$K(t) = \left(K_0 - \frac{\bar{I}}{\delta}\right)e^{-\delta t} + \frac{\bar{I}}{\delta}$$

where $\bar{I}/\delta$ is the steady-state value of the capital stock. Inspection of the solution indicates that $K(t)$ does converge to its steady state because the depreciation rate, δ, is positive. ■

Example 21.8 Let y stand for energy demand, and suppose that it grows according to

$$\dot{y} = 5y - 10$$

If energy demand has a value of 100 at time $t = 0$, determine whether it ever converges to a steady state.

Solution

Applying the techniques developed in this chapter gives the general solution

$$y(t) = Ce^{5t} + 2$$

At time $t = 0$ the solution must satisfy $y(0) = 100$. This means

$$100 = C + 2$$

Therefore $C = 98$. The solution becomes

$$y(t) = 98e^{5t} + 2$$

(Notice that we could have applied equation 21.8 to obtain this result directly.) Does energy demand in this model converge to its steady-state value $\bar{y} = 2$? Inspection reveals that $y(t)$ becomes infinitely large as t goes to infinity. Thus energy demand does not converge to its steady-state value in this model. ■

The Case of $a = 0$

If $a = 0$, the steady-state solution is undefined. What do we do in that case? Because the differential equation then becomes

$$\dot{y} = b$$

it can be integrated directly to get

$$y(t) = bt + C$$

A Walrusian Price Adjustment Model

In competitive markets, price is determined by supply and demand. Figure 21.1 shows a typical supply and demand diagram with the equilibrium price at p^e and the equilibrium quantity at q^e.

Is the point (q^e, p^e) a *stable* equilibrium? If it is, then even if the market is temporarily out of equilibrium for some reason, price and quantity will return to their equilibrium values. If it is an *unstable* equilibrium, then price and quantity will not return to the equilibrium if the market is ever put into disequilibrium.

For example, in figure 21.1 the equilibrium is stable because, if price were ever above p^e, there would be excess supply causing price to fall toward p^e, and if price were ever below p^e, there would be excess demand, causing price to rise toward p^e.

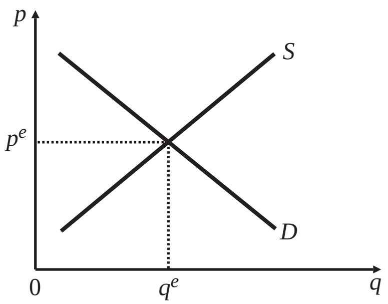


Figure 21.1 Stable equilibrium

In figure 21.2 we have allowed for the possibility of a downward-sloping supply curve. In this diagram the equilibrium is unstable: if price were ever above p^e, there would be excess demand, causing price to rise and therefore diverge from the equilibrium. Likewise, if price were ever below p^e, there would be excess supply, causing price to fall further away from the equilibrium.

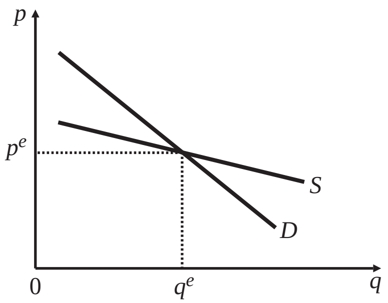


Figure 21.2 Unstable equilibrium

How can we determine in general whether an equilibrium is stable or not? An obvious way is to analyze the dynamics of the model of supply and demand. This is the task we turn to now.

Theory implies that price rises if there is excess demand and falls if there is excess supply. Let us suppose that the speed of price change is proportional to the supply-demand gap. This can be expressed algebraically as follows:

$$\dot{p} = \alpha(q^D - q^S), \qquad \alpha > 0 \tag{21.9}$$

where q^D and q^S are the quantities demanded and supplied, respectively, and α is a positive constant that determines the speed of price adjustments. We shall take the value of α as given. This formulation is consistent with economic theory because price rises ($\dot{p} > 0$) if quantity demanded exceeds quantity supplied and price falls ($\dot{p} < 0$) if quantity supplied exceeds quantity demanded.

Now let us suppose that the demand and supply functions are given by the following equations:

$$q^D = A + Bp$$
$$q^S = F + Gp$$

Setting $q^D = q^S$ and solving for the equilibrium price p^e gives us

$$p^e = \frac{A - F}{G - B}, \qquad G \neq B$$

We assume that $(A - F)/(G - B) > 0$. Normally we would expect demand to slope negatively, which requires that $B < 0$, and supply to slope positively, which requires that $G > 0$. However, we impose no restrictions on the signs of these coefficients at this point. Instead, we will see what restrictions we *must* place on them in order to ensure that price converges to the equilibrium price.

Substituting the supply and demand equations into equation (21.9) and rearranging gives the following linear differential equation with a constant coefficient and a constant term:

$$\dot{p} - \alpha(B - G)p = \alpha(A - F) \tag{21.10}$$

To solve, we begin with the homogeneous form, given by

$$\dot{p} - \alpha(B - G)p = 0$$

Rearranging gives

$$\frac{\dot{p}}{p} = \alpha(B - G)$$

Integrating both sides and taking antilogarithms gives

$$p_h(t) = Ce^{\alpha(B-G)t}$$

Next we use the steady-state solution as a particular solution. This is found by setting $\dot{p} = 0$ and solving. This gives

$$0 - \alpha(B - G)\bar{p} = \alpha(A - F)$$

and therefore

$$\bar{p} = \frac{A - F}{G - B}$$

which is the same as the equilibrium price. This is as it should be because $\dot{p} = 0$ when the quantities demanded and supplied are equal.

The general solution to the complete equation then is

$$p(t) = Ce^{\alpha(B-G)t} + \frac{A - F}{G - B}$$

Assuming that we know that at $t = 0$ the price is $p(0) = p_0$, the solution must satisfy

$$p_0 = C + \frac{A - F}{G - B}$$

After setting C to satisfy this condition, and using the expression for $\bar{p}$, the solution becomes

$$p(t) = [p_0 - \bar{p}]e^{-\alpha(G-B)t} + \bar{p} \tag{21.11}$$

Does market price converge to the steady-state equilibrium value? Inspection of the solution reveals that the necessary condition for price to converge is

$$G - B > 0$$

(since $\alpha > 0$). Only then will the exponent be negative, making the exponential term tend to zero as t tends to infinity.

Is this condition likely to be satisfied? It is if demand is negatively sloped ($B < 0$) and supply is positively sloped ($G > 0$), which is the usual case, depicted in figure 21.1. However, it is satisfied even if demand is positively sloped ($B > 0$) as long as G is also positive and is larger than B. It is even satisfied if supply is negatively sloped ($G < 0$), as long as demand is also negatively sloped and $G - B > 0$. It is not satisfied, however, if both curves are sloping opposite to their usual directions. That is, it is not possible to satisfy the stability condition if $B > 0$ and $G < 0$.

EXERCISES

1. Solve the following linear, first-order differential equations, and ensure that the initial conditions are satisfied.

(a) $\dot{y} - y = 0$ and $y(0) = 1$

(b) $\dot{y} + 3y = 12$ and $y(0) = 10$

(c) $2\dot{y} + \frac{1}{2}y = 12$ and $y(0) = 10$

(d) $\dot{y} = 5$ and $y(0) = 1$

(e) $\dot{y} = 6y - 6$ and $y(0) = 3$

2. Solve the following linear, first-order differential equations, and ensure that the initial conditions are satisfied.

(a) $10\dot{y} = 5y$ and $y(0) = 1$

(b) $4\dot{y} - 4y = -8$ and $y(0) = 10$

(c) $\dot{y} = 7$ and $y(0) = 2$

(d) $\dot{y} = 2y - 1$ and $y(0) = 5$

(e) $\dot{y} + 2y = 4$ and $y(0) = 3$

3. Let $p(t)$ represent the consumer price index. If the rate of inflation of the price index is constant at 5% (i.e., the growth rate of $p(t)$ is 5%), and the price index has a base value of 100 at time $t = 0$, solve for the expression showing the price index as a function of time.

4. If income per capita is growing at a constant rate of 3%, how long will it take to double?

5. Let $y(t)$ be the reserves of oil left in an oil pool at time t. Suppose that extraction reduces reserves at a constant rate equal to α. (The rate of decline of reserves is α.) If initial reserves at $t = 0$ were 500 million barrels, solve for the expression showing reserves as a function of time.

6. Use the information in exercise 5 and assume that $\alpha = 0.1$. Find the time at which 50% of oil reserves have been used up. Find the time at which 95% of oil reserves have been used up.

7. On the floor of the stock exchange, traders meet to buy and sell stock in various companies. Suppose that the change in the quantity sold of a particular stock depends on the gap between the offer price p^D and the asking price p^S. In particular, assume that $\dot{q} = \alpha(p^D - p^S)$. The inverse-demand function of the buyers is

$$p^D = a + bq$$

and the inverse-supply function of the sellers is

$$p^S = g + hq$$

If initial price is p_0 at $t = 0$, find the equilibrium quantity sold in this market and find the expression showing quantity sold as a function of time. What conditions on the parameters of the inverse demand and supply curves must hold for the equilibrium to be stable?

21.2 Nonautonomous Equations

If the coefficient, a, or the term, b, in a linear differential equation are functions of time, the equation is *nonautonomous*. In that case, the solution technique used in the previous section will not work in general. In this section, we explain a general solution technique that works for *any* linear, first-order differential equation.

Definition 21.4 The general form of the linear, first-order differential equation is

$$\dot{y} + a(t)y = b(t) \tag{21.12}$$

where $a(t)$ and $b(t)$ are known, continuous functions of t.

Notice that an equation of the form $g(t)\dot{y} + h(t)y = k(t)$ can always be put into the form in definition 21.4 by dividing through by $g(t)$, provided that $g(t)$ is never equal to zero.

One solution strategy for the general case is to use theorem 21.1, which applies to any linear, first-order differential equation, whether it is autonomous or not. We would begin by solving the homogeneous form, and then try to find a particular solution to the complete equation. The difficulty with this approach is that we can no longer use the easy-to-find steady-state solution as a particular solution. The reason is simply that when $a(t)$ and/or $b(t)$ are not constant, there does not exist, in general, any value of y at which $\dot{y} = 0$. Instead, we would have to use an alternative method for finding a particular solution, such as the *the method of undetermined coefficients*, first encountered in chapter 20.

An alternative solution strategy, and the one we adopt here, is to use the concept of an **integrating factor** that converts any linear, first-order differential equation into an equation that can be directly integrated to obtain the general solution. We begin by stating the general solution, and then explain how it is derived.

Theorem 21.5 The general solution to any linear, first-order differential equation is

$$y(t) = e^{-A(t)}\left[\int_0^t e^{A(t)}b(t)\,dt + C\right] \tag{21.13}$$

where $A(t) = \int a(t)\,dt$

Before working through some examples, let's see how the general solution is derived. The solution technique hinges on the fact that any linear, first-order differential equation can be multiplied by a known factor which converts it into a form that can be integrated directly.

Theorem 21.5 makes use of the term $A(t)$, which is defined as the integral of the coefficient $a(t)$. To see how the general solution is obtained, take the derivative of the function

$$e^{A(t)}y(t)$$

This gives

$$e^{A(t)}\left[\frac{dA(t)}{dt}y(t) + \dot{y}\right]$$

Since $a(t) = dA(t)/dt$, we have just shown that

$$\frac{d}{dt}\left[e^{A(t)}y(t)\right] = e^{A(t)}[a(t)y(t) + \dot{y}]$$

This result suggests we use the following technique for solving the differential equation: multiply through the equation by the term $e^{A(t)}$. This gives

$$e^{A(t)}[\dot{y} + a(t)y] = e^{A(t)}b(t)$$

We just showed, however, that the left-hand side is equal to

$$\frac{d}{dt}\left[e^{A(t)}y\right]$$

Therefore an equivalent way of expressing the differential equation is

$$\frac{d}{dt}\left[e^{A(t)}y(t)\right] = e^{A(t)}b(t)$$

In this form, the differential equation can be solved by direct integration. Doing so gives

$$e^{A(t)}y(t) = \int e^{A(t)}b(t)\,dt + C$$

Dividing by $e^{A(t)}$ gives

$$y(t) = e^{-A(t)}\left[\int e^{A(t)} b(t)\,dt + C\right]$$

The Integrating Factor

The trick we introduced above was to multiply both sides of the differential equation by a specific term. This made the differential equation amenable to direct integration. Terms that do this are called *integrating factors*. It was discovered long ago that there is an integrating factor for any linear differential equation, and it always takes the same form. We state this in the form of a theorem:

Theorem 21.6 The general form of the integrating factor for the linear, first-order differential equation is

$$e^{A(t)} \tag{21.14}$$

where $A(t) = \int a(t)\,dt$.

In the special case where the coefficient $a(t) \equiv a$, a constant, then the integrating factor becomes e^{at}.

Example 21.9 Solve the differential equation

$$\dot{y} - 2ty = bt$$

Solution

In this case we have $a(t) = -2t$. Therefore

$$A(t) = -\int 2t\,dt = -t^2$$

(We ignore the constant of integration in calculating $A(t)$ because it will be subsumed anyway in the overall constant of integration C.) Multiplying both sides of the differential equation by the integrating factor gives

$$e^{-t^2}[\dot{y} - 2ty] = e^{-t^2} bt$$

The equivalent expression is

$$\frac{d}{dt}e^{-t^2}y = e^{-t^2}bt$$

Integrating both sides gives

$$e^{-t^2}y = \int e^{-t^2}bt\,dt$$

which gives

$$y(t) = e^{t^2}\left[\int e^{-t^2}bt\,dt + C\right]$$

Careful inspection reveals that the solution can be carried a step further to give

$$\begin{aligned} y(t) &= e^{t^2}\left[-\frac{be^{-t^2}}{2} + C\right] \\ &= -\frac{b}{2} + Ce^{t^2} \end{aligned}$$

■

An Aggregate Growth Model with Technological Change

Consider a simple model of an economy that produces output using only capital. The amount produced is given by the production function which, we assume, shifts out over time because of technological change. Letting y denote output, and k denote capital, we assume that the technical relationship is

$$y = (a + \alpha k)t^{1/2}$$

This says that output is a linear function of the amount of capital in the economy k but that, over time, the entire function increases.

We further assume that a constant share s of output is saved, where $0 < s < 1$. Capital accumulation in the economy is equal to savings. Therefore

$$\dot{k} = sy$$

Substituting for y gives

$$\dot{k} = s(a + \alpha k)t^{1/2}$$

Rearranging gives

$$\dot{k} - s\alpha t^{1/2} k = sat^{1/2}$$

We wish to solve this to obtain an expression showing k as a function of t. The coefficient is $-s\alpha t^{1/2}$. Therefore

$$A(t) = -\int s\alpha t^{1/2}\, dt$$

which can be integrated to obtain

$$A(t) = -\frac{2}{3} s\alpha t^{3/2}$$

An equivalent expression for the differential equation then is

$$\frac{d}{dt}\left[e^{-2s\alpha t^{3/2}/3} k\right] = sat^{1/2} e^{-2s\alpha t^{3/2}/3}$$

Integrating both sides gives

$$e^{-2s\alpha t^{3/2}/3} k = sa \int t^{1/2} e^{-2s\alpha t^{3/2}/3}\, dt + C$$

The integration on the right-hand side can be taken further to get

$$e^{-2s\alpha t^{3/2}/3} k = sa\left[-\frac{e^{-2s\alpha t^{3/2}/3}}{s\alpha}\right] + C$$

Solving for $k(t)$ gives

$$k(t) = -\frac{a}{\alpha} + Ce^{2s\alpha t^{3/2}/3}$$

Assuming an initial condition of $k(0) = k_0$, we must set

$$C = \frac{a}{\alpha} + k_0$$

After substituting, we get

$$k(t) = -\frac{a}{\alpha} + \left(\frac{a}{\alpha} + k_0\right)e^{2s\alpha t^{3/2}/3}$$

as the explicit solution for capital in this model.

Although we would not expect $k(t)$ to converge to a steady state, given the nonautonomous nature of the model, it is reasonable to ask whether it converges to any particular growth path. Inspection of the solution reveals that this is not the case in this model because the exponential term in the solution grows without limit.

EXERCISES

1. Solve $\dot{y} - y = e^{2t}$ and $y(0) = 1$

2. Solve $\dot{y} + 4ty = 0$

3. Solve $\dot{y} - 3t^{-2}y = t^{-2}$

4. Suppose that an economics professor decides to deduct grade points for assignments turned in late in order to provide a stronger incentive to complete assignments on time. There is an initial penalty of 10 grade points for being late with an assignment. After that, the number of marks deducted, m, increases by the amount of the time late. In a continuous-time framework, this means that

$$\dot{m} = t$$

where $\dot{m}$ is the increase in the grade-point penalty and t is the amount of time late. Solve the differential equation to obtain an expression for the grade points deducted as a function of the time late.

5. For exercise 4, let one 24-hour period represent 1 unit of elapsed time. Find the value of the late penalty after 4 days have elapsed, and after 6 have elapsed.

6. Suppose that an instructor has adopted the practice of deducting grade points for late assignments. There is an automatic 5-grade-point penalty for being late and the grade points deducted increase by the square root of the amount of time late. Solve for the grade points deducted as a function of the time late.

7. Make the following two changes to the economic growth model. First, let the production function be $y = \alpha tk$. Second, let savings be given by $S = y - \beta t$. Assuming that $k_0 > \beta/\alpha$, solve to find the expression for the economy's capital stock as a function of time. What is the steady-state capital stock size? Does the capital stock converge to the steady state?

CHAPTER REVIEW

Key Concepts

convergence
general solution
homogeneous form
initial-value problem
integrating factor
particular solution
steady-state value

Review Questions

1. Describe the two-step procedure for obtaining the general solution to the complete, autonomous, linear, first-order differential equation.

2. Explain what is meant by the steady state of a linear, first-order differential equation.

3. Under what conditions is the particular solution equal to the steady-state solution?

4. State and explain the necessary and sufficient condition for convergence in an autonomous, linear, first-order differential equation.

5. Explain how to find the integrating factor for a first-order differential equation.

6. Explain how to use the integrating factor to help solve a linear, first-order differential equation.

Review Exercises

1. Suppose that energy consumption E grows at the rate of 2% and was equal to 2 units at time t_0. Solve for energy consumption as a function of time.

2. Suppose that national income Y grows at a rate of g and national population P grows at a rate of α. Define income per capita as $y = Y/P$. Solve for income per capita as a function of time.

3. Let $K(t)$ be the quantity of capital available in an industry at time t. If $K(0) = 500$ and if the depreciation rate is 5% and the investment rate is a constant 100 units, solve for the expression showing the quantity of capital available as a function of time. Find the steady-state capital stock, and show that $K(t)$ converges to the steady state.

4. A perfectly competitive firm maximizes profits by producing the quantity of output at which marginal cost equals price. Assuming that it takes time for the firm to change the quantity of output it produces, let it adjust its output level in proportion to the gap between price and marginal cost. That is, assume that

$$\dot{q} = \alpha[p - \text{MC}(q)]$$

where q is the quantity of output, p is the price of output, and $\text{MC}(q)$ is the marginal-cost function. Let the marginal-cost function be $\text{MC}(q) = aq$, where a is a positive constant. Solve this differential equation for $q(t)$, find the steady state for q, and determine whether $q(t)$ converges to the steady state.

5. Let K^* be a firm's desired capital stock. Although K^* will depend in general on a number of factors, such as the desired output level of the firm, the price of output, and the price of capital, assume these factors are constant so that K^* is also constant. Because it is costly for the firm to adjust its capital stock (e.g., workers have to be diverted from producing output to installing additional machinery), the firm adjusts gradually towards K^*. Specifically, suppose that the change in the capital stock K is proportional to the gap between the desired and the current capital stock, where α is the factor of proportionality that determines the speed of adjustment. Assuming that there is no depreciation, and assuming that the initial capital stock at $t = 0$ is K_0, write out the differential equation for the capital stock and solve it. What is the steady-state stock size? Does $K(t)$ converge to the steady-state stock size?

6. Use the information in review exercise 3, but now let investment grow at the rate g from an initial level of I_0 at time $t = 0$. Solve for the expression that shows capital as a function of time. Does $K(t)$ converge to a finite limit?

7. In the growth model of section 21.2, assume that $a = 0$ and solve the new differential equation without using the integrating factor.

8. Suppose that the demand for wheat is given by $q^D = A + Bp$ but the supply is given by

$$q^S = F + Gp + H(1 - e^{-\mu t}), \qquad \mu > 0$$

where the last term reflects productivity growth over time. That is, the supply curve for wheat shifts (smoothly) out over time because of continuous technological improvements in production. Assume that price adjusts if there is excess demand or supply according to $\dot{p} = \alpha(q^D - q^S)$. Solve this differential equation for $p(t)$ given $p(0) = p_0$.

Chapter 22 Nonlinear, First-Order Differential Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- Nonlinear Differential Equation Examples
- A Fishery Model with the Harvest Rate Proportional to Stock Size: Example
- An Explicit Solution of the Fishery Model: Example
- The Aggregate Growth Model with Technological Change and Zero Population Growth: Example
- Practice Exercises

In chapter 21 we saw that we could apply a single solution technique to solve any first-order differential equation that is *linear*. When the differential equation is *nonlinear*, however, no single solution technique will work in all cases. In fact only a few special classes of nonlinear, first-order differential equations can be solved at all. We will examine two of the more common classes in section 22.2. Even though solutions are known to exist for any nonlinear differential equation of the first order that satisfies certain continuity restrictions, it is simply not possible to find that solution in many cases. An alternative commonly used in economics is to do a qualitative analysis with the aid of *phase diagrams*. This technique is examined in section 22.1.

22.1 Autonomous Equations and Qualitative Analysis

Before attempting a qualitative analysis of an autonomous, nonlinear differential equation, it is essential to know the conditions under which a solution exists. We begin by defining the *initial-value problem* and then state the conditions for the existence of a solution.

Definition 22.1 The **initial-value problem** for an autonomous, nonlinear, first-order differential equation is expressed as

$$\dot{y} = g(y) \tag{22.1}$$

$$y(t_0) = y_0 \tag{22.2}$$

The function $g(y)$ must satisfy the properties stated in theorem 22.1 if we are to be guaranteed a solution to the initial-value problem.

Theorem 22.1 If the function g and its partial derivative $\partial g/\partial y$ are continuous in some closed rectangle containing the point (t_0, y_0), then in a neighborhood around t_0 contained in the rectangle, there is a unique solution $y = \xi(t)$ satisfying equations (22.1) and (22.2).

Theorem 22.1 assures us that a solution to the initial-value problem for nonlinear, first-order differential equations does indeed exist if $g(y)$ satisfies the continuity conditions stated. However, the knowledge that a solution exists by no means ensures that we will be able to find it. In fact, it is rare in economic applications to be able to find explicit solutions to nonlinear differential equations. Instead, it is common to conduct a **qualitative analysis,** often with the aid of a **phase diagram.**

Consider the nonlinear, first-order differential equation

$$\dot{y} = y - y^2 \tag{22.3}$$

It turns out that equation (22.3) is a member of one of the classes of nonlinear differential equations that we can solve using some specialized techniques, as we will later see. However, for now, let us suppose that we do not know these specialized techniques. It quickly becomes clear that our knowledge of solving *linear* differential equations will not help us find a solution to equation (22.3). Instead, we will attempt a qualitative analysis of the solution with the aid of a phase diagram. To this end, rearrange equation (22.3) to get

$$\dot{y} = y(1 - y) \tag{22.4}$$

For an autonomous, first-order differential equation, we can find the steady-state values of y by setting $\dot{y} = 0$. Doing this in equation (22.4) reveals that the system will be at rest ($\dot{y} = 0$) at

$$y = 0 \quad \text{and} \quad y = 1$$

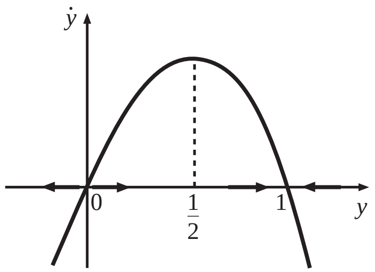


Figure 22.1 Phase diagram for equation (22.4)

This information is used to construct the phase diagram for equation (22.4) in figure 22.1.

The phase diagram of a single differential equation shows $\dot{y}$ as a function of y. We are particularly interested in finding the range of y values over which y is increasing over time ($\dot{y} > 0$) and the range over which y is decreasing over time ($\dot{y} < 0$). We can construct the diagram by graphing equation (22.4) with $\dot{y}$ as the variable on the vertical axis and y as the variable on the horizontal axis. For example, we have just determined that the value of $\dot{y}$ is zero when $y = 0$ and $y = 1$. This gives us two points on the graph. We now want to use equation (22.4) to plot the rest of the relationship. We can use standard qualitative graphing techniques to do this. First, determine the value of y at which the curve reaches an extreme value by taking the derivative of $\dot{y}$ with respect to y and setting it equal to zero:

$$\frac{d\dot{y}}{dy} = 1 - 2y = 0$$

This gives us $y = 1/2$. Determine whether the curve reaches a maximum or a minimum at $y = 1/2$ by taking the second derivative of $\dot{y}$ with respect to y:

$$\frac{d^2\dot{y}}{dy^2} = -2$$

The negative value of the second derivative tells us that the curve reaches a maximum at $y = 1/2$. Accordingly we know the curve must have the shape shown in figure 22.1.

A phase diagram helps us to determine the qualitative properties of a solution to a differential equation without actually having to solve it. In figure 22.1 we see that if the value of y ever reaches zero or one, it will remain at that value forever because $\dot{y} = 0$ at each of those values. What happens if y has a value between zero and one? The phase diagram shows that $\dot{y} > 0$ for $0 < y < 1$. This means that whenever the value of y is between zero and one, y will tend to *increase* over time. Of course, we already know that if it ever reaches one, the value will remain there.

What happens when $y > 1$? We see from figure 22.1 that $\dot{y} < 0$ for $y > 1$. If ever y exceeds one, its value will tend to *decrease* toward one. Finally, what happens if $y < 0$? We see that $\dot{y} < 0$ for $y < 0$, which means that if ever the value of y is less than zero, its value will continue to *decrease*.

All this information is placed in the diagram in the form of arrows, to remind us of the direction of motion of the variable y in the different regions. Our analysis, summarized in the **arrows of motion** in figure 22.1, indicates that the tendency is for y to converge to the value one if it starts at a value exceeding zero, diverge to $-\infty$ if it starts at a value less than zero, and to remain at zero if it starts at zero.

Thus, without solving the differential equation, we have obtained a great deal of insight into its solution by relying on a qualitative analysis with the aid of a phase diagram. Once we are given an initial condition for y, we can say with confidence how y will behave over time (increase or decrease) and to what value it will converge. In particular, provided that $y_0 > 0$, we are able to conclude that $y(t)$ will converge to the value $y = 1$.

Stability Analysis

We required two types of information in the above qualitative analysis. First, we needed to know the steady-state values for y (i.e., the values at which $\dot{y} = 0$). Second, we needed to know the arrows of motion, or the *motion* of the system around the steady-state values, to determine whether or not the system would converge to one of these steady-state values.

We found that the system converged to one of the steady-state values $(y = 1)$ but not the other $(y = 0)$. We could say that $y = 1$ is a *stable* equilibrium and $y = 0$ is an *unstable* equilibrium. What makes one stable and the other unstable? In terms of figure 22.1, the arrows of motion point toward the stable equilibrium but away from the unstable equilibrium. But what makes the arrows of motion do this? It turns out that the *slope* of the $\dot{y}$ line in figure 22.1, as it cuts through the equilibrium points, determines whether those equilibrium points will be stable. The slope is negative at the point $y = 1$ but it is positive at the point $y = 0$. Because the slope of the $\dot{y}$ line as it cuts through the equilibrium points is just the derivative of $\dot{y}$ with respect to y evaluated at the equilibrium points, we can state this new result as the following:

Theorem 22.2 A steady-state **equilibrium point** of a nonlinear, first-order differential equation is **stable** if the derivative $d\dot{y}/dy$ is negative at that point and **unstable** if the derivative is positive at that point.

Let us apply theorem 22.2 to the differential equation (22.4).

$$\text{At } y = 0 \quad \frac{d\dot{y}}{dy} = 1 - 2(0) = 1$$

$$\text{At } y = 1 \quad \frac{d\dot{y}}{dy} = 1 - 2(1) = -1$$

We conclude that $y = 0$ is an unstable equilibrium and $y = 1$ is a stable equilibrium.

The stability condition ($d\dot{y}/dy < 0$ at equilibrium values) is intuitively sensible. It says that if the system is at equilibrium and is somehow pushed away from the equilibrium point, the *motion* of y will be in the opposite direction (negative derivative) of the push. This means that the system will tend to return to equilibrium. If the motion of y is in the same direction (positive derivative) as the push (as it is at $y = 0$ in figure 22.1), the system will tend to move further away from equilibrium.

Finally it is worth noting that this stability condition does not apply only to *nonlinear*, first-order differential equations. It also applies to the *linear*, first-order differential equations with constant coefficients: $\dot{y} + ay = b$. Here the derivative $d\dot{y}/dy = -a$ is a constant. Thus the value of y will converge to the equilibrium (which is b/a) only if $a > 0$, which is the same convergence condition we derived in chapter 21.

Example 22.1 Find the steady-state points and determine their stability properties for the following:

$$\dot{y} = 3y^2 - 2y$$

Solution

The steady-state points occur where $\dot{y} = 0$. This gives

$$y(3y - 2) = 0$$

Therefore $y = 0$ and $y = 2/3$ are the steady-state points. Applying theorem 22.2 gives

$$\frac{d\dot{y}}{dy} = 6y - 2 = \begin{cases} -2 & \text{at } y = 0 \\ 2 & \text{at } y = 2/3 \end{cases}$$

Therefore $y = 0$ is stable, but $y = 2/3$ is unstable. ■

A Fishery Model with a Constant Harvest Rate

Suppose that a fish population grows according to the function

$$g(y) = 2y\left(1 - \frac{y}{2}\right)$$

where y is the stock of fish. The fish population is subjected to a constant level of harvesting by a fishing industry. If the harvest is a constant amount equal to $3/4$, will the fish population reach a steady-state (positive) size, in which case the harvest is a sustainable activity, or will the fish population decline and become extinct?

To answer these questions, we conduct a qualitative analysis of the dynamics of the population. The growth of the fish population is reduced by $3/4$ at each point in time, owing to the fishing industry. The change in the stock of fish is

$$\dot{y} = 2y\left(1 - \frac{y}{2}\right) - \frac{3}{4}$$

The steady-state value of the population occurs where $\dot{y} = 0$ which, after multiplying through the brackets, implies that

$$y^2 - 2y + \frac{3}{4} = 0$$

for which the solutions are

$$y = \frac{2 \pm \sqrt{4 - 4\frac{3}{4}}}{2}$$

Solving gives $y = 1/2$ and $y = 3/2$ as the two steady-state values of the fish population. Are they stable? We can determine the answer by applying theorem 22.2. First, take the derivative of $\dot{y}$ with respect to y. This gives us

$$\frac{d\dot{y}}{dy} = 2 - 2y$$

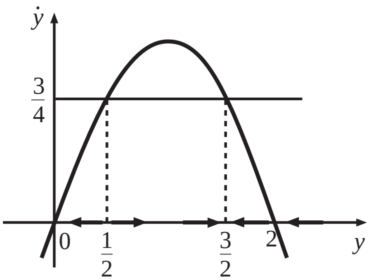


Figure 22.2 Phase diagram for the fishery model with a constant harvest rate

Evaluated at $y = 1/2$, the derivative equals 1, a positive value, indicating instability. Evaluated at $y = 3/2$, the derivative equals -1, a negative value, indicating stability. We conclude that $y = 1/2$ is an unstable steady-state stock size and $y = 3/2$ is a stable one.

How can we determine which equilibrium is likely to arise and whether the fish population is likely to become extinct? A qualitative analysis using the phase diagram in figure 22.2 will answer these questions.

Rather than graphing the curve for $\dot{y}$, we have graphed the curve for $g(y)$ and the line 3/4 separately. We need only remember that when $g(y) > 3/4$, the fish population grows; when $g(y) < 3/4$, the fish population declines; and when $g(y) = 3/4$, the fish population has reached a steady state.

The two steady-state values for y are shown in figure 22.2. What is the *motion* of the dynamic system? That is, in which direction does y move when it is not at one of the steady-state values? The phase diagram makes it clear that when y is between 0 and 1/2, then $g(y) < 3/4$, so the population declines, as indicated by the arrow of motion. When y is between 1/2 and 3/2, then $g(y) > 3/4$, so the population grows. Finally, when y is larger than 3/2, then $g(y) < 3/4$, so the population declines. We conclude that the fish population will reach the stable steady-state value of 3/2 and remain there forever, provided that $y_0 > 1/2$. On the other hand, if the initial population were less than 1/2, the population would decline to zero.

The Neoclassical Model of Economic Growth

What determines the long-run growth rate of an economy? The theory of economic growth was developed to answer this question and is still evolving. One of the important building blocks for modern refinements of the theory is the model that Nobel laureate Robert Solow developed in the 1950s.

In this model we assume that output per person in an economy can be expressed as a concave function of the capital–labor ratio

$$y = f(k)$$

where y is output per person and $k = K/L$ is the capital–labor ratio. Here K is the aggregate capital stock and L is total labor, which is equal to the total number of persons, assuming, for simplicity, that everyone works. Concavity implies that $f'(k) > 0$ and $f''(k) < 0$.

The economy's output can be consumed or saved. The economy's capital stock, K, increases by the amount of investment, which is by definition equal to the amount of output saved. We assume a constant savings rate, s. Hence

$$\dot{K} = sY$$

is the change in the capital stock, where Y is aggregate output. Since $k = K/L$, then

$$\begin{aligned} \dot{k} &= \frac{d}{dt}\left(\frac{K}{L}\right) \\ &= \frac{L\dot{K}}{L^2} - \frac{K\dot{L}}{L^2} \end{aligned}$$

Rearrange this to obtain

$$\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}$$

The labor force is assumed to grow at the constant rate n. Making this substitution and the substitution for $\dot{K}$ gives

$$\dot{k} = s\frac{Y}{L} - nk$$

Since $y = Y/L$ and using $y = f(k)$, this becomes

$$\dot{k} = sf(k) - nk$$

This nonlinear differential equation for the capital–labor ratio describes the growth of the economy in this model. We want to analyze this differential equation to see what the assumptions of the Solow model imply about the properties of the growth path of the economy. Since the labor force is assumed to grow exogenously, does this mean that the capital–labor ratio, k, and output per person, y, will fall over time, rise over time, or reach a steady state?

To construct the phase diagram, we first find the steady-state points by setting $\dot{k} = 0$. This implies that

$$\frac{f(k)}{k} = \frac{n}{s}, \qquad k \neq 0$$

Because $f(k)$ is a monotonic function, we know there is only one value of $k > 0$ that solves this equation. Call this value k^*. However, we also assume that $f(0) = 0$, so that $k = 0$ also is a steady-state equilibrium. Next we find the slope of the phase line and the points where the slope is zero:

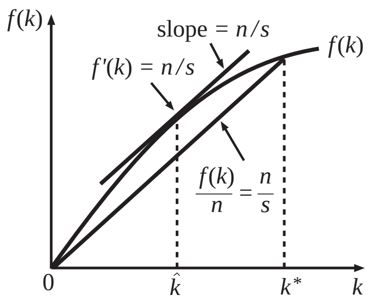


Figure 22.3 Concavity of $f(k)$ implies that $k^* > \hat{k}$

$$\frac{d\dot{k}}{dk} = sf'(k) - n$$

The slope equals zero at the point $\hat{k}$ where

$$f'(\hat{k}) = \frac{n}{s}$$

The second derivative

$$\frac{d^2\dot{k}}{dk^2} = sf''(k) < 0$$

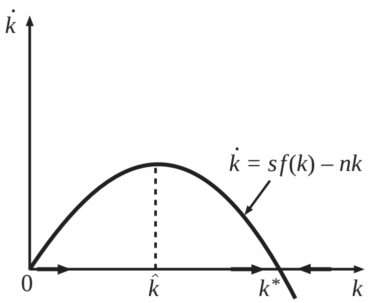


Figure 22.4 Phase diagram for the neoclassical growth model

tells us that the curve reaches a maximum at $\hat{k}$. Comparing the equations implicitly defining k^* and $\hat{k}$, we can conclude that $k^* > \hat{k}$. This follows from the fact that $f(k)$ is a concave function (see figure 22.3). The *average* value of the function, $f(k)/k$, is equal to n/s at k^*; the *marginal* value of the function, $f'(k)$, is equal to n/s at $\hat{k}$.

We can put all of this information together to generate the phase diagram shown in figure 22.4. The $\dot{k}$ curve equals zero at $k = 0$ and $k = k^*$ and reaches a maximum at $k = \hat{k}$. The arrows of motion shown follow from the fact that the curve indicates $\dot{k} > 0$ for $k < k^*$ and $\dot{k} < 0$ for $k > k^*$. We conclude that the point $k = 0$ is an unstable steady-state equilibrium point and the point $k = k^*$ is a stable steady-state equilibrium point. Theorem 22.2 confirms this conclusion: the slope of the $\dot{k}$ function is negative at k^*, indicating stability, and positive at zero, indicating instability.

The Solow growth model predicts the convergence of the capital–labor ratio to a constant k^*. Because the labor force is assumed to grow at the rate n, the model also predicts the convergence of the economy to a steady-state growth path on which output, capital, and the labor force all grow at the rate n. Empirical tests of this model indicate that it does a good job of explaining the growth rates of many countries but suggest that it is not able to account for *all* of the growth experienced in most countries. A response to this observation has been to augment the model by assuming exogenous technological change. Solving this augmented model is posed as an exercise at the end of this chapter. Although this augmented model is even better at explaining observed growth rates, many economists believe that relying on exogenous technological change is an unsatisfactory way to make the model do a better job of explaining observed growth rates. As a result recent research has been directed at making technological change endogenous in the growth model.

EXERCISES

1. Use a phase diagram and theorem 22.2 to conduct a qualitative analysis of $\dot{y} = -y + y^2 + 3/16$.

2. Use a phase diagram and theorem 22.2 to conduct a qualitative analysis of $\dot{y} = y - y^2 + 3/16$.

3. Use a phase diagram and theorem 22.2 to conduct a qualitative analysis of $\dot{y} = y - y^{1/2}$.

4. Use a phase diagram and theorem 22.2 to conduct a qualitative analysis of $\dot{y} = y^{1/2} - y$.

5. Quantity demanded in a market is given by

$$q^d = p^{-2}$$

and quantity supplied is given by

$$q^s = 8p$$

If price adjusts according to $\dot{p} = \alpha(q^d - q^s)$, where $\alpha > 0$ is a constant, conduct a qualitative analysis of the dynamics of market price.

6. Increases in carbon dioxide in the earth's atmosphere have been cited as a probable cause of "global warming." Let y represent the *stock* of carbon dioxide and let $x > 0$ (a constant) represent the *flow* of carbon dioxide emissions that come from industrial activity. Assume that the dynamics of y are given by

$$\dot{y} = x - y^a$$

where the term y^a represents the earth's capacity to remove carbon dioxide from the atmosphere and allow its absorption elsewhere (i.e., in trees, oceans). Conduct a qualitative analysis of this model, first for the case $a > 0$ and then for the case $a < 0$. Comment on your results.

22.2 Two Special Forms of Nonlinear, First-Order Differential Equations

We now turn to two classes of *nonautonomous*, nonlinear, first-order differential equations for which we can obtain explicit solutions.

Definition 22.2 The general form of the nonautonomous, first-order differential equation is

$$\dot{y} = f(t, y) \tag{22.5}$$

In this general form, the equation can be a nonlinear function of both y and t.

Bernoulli's Equation

The differential equation

$$\dot{y} + a(t)y = b(t)y^n \tag{22.6}$$

where $n \neq 0$ or 1, is known as **Bernoulli's equation.** (If $n = 0$ or 1, it is a linear differential equation, which we have already examined.) Note that the differential equation (22.3) is a Bernoulli equation with $n = 2$, $b(t) = -1$, and $a(t) = -1$.

If we assume that $a(t)$ and $b(t)$ are continuous on some interval $\mathcal{T}$, then we can transform equation (22.6) into a linear equation through a judicious *change of variable*. Provided that $y(t) \neq 0$, we can multiply through by y^{-n} to obtain

$$y^{-n}\dot{y} + a(t)y^{1-n} = b(t)$$

Now define a new variable $x = y^{1-n}$. This means that $\dot{x} = (1-n)y^{-n}\dot{y}$, and equation (22.6) becomes

$$\frac{\dot{x}}{1-n} + a(t)x = b(t)$$

which is now a linear equation that we can solve with linear techniques. Once we obtain the solution for $x(t)$, we can transform it back into $y(t)$. However, a cautionary note is necessary: this procedure is valid only if $y(t) \neq 0$ at every $t \in \mathcal{T}$.

An illustration of this technique in a fishery model can be found at http://mitpress.mit.edu/math_econ3.

Separable Equations

We can always write a function $f(t, y)$ in $\dot{y} = f(t, y)$ as the ratio of two other functions, $M(t, y)$ and $-N(t, y)$ (where the minus sign is for convenience, as we shall see). We can then write the differential equation (22.5) as

$$M(t, y) + N(t, y)\dot{y} = 0 \tag{22.7}$$

Definition 22.3 A nonlinear, first-order differential equation is **separable** if $M(t, y) = A(t)$, a function only of t, and $N(t, y) = B(y)$, a function only of y. A separable, nonlinear, first-order differential equation can therefore be written as

$$A(t) + B(y)\dot{y} = 0 \tag{22.8}$$

Why are separable equations singled out for special attention? The reason is that they can be solved by direct integration, as we now show. Rewrite equation (22.8) as

$$A(t)\,dt + B(y)\,dy = 0$$

This equation can be integrated directly to obtain

$$\int A(t)\,dt + \int B(y)\,dy = C \tag{22.9}$$

which is the solution to the separable differential equation. If, in addition, there is an initial condition $y(t_0) = y_0$ to be satisfied, then we can solve the initial-value problem in the usual way by applying the initial condition to equation (22.9) to evaluate C.

In practice, it may be impossible to perform the required integration in equation (22.9), which means we cannot obtain an *explicit* solution for $y(t)$, but the growth model we provide at http://mitpress.mit.edu/math_econ3 is an example where an explicit solution is possible.

EXERCISES

1. Solve

$$\dot{y} + 2y = \frac{3}{y}$$

2. Suppose that there are X farms in a large geographical region. At time 0, a technological innovation is introduced to the region and spreads gradually from farm to farm. Let $N(t)$ be the number of farms that have adopted the innovation by time t, and asume that the rate of adoption is proportional to the product of $N(t)$ and $X - N(t)$:

$$\dot{N} = \alpha N(X - N)$$

Solve for $N(t)$.

3. Solve

$$\dot{y} = \frac{t^2}{y}$$

4. Solve

$$\dot{y} = \frac{t^2}{y(1 + t^3)}$$

5. Solve

$$\dot{y} = \frac{-t}{y^2}$$

6. Suppose that the growth rate of a population falls over time so that the population growth is given by

$$\dot{y} = \frac{t}{t^2 + 1} y$$

Solve for $y(t)$.

CHAPTER REVIEW

Key Concepts

arrows of motion
Bernoulli's equation
initial-value problem
phase diagram
qualitative analysis
separable equation
stable steady-state equilibrium point
unstable steady-state equilibrium point

Review Questions

1. Explain how a phase diagram for a differential equation differs from a phase diagram for a difference equation.
2. What is the relationship between theorem 22.2 and the phase diagram?
3. Why is it important to know the convergence property of a steady-state equilibrium point?

Review Exercises

1. Draw a phase diagram for *and* solve $\dot{y} = 2y - 6y^2$.
2. Draw a phase diagram for *and* solve $\dot{y} = ry(1 - ky)$.
3. Solve the separable, nonlinear differential equation $\dot{y} = t/y$.
4. Solve, as far as possible, the separable, nonlinear differential equation

$$\dot{y} = \frac{t + 1}{y^4 + 1}$$

5. A perfectly competitive firm maximizes profits by producing the quantity of output at which marginal cost equals price. Assuming that it takes time for the firm to change the quantity of output it produces, let the firm adjust its output level in proportion to the gap between price and marginal cost. That is, assume that

$$\dot{q} = \alpha[p - \text{MC}(q)]$$

where q is the quantity of output, p is the price of output, and $mc(q)$ is the marginal-cost function. Let the marginal-cost function be $\text{MC}(q) = aq^2$, where a is a positive constant. Draw the phase diagram for this nonlinear differential equation for $q(t)$. Determine whether the steady-state equilibrium point is stable using the phase diagram. Confirm your finding using theorem 22.2.

6. Using the same model as in exercise 5, but now assuming that the marginal-cost function is given by $\text{MC}(q) = aq^{-1}$, draw a phase diagram and determine whether the steady-state equilibrium point is stable using the phase diagram and theorem 22.2.

Chapter 23 Linear, Second-Order Differential Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Walrusian Price-Adjustment Model with Entry and Exit
- Practice Exercises

Until now we have confined our analysis of differential equations to those of the first order. In this chapter we will examine linear, second-order differential equations with constant coefficients. We focus our attention on the *autonomous* case in section 23.1 and consider a special *nonautonomous* case in section 23.2.

23.1 The Linear, Autonomous, Second-Order Differential Equation

We begin by explaining how to solve a linear, autonomous, second-order differential equation.

Definition 23.1

The **linear, autonomous, second-order differential equation** (constant coefficients and a constant term) is expressed as

$$\ddot{y} + a_1 \dot{y} + a_2 y = b \tag{23.1}$$

Equation (23.1) is linear because y, $\dot{y}$, and $\ddot{y}$ are not raised to any power other than one. It is autonomous because it has constant coefficients, a_1 and a_2, and a constant term, b. If the coefficients or the term vary with t, then the equation is nonautonomous. In section 23.2 we consider the case of a variable term.

Rather than try to solve the complete equation in one step, we exploit the fact that the complete solution to a linear differential equation is equal to the sum of the solution to its homogeneous form and a particular solution to the complete equation. In symbols

$$y = y_h + y_p \tag{23.2}$$

where y is the complete solution, y_h is the solution to the homogeneous form, and y_p is a particular solution. Many readers are probably familiar with this technique because we used it in earlier chapters to solve linear, first-order differential equations and linear difference equations.

The General Solution to the Homogeneous Equation

The first step in solving the linear, autonomous, second-order differential equation is to solve the homogeneous form of equation (23.1).

Definition 23.2 The *homogeneous* form of the linear, second-order differential equation with constant coefficients is

$$\ddot{y} + a_1\dot{y} + a_2 y = 0 \tag{23.3}$$

To solve this *second-order* differential equation, we will make use of what we already know about the solution to its *first-order* counterpart, the linear, homogeneous, first-order differential equation with a constant coefficient. In chapter 21 we learned that solutions to equations of this kind are of the form

$$y(t) = Ae^{rt} \tag{23.4}$$

where the values for A and r are determined by initial conditions and the coefficient of the equation. A reasonable hypothesis is that solutions to second-order equations are of the same form. Specifically, let us conjecture that equation (23.4) is a solution to equation (23.3). If we are right, then equation (23.4) must *satisfy* equation (23.3). To see if it does, differentiate equation (23.4) to get

$$\dot{y} = rAe^{rt} \tag{23.5}$$

and differentiate again to get

$$\ddot{y} = r^2Ae^{rt} \tag{23.6}$$

Substitute the hypothesized solution and its derivatives, equations (23.4) to (23.6), into the left-hand side of equation (23.3) and check that it satisfies the equality in equation (23.3). The left-hand side becomes

$$\begin{aligned}\ddot{y} + a_1\dot{y} + a_2 y &= r^2 Ae^{rt} + a_1 r Ae^{rt} + a_2 Ae^{rt}\\ &= Ae^{rt}(r^2 + a_1 r + a_2)\end{aligned}$$

Ruling out the special (and trivial) case of $A = 0$, our conjecture is correct if the expression in brackets is equal to zero, for then our conjectured solution satisfies equation (23.3). But since r is, as yet, an unspecified parameter in the solution, we are free to choose it to make the expression in brackets identically equal to zero. In other words, if we choose r to satisfy

$$r^2 + a_1 r + a_2 = 0 \tag{23.7}$$

then equation (23.4) is indeed a solution to equation (23.3).

Equation (23.7), known as the characteristic equation, plays an important role in finding the solution to equation (23.3).

Definition 23.3 The **characteristic equation** of the linear second-order differential equation with constant coefficients is

$$r^2 + a_1 r + a_2 = 0$$

The values of r that solve the characteristic equation are known as the **characteristic roots** (or just the *roots*) or **eigenvalues** of the characteristic equation. Since the characteristic equation is a quadratic in the case of a second-order differential equation, there are two roots: we will call these r_1 and r_2. Their values are

$$r_1,\ r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{23.8}$$

If the two roots that solve the characteristic equation are different, we actually have *two* different solutions to equation (23.3). They are

$$y_1 = A_1 e^{r_1 t} \quad \text{and} \quad y_2 = A_2 e^{r_2 t} \tag{23.9}$$

Let us pause to verify that each of these is a solution to equation (23.3). We do this for the first solution, y_1, and leave it to the reader to verify that y_2 is also a solution. Substitute y_1 and its first and second derivatives into equation (23.3) and check that equation (23.3) is then satisfied (i.e., check that the left-hand side does equal zero). The first derivative is

$$\dot{y}_1 = r_1 A_1 e^{r_1 t}$$

and the second derivative is

$$\ddot{y} = r_1^2 A_1 e^{r_1 t}$$

After making substitutions, the left-hand side of equation (23.3) becomes

$$\begin{aligned} \ddot{y} + a_1 \dot{y} + a_2 y &= r_1^2 A_1 e^{r_1 t} + a_1 \left(r_1 A_1 e^{r_1 t}\right) + a_2 \left(A_1 e^{r_1 t}\right) \\ &= \left(r_1^2 + a_1 r_1 + a_2\right) A_1 e^{r_1 t} \end{aligned}$$

But this is equal to zero, and therefore satisfies equation (23.3), because the expression in brackets is the characteristic equation and r_1 is chosen to set this expression equal to zero. Therefore we are certain that y_1 is a solution to equation (23.3).

We have found there are two distinct solutions to the homogeneous form of a linear, second-order differential equation with constant coefficients. However, we are trying to find one *general* solution, namely a solution that represents every possible solution. It turns out that we actually have found it. Theorem 23.1 provides the explanation.

Theorem 23.1 Let y_1 and y_2 be two distinct solutions to the differential equation (23.3). If c_1 and c_2 are any two constants, then the function $y = c_1 y_1 + c_2 y_2$ is a solution to equation (23.3). Conversely, if y is *any* solution to equation (23.3), then there are unique constants, c_1 and c_2, such that $y = c_1 y_1 + c_2 y_2$.

Proof

We prove only the first part of the theorem. If y_1 and y_2 are solutions to equation (23.3), then it follows that

$$\ddot{y}_1 + a_1 \dot{y}_1 + a_2 y_1 = \ddot{y}_2 + a_1 \dot{y}_2 + a_2 y_2 = 0$$

We are given that $y = c_1 y_1 + c_2 y_2$. If y is a solution, then it must be true that

$$\ddot{y} + a_1 \dot{y} + a_2 y = 0$$

Let us see if this is true. Use the definition of y to obtain $\ddot{y}$ and $\dot{y}$, then substitute to see if this equation is satisfied. Differentiating the definition of y gives

$$\dot{y} = c_1 \dot{y}_1 + c_2 \dot{y}_2$$

and differentiating again gives

$$\ddot{y} = c_1\ddot{y}_1 + c_2\ddot{y}_2$$

Substituting gives

$$\begin{aligned}\ddot{y} + a_1\dot{y} + a_2 y &= (c_1\ddot{y}_1 + c_2\ddot{y}_2) + a_1(c_1\dot{y}_1 + c_2\dot{y}_2) + a_2(c_1 y_1 + c_2 y_2)\\ &= c_1(\ddot{y}_1 + a_1\dot{y}_1 + a_2 y_1) + c_2(\ddot{y}_2 + a_1\dot{y}_2 + a_2 y_2)\\ &= 0\end{aligned}$$

Therefore y is a solution to equation (23.3). The second part of the theorem says that any solution to the differential equation (e.g., one satisfying specific initial conditions) can be expressed as a linear combination of y_1 and y_2 through a suitable choice of the constants c_1 and c_2. Doing this requires that y_1 and y_2 be *distinct*, by which we mean they must be *linearly independent*. ■

The implication of theorem 23.1 is that the general solution to the homogeneous form in equation (23.3) is

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{23.10}$$

where we have defined new constants: $C_1 = c_1 A_1$ and $C_2 = c_2 A_2$. It is apparent now that we actually need two distinct solutions, y_1 and y_2, in order to form the general solution. A rationale for this is that because two constants are lost in going from a function $y(t)$ to its second derivative, we actually need *two* distinct solutions so that these two constants can be recovered in solving the second-order differential equation.

If $r_1 = r_2$ (the case of repeated roots that occurs if $a_1^2 - 4a_2 = 0$), we do not have two distinct solutions, so theorem 23.1 would not seem to provide the general solution. However, it is still possible to find two distinct solutions. Rather than derive a second distinct solution, we will simply state the result and then verify that it is correct.

If $r_1 = r_2 = r$, the two distinct solutions to equation (23.3) are given by

$$y_1 = A_1 e^{rt} \quad \text{and} \quad y_2 = tA_2 e^{rt} \tag{23.11}$$

These solutions are *distinct* because they are linearly independent. ($A_1 e^{rt}$ cannot be made equal to $tA_2 e^{rt}$ by multiplying it by a constant coefficient.) It is also possible to verify that the second solution will satisfy equation (23.3). To see this, first note that the case of repeated roots arises only when

$$a_1^2 - 4a_2 = 0$$

which means the solution to the characteristic equation is $r_1 = r_2 = r = -a_1/2$. Next differentiate y_2 to get

$$\dot{y}_2 = A_2 e^{rt} + rtA_2 e^{rt}$$

Differentiate again to get

$$\ddot{y}_2 = rA_2 e^{rt} + rA_2 e^{rt} + r^2 tA_2 e^{rt}$$

Substitute y_2 and its two derivatives (and the value of r) into equation (23.3) and check. The left-hand side of equation (23.3) becomes

$$\begin{aligned}
\ddot{y} + a_1\dot{y} + a_2 y &= 2rA_2 e^{rt} + r^2 tA_2 e^{rt} + a_1(A_2 e^{rt} + rtA_2 e^{rt}) + a_2(rtA_2 e^{rt}) \\
&= A_2 e^{rt}[t(r^2 + a_1 r + a_2) + 2r + a_1] \\
&= A_2 e^{rt}\left[t\left(a_1^2/4 - a_1^2/2 + a_2\right) - a_1 + a_1\right] \\
&= A_2 e^{rt}\left[\frac{t}{4}\left(4a_2 - a_1^2\right)\right]
\end{aligned}$$

The last expression is equal to zero because $a_1^2 - 4a_2 = 0$ in the case of repeated roots. This proves that y_2 is a solution for equation (23.3).

The results we have obtained to this point are summarized in theorem 23.2.

Theorem 23.2 The solution to the homogeneous form of the linear, second-order differential equation with constant coefficients is

$$y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad \text{if } r_1 \neq r_2 \tag{23.12}$$

$$y_h(t) = C_1 e^{rt} + C_2 te^{rt} \quad \text{if } r_1 = r_2 = r \tag{23.13}$$

where C_1 and C_2 are arbitrary constants of integration, the h subscript indicates the solution to the homogeneous equation, and r_1 and r_2 are given by

$$r_1,\ r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{23.14}$$

Example 23.1 Solve the following homogeneous differential equation:

$$\ddot{y} + \frac{1}{2}\dot{y} + \frac{3}{64} = 0$$

Solution

The characteristic equation is

$$r^2 + \frac{1}{2}r + \frac{3}{64} = 0$$

for which the roots are

$$r_1,\ r_2 = \frac{-1/2 \pm \sqrt{1/4 - 4 \times (3/64)}}{2}$$

The roots are $r_1 = -1/8$ and $r_2 = -3/8$. By theorem 23.2, the solution to the differential equation is

$$y_h(t) = C_1 e^{-t/8} + C_2 e^{-3t/8}$$

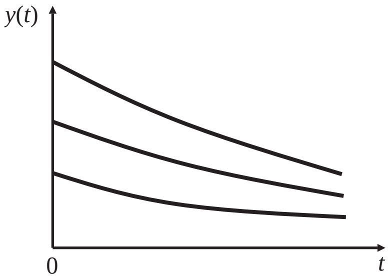


Figure 23.1 Representative trajectories for the solution to example 23.1

Figure 23.1 shows some representative graphs of this solution for different values of the constants of integration. After explaining how to obtain the complete solution, we will explain how the constants of integration are determined when initial conditions are specified as part of the problem. ■

Example 23.2 Solve the following homogeneous differential equation:

$$4\ddot{y} - 8\dot{y} + 3 = 0$$

Solution

After dividing through by 4, the characteristic equation is

$$r^2 - 2r + \frac{3}{4} = 0$$

for which the roots are $r_1 = 1/2$ and $r_2 = 3/2$. By theorem 23.2, the solution to the differential equation is

$$y_h(t) = C_1 e^{t/2} + C_2 e^{3t/2}$$

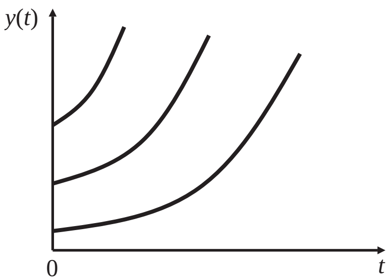


Figure 23.2 Representative trajectories for the solution to example 23.2

Figure 23.2 shows some representative graphs of this solution for different values of the constants of integration. ■

Example 23.3 Solve the following homogeneous differential equation:

$$\ddot{y} - 4\dot{y} + 4 = 0$$

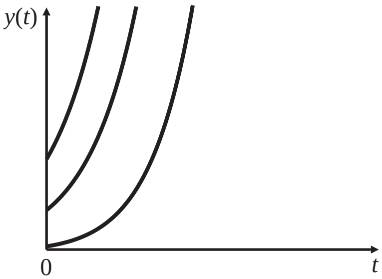


Figure 23.3 Representative trajectories for the solution to example 23.3

Solution

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

for which the roots are $r_1 = r_2 = 2$. Using the solution for the case of equal roots in theorem 23.2, the solution to the differential equation is

$$y_h(t) = C_1 e^{2t} + C_2 t e^{2t}$$

Figure 23.3 shows graphs of this solution. ■

Complex Roots

When $a_1^2 - 4a_2 < 0$, the characteristic roots are complex numbers. In this case the solution in theorem 23.2 still applies but needs to be expressed differently. To see how this is done, first write the algebraic solution to the characteristic equation as

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{(-1)\left(4a_2 - a_1^2\right)}}{2}$$
$$= \frac{-a_1 \pm \sqrt{-1}\sqrt{4a_2 - a_1^2}}{2}$$

Using the concept of the imaginary number, $i = \sqrt{-1}$, we can then write the roots of the characteristic equation as the conjugate complex numbers

$$r_1, r_2 = h \pm vi$$

where

$$h = \frac{-a_1}{2} \quad \text{and} \quad v = \frac{\sqrt{4a_2 - a_1^2}}{2}$$

We can now express the solution as

$$y_h = C_1 e^{(h+vi)t} + C_2 e^{(h-vi)t} = e^{ht}(C_1 e^{vit} + C_2 e^{-vit})$$

As shown in the review section on complex numbers and circular functions at http://mitpress.mit.edu/math_econ3, we can use *Euler's formula* to express the imaginary exponential functions, e^{vit} and e^{-vit}, as

$$e^{i(vt)} = \cos vt + i \sin vt$$
$$e^{-i(vt)} = \cos vt - i \sin vt$$

Using these relationships, we write the solution to the homogeneous form of the differential equation as

$$y_h = e^{ht}[C_1(\cos vt + i \sin vt) + C_2(\cos vt - i \sin vt)]$$

or as

$$y_h = e^{ht}(C_1 + C_2) \cos vt + e^{ht}(C_1 - C_2)i \sin vt$$

Since $(C_1 + C_2)$ and $(C_1 - C_2)i$ are arbitrary constants, we can rename them as A_1 and A_2

$$A_1 = C_1 + C_2 \quad \text{and} \quad A_2 = (C_1 - C_2)i$$

An important point is that A_1 and A_2 are real-valued. The reason is that C_1 and C_2, like the roots, are conjugate complex numbers. As shown in the review section at http://mitpress.mit.edu/math_econ3, the sum of two conjugate complex numbers is always a real number; and the product of i and the difference between two conjugate complex numbers is also a real number. Consequently we obtain a real-valued solution to the differential equation even though the roots are complex numbers.

Theorem 23.3 If the roots of the characteristic equation are complex numbers, the solution to the homogeneous form of the linear, second-order differential equation with constant coefficients can be expressed as

$$y_h = A_1 e^{ht} \cos vt + A_2 e^{ht} \sin vt \tag{23.15}$$

where

$$h = \frac{-a_1}{2} \quad \text{and} \quad v = \frac{\sqrt{4a_2 - a_1^2}}{2}$$

When the roots of the characteristic equation are complex-valued, the solution involves circular, or trigonometric, functions (sine and cosine) of t. This has interesting applications to economics because the circular functions are oscillating functions of t, which leads to cyclical behavior for $y(t)$, much like many real-world economic variables.

Example 23.4 Solve the following homogeneous differential equation:

$$\ddot{y} + 2\dot{y} + 5y = 0$$

Solution

The characteristic equation is

$$r^2 + 2r + 5 = 0$$

for which the roots are

$$\begin{aligned} r_1, r_2 &= \frac{-2}{2} \pm \frac{1}{2}\sqrt{4 - 4 \times 5} \\ &= -1 \pm 2i \end{aligned}$$

In this case $h = -1$ and $v = 2i$. By theorem 23.3, the solution is

$$y_h(t) = A_1 e^{-t} \cos 2t + A_2 e^{-t} \sin 2t$$

Figure 23.4 shows a representative trajectory for this solution. ■

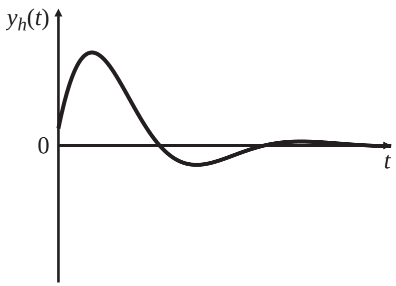


Figure 23.4 Representative trajectory for example 23.4

The Particular Solution

We have found the solution to the homogeneous form of equation (23.1). If we can now find a *particular* solution to equation (23.1), we will be able to obtain the complete solution by adding these two solutions together.

As in earlier chapters, the **particular solution** we look for in the case of an autonomous equation is the *steady-state solution* for y. If $\bar{y}$ is a steady-state value, it must be true that $\ddot{y} = \dot{y} = 0$ at this value. Set $\ddot{y} = \dot{y} = 0$ in the complete differential equation in equation (23.1) to solve for $\bar{y}$. This gives

$$\bar{y} = \frac{b}{a_2}, \qquad a_2 \neq 0$$

The steady-state solution exists as long as $a_2 \neq 0$. For the remainder of this section, we assume $a_2 \neq 0$. The steady-state solution then serves as the particular solution we require:

$$y_p = b/a_2$$

where the p subscript reminds us that this is the particular solution. If $a_2 = 0$, the steady-state value is undefined and we must instead use a different technique, explained in section 23.2, to find a particular solution.

Example 23.5 Find the particular solution for the following differential equation:

$$\ddot{y} - 5\dot{y} + 2y = 10$$

Solution

Find the steady-state value by setting $\ddot{y} = \dot{y} = 0$, and use this as the particular solution. This gives

$$y_p = 5$$ ■

Example 23.6 Find the particular solution for the following differential equation:

$$2\ddot{y} - 3y = -21$$

Solution

Find the steady-state value by setting $\ddot{y} = 0$, and use this as the particular solution. This gives

$$y_p = 7$$ ■

The Complete Solution

The **complete solution** to a second-order differential equation is the *sum* of the homogeneous solution and a particular solution

$$y = y_h + y_p$$

In theorem 23.4 we use this result to pull together all we have derived so far in this chapter.

Theorem 23.4 The complete solution to the linear, autonomous, second-order linear differential equation (23.1) (constant coefficients and a constant term) is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{b}{a_2} \quad \text{if } r_1 \neq r_2$$

$$y(t) = C_1 e^{rt} + tC_2 e^{rt} + \frac{b}{a_2} \quad \text{if } r_1 = r_2 = r$$

$$y(t) = e^{ht}\,(A_1 \cos vt + A_2 \sin vt) + \frac{b}{a_2} \quad \text{if roots are complex numbers}$$

where $r_1,\ r_2 = -a_1/2 \pm \sqrt{a_1^2 - 4a_2}/2$, $h = -a_1/2$, and $v = \sqrt{4a_2 - a_1^2}/2$.

A Price-Adjustment Model with Inventories

In chapter 21 we studied the dynamics of price adjustments in a model of a competitive market. We supposed that price adjusts in response to the demand-supply gap as follows:

$$\dot{p} = \alpha(q^D - q^S); \quad \alpha > 0$$

where q^D and q^S are the quantities demanded and supplied. However, this model neglects the inventory of unsold merchandise that arises when there is excess supply. How will the dynamics of price adjustment be affected if we take account of this inventory? To answer this question, we assume there is downward pressure on price not only when there is excess supply being produced at the current price, but also when there is an inventory of unsold merchandise. This idea may be expressed algebraically as

$$\dot{p} = \alpha(q^D - q^S) - \beta \int_0^t [q^S(s) - q^D(s)]\, ds, \qquad \alpha > 0, \beta > 0 \qquad (23.16)$$

The first term is the demand-supply gap at the current price. With $\alpha > 0$, this term causes price to adjust upward when there is excess demand and downward when there is excess supply. The second term is the integral of (accumulated stock of) past differences between quantity supplied and demanded. As such, it is the inventory of unsold merchandise. With $\beta > 0$, this term causes price to adjust downward when the inventory is greater than zero. To simplify our analysis of this problem, we assume that the inventory of unsold merchandise is always non-negative so we can avoid having to introduce a nonnegativity constraint on the inventory. Doing so would make the problem more complicated and would divert our attention from the main task. As a result equation (23.16) is the differential equation we wish to analyze. Although it appears to be a first-order differential equation, it really involves two orders of differentials, owing to the presence of the time integrals of supply and demand. To see how this leads to the second-order differential equation we wish to solve, take the time derivative of both sides of equation (23.16). This gives

$$\ddot{p} = \alpha(\dot{q}^D - \dot{q}^S) - \beta[q^S(t) - q^D(t)]$$

If, as before, we assume that the demand curve is given by $q^D = A + Bp$ and the supply curve is given by $q^S = F + Gp$, then this model of price adjustment is described by the following linear, second-order differential equation:

$$\ddot{p} + \alpha(G - B)\dot{p} + \beta(G - B)p = \beta(A - F)$$

To solve it, we begin with the homogeneous form

$$\ddot{p} + \alpha(G - B)\dot{p} + \beta(G - B)p = 0$$

The characteristic equation is

$$r^2 + \alpha(G - B)r + \beta(G - B) = 0$$

and the characteristic roots are

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

where $a_1 = \alpha(G - B)$ and $a_2 = \beta(G - B)$.

Assuming, for now, that the roots are real-valued and different, we write the solution to the homogeneous form as

$$p_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Now we can find the particular solution. The particular solution we use is the steady-state value of price. This is found by setting $\ddot{p} = \dot{p} = 0$ in the complete equation, then solving. This gives

$$p_p = \bar{p} = \frac{A - F}{G - B}, \qquad G - B \neq 0$$

By theorem 23.3, the complete solution to the linear, second-order differential equation for price in this model, for the case of distinct, real-valued roots, is

$$p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{A - F}{G - B}$$

Example 23.7 **Roots Real and Distinct**

Use the following parameter values to solve the differential equation for price in the price-adjustment model: $\alpha = 0.25$, $\beta = 0.2$, $G - B = 20$, and $A - F = 100$.

Solution

We obtain $a_1 = 5$, $a_2 = 4$, and $\bar{p} = 5$. The roots then are $r_1 = -1$ and $r_2 = -4$, and the complete solution is

$$p(t) = C_1 e^{-t} + C_2 e^{-4t} + 5$$ ■

Example 23.8 **Roots Real and Equal**

Use the following parameter values to solve the differential equation for price in the price-adjustment model: $\alpha = 0.2$, $\beta = 0.2$, $G - B = 20$, and $A - F = 100$.

Solution

The roots are equal, $r = -2$, and the complete solution is

$$p(t) = C_1 e^{-2t} + C_2 t e^{-2t} + 5$$ ■

Example 23.9 **Complex Roots**

Use the following parameter values to solve the differential equation for price in the price-adjustment model: $\alpha = 0.05$, $\beta = 0.5$, $G - B = 20$, and $A - F = 100$.

Solution

We obtain $a_1 = 1$ and $a_2 = 10$. The roots are $r_1, r_2 = -0.5 \pm 1.5i$. The complete solution is

$$p(t) = 5 + e^{-0.5t}[A_1 \cos(1.5t) + A_2 \sin(1.5t)]$$ ■

The Constants of Integration

We can determine the constants of integration in the complete solution from initial conditions. Since there are two constants, we require two initial conditions. Usually initial conditions given are the values of y and $\dot{y}$ at $t = 0$.

Example 23.10 If $p(0) = 30$ and $\dot{p}(0) = -3$, solve for the constants of integration in the following complete solution for price in the price-adjustment model:

$$p(t) = C_1 e^{-t} + C_2 e^{-4t} + 5$$

Solution

To ensure the solution satisfies the given initial conditions, evaluate the solution at $t = 0$ to get

$$p(0) = C_1 + C_2 + 5$$

We therefore set $C_1 = 30 - 5 - C_2$. To determine the value of the second constant, first differentiate the solution to get

$$\dot{p} = -C_1 e^{-t} - 4C_2 e^{-4t}$$

and then evaluate it at $t = 0$ to get

$$\dot{p}(0) = -C_1 - 4C_2$$

This gives a second equation for C_1: $C_1 = -4C_2 + 3$. Using these two equations to solve for C_1 and C_2 gives $C_1 = 97/3$ and $C_2 = -22/3$. ■

Example 23.11 If $p(0) = 30$ and $\dot{p}(0) = -3$, solve for the constants of integration in the following complete solution for price in the price-adjustment model:

$$p(t) = C_1 e^{-2t} + C_2 t e^{-2t} + 5$$

Solution

We can determine the constants of integration using the same procedure as shown in example 23.10. In this case they are $C_1 = 25$ and $C_2 = 47$. ■

Example 23.12 If $p(0) = 30$ and $\dot{p}(0) = -3$, solve for the constants of integration in the following complete solution for price in the price-adjustment model:

$$p(t) = 5 + e^{-0.5t}[A_1 \cos(1.5t) + A_2 \sin(1.5t)]$$

Solution

Evaluating the constants of integration in the case of complex roots requires a knowledge of some of the properties of circular functions. Evaluating the solution at $t = 0$ gives

$$p(0) = 5 + A_1$$

where we have used the properties that $\cos 0 = 1$ and $\sin 0 = 0$. Since $p(0) = 30$, we set $A_1 = 25$. Next, using the properties that $d \sin x/dx = \cos x$ and $d \cos x/dx = -\sin x$, we obtain

$$\dot{p}(0) = -0.5A_1 + 1.5A_2$$

Since $\dot{p}(0) = -3$, we set $A_2 = 19/3$. ■

The Steady State and Convergence

It is natural to wonder whether a solution to a linear, second-order differential equation will converge to the steady-state equilibrium value. If it does, we say the steady state is a *stable* equilibrium. On the other hand, if the solution diverges, we say the steady state is an *unstable* equilibrium. It is crucial in economic applications to determine the stability properties of an equilibrium. Only if an equilibrium is stable can we predict that the relevant economic variables will tend to converge toward equilibrium values. The importance of this is that we can then use economic theory to help explain the behavior of economic variables over time. Theorem 23.5 states the conditions for **convergence.**

Theorem 23.5 The solution to the linear, second-order differential equation with constant coefficients and constant term converges to the steady-state equilibrium if and only if the real parts of the roots of its characteristic equation are negative. In the case of real roots, the real parts are the roots themselves. In the case of complex roots, the real part is h.

Proof

We consider the three possible cases in turn.

Case 1 *Roots real and distinct.* The complete solution in this case is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{b}{a_2}$$

To determine if $y(t)$ converges, take the limits as t goes to infinity on both sides to get

$$\lim_{t\to\infty} y(t) = C_1 \lim_{t\to\infty}(e^{r_1 t}) + C_2 \lim_{t\to\infty}(e^{r_2 t}) + \frac{b}{a_2}$$

If r_1 and r_2 are both negative, the two exponential terms involving t converge to 0 in the limit so that $y(t)$ converges to b/a_2. If r_1 and r_2 are both positive, then the two terms involving t diverge to infinity so that $y(t)$ diverges to $+\infty$ or $-\infty$ depending on the signs of C_1 and C_2. If one root is positive and the other is negative, the term with the negative root converges to zero, but the other term diverges to infinity, except in the special case of a zero constant of integration on that term. As a result $y(t)$ diverges, except in that special case. Thus $y(t)$ converges to its steady-state equilibrium value of b/a_2 for all values of the constants of integration if and only if both roots are negative.

Case 2 *Roots real and equal.* The complete solution in this case is

$$y(t) = C_1 e^{rt} + C_2 t e^{rt} + \frac{b}{a_2}$$

Taking limits on both sides gives

$$\lim_{t\to\infty} y(t) = \frac{b}{a_2} + C_1 \lim_{t\to\infty}(e^{rt}) + C_2 \lim_{t\to\infty}(te^{rt})$$

If the repeated root, r, is positive, then $y(t)$ will diverge to positive or negative infinity. If the root is negative, then $y(t)$ will converge to b/a_2; however, this is a bit more difficult to see than in Case 1. The reason is that the limit of the term (te^{rt}) is of the form $(\infty \cdot 0)$ when $r < 0$. To take this limit, we must first convert it to the form (∞/∞) by writing it as (t/e^{-rt}). We can then apply l'Hôpital's rule, which says that the limit of (t/e^{-rt}) when $r < 0$ is equal to the limit of the derivative of the numerator divided by the derivative of the denominator (with respect to t). Taking these derivatives gives us $(-1/r)e^{rt}$, the limit of which is zero for $r < 0$.

Case 3 *Complex roots.* The solution in this case is

$$y(t) = e^{ht}(A_1 \cos vt + A_2 \sin vt) + \frac{b}{a_2}$$

where $h = -a_1/2$ and $v = \sqrt{4a_2 - a_1^2}$.

What is the limit of $y(t)$ as $t \to \infty$ in this case? First, as is shown in the appendix, the cosine and sine functions are circular functions bounded between $-A_i$ and A_i. Thus the term inside the brackets is an oscillating function that is bounded as $t \to \infty$. This term is multiplied by e^{ht}, which will grow without limit if $h > 0$. On the other hand, if $h < 0$, then e^{ht} converges to zero. As a result $y(t)$ diverges in ever-increasing oscillations if $h > 0$ and converges to b/a_2 in ever-decreasing oscillations if $h < 0$. Since h is often referred to as the real part of a complex root, we conclude that $y(t)$ converges to b/a_2 if the *real part of the complex root is negative*.

An interesting special case arises in the case of complex roots when the coefficient $a_1 = 0$. In this case, $e^{ht} = 1$ and $y(t)$ neither converges to nor diverges from b/a_2. Instead, $y(t)$ itself becomes a circular function that permanently fluctuates with a regular period and amplitude around the value b/a_2. ■

Example 23.13 Convergence in the Price-Adjustment Model with Inventories

Determine the restrictions on the parameters of the price-adjustment model that guarantee that all price trajectories (from any initial conditions) converge to the steady-state equilibrium price.

Solution

The roots of the characteristic equation for this model are

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

where $a_1 = \alpha(G - B)$ and $a_2 = \beta(G - B)$. To ensure that both r_1 and r_2 are negative requires $a_1 > 0$ and $a_2 > 0$. To see why, note that if $a_1 < 0$, then one or both of the roots could be positive. If $a_1 > 0$ but $a_2 < 0$, then one of the roots ($r_1 = -a_1/2 - \sqrt{a_1^2 - 4a_2/2}$) is negative but the other one ($r_2 = -a_1/2 + \sqrt{a_1^2 - 4a_2/2}$) could be positive.

Since $\alpha, \beta > 0$, the restrictions that $a_1 > 0$ and $a_2 > 0$ amount to requiring $G - B > 0$. This is the same condition required for convergence in the price-adjustment model examined in chapter 21, where we ignored the effect of inventories on price adjustment. Therefore adding an inventory effect to the model does not affect the stability of the model.

If $a_1 > 0$, $a_2 > 0$, and a_2 is large enough that $a_1^2 - 4a_2 \leq 0$, then the roots are complex-valued. In this case we require only the real part of the root to be negative, which means $a_1 > 0$. However, since this case can only arise when $a_2 > 0$, the condition for convergence is no different in the complex-root case.

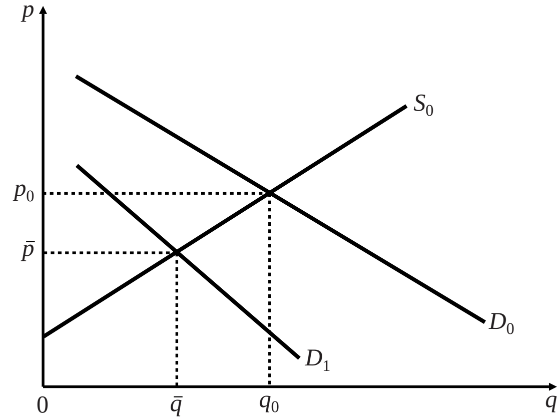


Figure 23.5 Price must adjust from the initial equilibrium at p_0 to the new equilibrium at $\bar{p}$

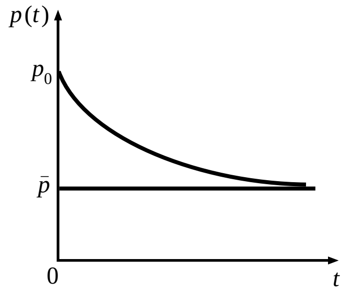


Figure 23.6 Price adjustment is monotonic if the roots are real valued

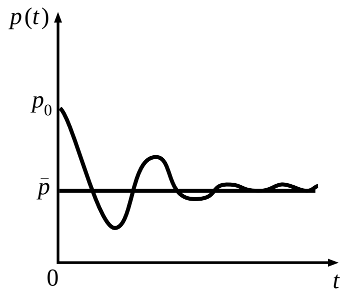


Figure 23.7 Price adjustment is oscillatory if the roots are complex valued

If the stability condition is satisfied, then $p(t)$ converges to $\bar{p}$ regardless of whether the roots are real or complex-valued; however, the *path* $p(t)$ takes toward $\bar{p}$ will be very different. We demonstrate this in figures 23.5 to 23.7. Figure 23.5 depicts the market initially in equilibrium at the point (q_0, p_0), which is where the initial demand curve, D_0, intersects the supply curve. Now suppose that the demand curve shifts to the left. The new equilibrium is at the point $(\bar{p}, \bar{q})$.

If the roots of the characteristic equation are real, distinct, and negative, then the path that price takes from its initial value, p_0, to its new equilibrium value, $\bar{p}$, will be **monotonic** as depicted in figure 23.6. If the roots of the characteristic equation are complex-valued with the real part negative, then the path that price takes to its new equilibrium value will display oscillations that are dampened over time, as depicted in figure 23.7. As is apparent in figure 23.7, this price-adjustment model can generate some rather complicated-looking price behavior if the characteristic roots are complex-valued: as a result of an exogenous shock (shift of the demand curve) to the market, cycles of economic activity arise that are gradually dampened over time as the market converges to the new equilibrium. ■

EXERCISES

1. Solve the following linear, second-order differential equations, including the constants of integration, using initial conditions $y_0 = 10$ and $\dot{y}_0 = 8$:

(a) $3\ddot{y} + 6\dot{y} - 9y = -18$

(b) $\ddot{y} + \dot{y} + \frac{1}{4}y = 2$

(c) $2\ddot{y} - \frac{1}{3}\dot{y} - \frac{1}{3}y = 10$

(d) $3\ddot{y} + 6\dot{y} + 3y = 9$

2. Solve the following linear, second-order differential equations. Also solve for the constants of integration in (a) and (b) using initial conditions $y_0 = 10$ and $\dot{y}_0 = 8$.

(a) $\ddot{y} + 5\dot{y} - 6y = 36$

(b) $\ddot{y} - 5\dot{y} + 6y = 3$

(c) $\ddot{y} - 2\dot{y} + 2y = 1$

(d) $\ddot{y} + \dot{y} + \frac{5}{4}y = 20$

3. In the *price-adjustment model with inventories*, derive the restriction on the value of B that ensures price converges to the steady state if $\alpha = 0.25$, $\beta = 0.2$, $A - F = 10$, and $G = 10$.

4. In the *price-adjustment model with inventories*, derive the restriction on the value of G that ensures price converges to the steady state if $\alpha = 0.05$, $\beta = 0.5$, $A - F = 10$, and $B = -2$.

23.2 The Linear, Second-Order Differential Equation with a Variable Term

If the *term* $b(t)$ in the differential equation is not a constant, then a steady-state solution will usually not exist. As a result, we have to look for an alternative to the steady-state solution to serve as a particular solution. This is not difficult if $b(t)$ is a fairly simple function, but it can become very difficult otherwise. Various techniques have been developed to find a particular solution but most of these are beyond the scope of this book. However, the technique introduced in chapter 20 for finding particular solutions to difference equations applies equally well to differential equations. This technique, called the method of undetermined coefficients, is applied below to linear, second-order differential equations with a **variable term** to demonstrate its use.

As we explained in chapter 20, this method relies on one's ability to "guess" the form of the particular solution. The following are two guidelines that facilitate this procedure:

Case 1 If $b(t)$ is an nth degree polynomial in t, say $p_n(t)$, then assume that the particular solution is also a polynomial. That is, assume that

$$y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$$

where the A_i are constants, the values of which are determined by substituting the assumed particular solution into the differential equation and then by equating coefficients of like terms, as shown in the examples below.
Case 2 If $b(t)$ is of the form $e^{\alpha t} p_n(t)$, where $p_n(t)$ is a polynomial in t as in case 1, and α is a known constant, then assume that the solution is given by

$$y_p = e^{\alpha t}\big(A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0\big)$$

where the A_i are determined as in case 1.

We should note an important exception to these rules for guessing solutions. If any term in the assumed solution is also a term of y_h (the homogeneous solution) disregarding multiplicative constants, then the assumed solution must be modified as follows: multiply the assumed solution by t^k, where k is the smallest positive integer such that the common terms are then eliminated. See example 23.15 below.

Example 23.14 Solve $\ddot{y} + 3\dot{y} - 4y = t^2$.

Solution

First, we solve for the homogeneous form

$$\ddot{y} + 3\dot{y} - 4y = 0$$

which has the characteristic equation $(r^2 + 3r - 4) = 0$. The roots of this equation are -4 and 1. Therefore

$$y_h = C_1 e^{-4t} + C_2 e^t$$

To find a particular solution, we note that $b(t)$ is a second-degree polynomial in t. We therefore guess that

$$y_p = A_2 t^2 + A_1 t + A_0$$

To determine the values of A_0, A_1, and A_2, differentiate this expression to obtain

$$\dot{y}_p = 2A_2 t + A_1$$

Differentiate again to obtain

$$\ddot{y}_p = 2A_2$$

Since the particular solution must satisfy the differential equation, substitute these expressions back into the complete differential equation to get

$$2A_2 + 3(2A_2t + A_1) - 4\big(A_2t^2 + A_1t + A_0\big) = t^2$$

Collect like terms to get

$$-(4A_2 + 1)t^2 + (6A_2 - 4A_1)t + (2A_2 + 3A_1 - 4A_0) = 0$$

For this to hold for all t, each of the expressions in brackets must be identically equal to zero. This allows us to determine the values of the constants in our assumed solution. Specifically, we must have

$$A_2 = -\frac{1}{4}, \quad A_1 = -\frac{3}{8}, \quad A_0 = -\frac{13}{32}$$

Using the fact that $y = y_h + y_p$, we have solved the problem as follows:

$$y(t) = -\frac{1}{4}t^2 - \frac{3}{8}t - \frac{13}{32} + C_1e^{-4t} + C_2e^t$$

■

Example 23.15 Solve $\ddot{y} + 3\dot{y} - 4y = 5e^t$.

Solution

The homogeneous solution is the same as in the previous example. The particular solution is found by guessing that

$$y_p = A_0e^t$$

However, this has the same form (up to a multiplicative constant) as one of the terms in y_h. We therefore must modify our guess for this procedure to work. We do this by multiplying our first guess by t ($k = 1$ here). This gives us

$$y_p = tA_0e^t$$

which no longer is of the same form as any of the terms in y_h. We now proceed as usual by differentiating y_p to get

$$\dot{y}_p = A_0 e^t + t A_0 e^t$$

and

$$\ddot{y}_p = 2A_0 e^t + t A_0 e^t$$

Substituting these expressions back into the full differential equation, collecting like terms, and solving gives us $A_0 = 1$. Therefore the solution is

$$y(t) = te^t + C_1 e^{-4t} + C_2 e^t$$ ■

A Nonautonomous Pollution Model

The stock of carbon dioxide in the global atmosphere is believed to have a positive influence on global warming. This stock grows with industrial emissions of carbon dioxide resulting from the burning of fossil fuels, but it is reduced by a certain amount because of natural absorption by oceans and vegetation. In this model we will examine the dynamics of the stock of global carbon dioxide. The stock tends to increase as industrial emissions grow over time due to growth in global economic activity. On the other hand, the stock is negatively influenced by stricter pollution controls being imposed as the global warming problem worsens.

Let y represent the stock of carbon dioxide. Assume that the stock changes according to

$$\dot{y} = x - \alpha y \tag{23.17}$$

where x represents industrial emissions of carbon dioxide and $\alpha > 0$ is the parameter that determines the rate of carbon assimilation by the natural environment. Assume further that industrial emissions change over time according to

$$\dot{x} = ae^{bt} - \beta y \tag{23.18}$$

where a, b, and β are positive constants. The first term causes emissions to grow with t. This term represents the effect of growth in the level of economic activity on industrial emissions of carbon dioxide. The second term causes emissions to be negatively influenced by the stock of carbon dioxide. This term represents the assumption that governments introduce stricter controls on industry's emissions of carbon dioxide as the pollution problem worsens.

Differentiating equation (23.17) gives

$$\ddot{y} = \dot{x} - \alpha \dot{y}$$

Using equation (23.18) to substitute for $\dot{x}$ and rearranging gives a second-order differential equation for y

$$\ddot{y} + \alpha \dot{y} + \beta y = ae^{bt}$$

This is a nonautonomous, linear, second-order differential equation that we can solve. Starting with the homogeneous form, which is

$$\ddot{y} + \alpha \dot{y} + \beta y = 0$$

the characteristic equation is

$$r^2 + \alpha r + \beta = 0$$

with roots

$$r_1, r_2 = \frac{-\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$$

Since α and β are assumed positive, then both roots are negative if real, and the real part is negative if both roots are complex. The solution to the homogeneous form is

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

To find a particular solution, we note the term is an exponential function of t. Therefore we try a particular solution of the same form

$$y_p = A_0 e^{bt}$$

To determine the coefficient A_0, differentiate the trial solution to get

$$\dot{y}_p = bA_0 e^{bt}$$

and differentiate again to get

$$\ddot{y}_p = b^2 A_0 e^{bt}$$

Now substitute the trial y_p and its two derivatives back into the complete differential equation. This gives

$$b^2 A_0 e^{bt} + \alpha b A_0 e^{bt} + \beta A_0 e^{bt} = a e^{bt}$$

Solving for A_0 gives

$$A_0 = \frac{a}{b^2 + \alpha b + \beta}$$

The particular solution therefore is

$$y_p = \frac{a}{b^2 + \alpha b + \beta} e^{bt}$$

and the complete solution to the differential equation is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{a}{b^2 + \alpha b + \beta} e^{bt}$$

Inspection of this solution tells us how the stock of carbon dioxide behaves over time in this model. We have already determined that both roots are negative. Therefore, as $t \to \infty$, the first two terms in the solution go to zero. If $b = 0$, then the third term becomes a/β, and we conclude that the stock of carbon dioxide converges to this value over time. This would be good news for future generations. If, on the other hand, $b > 0$, then the third term is an increasing function of t; it therefore grows without limit causing y to also grow without limit.

EXERCISES

Solve the following linear, second-order differential equations:

1. $3\ddot{y} + 6\dot{y} - 9y = -18e^{2t}$

2. $\ddot{y} + 5\dot{y} - 6y = 2e^{t}$

3. $\ddot{y} - 5\dot{y} + 6y = e^{-t/2}$

4. $\ddot{y} + \dot{y} = 2$

5. $\ddot{y} + \frac{1}{2}\dot{y} = 4$

6. Solve the *nonautonomous pollution model* using the following parameter values: $\alpha = 0.5$, $\beta = 1/18$, $a = b = 1$, $y_0 = 100/14$, and $\dot{y}_0 = -1$.

CHAPTER REVIEW

Key Concepts

characteristic equation
characteristic root
complex roots
complete solution
convergence
eigenvalues
homogeneous solution
monotonic trajectory
oscillating trajectory
particular solution
variable term

Review Questions

1. What is the characteristic equation and what role does it play in the solution to a linear, second-order differential equation?
2. Under what conditions is the *particular* solution given by the steady-state solution?
3. Under what conditions will a linear, autonomous, second-order differential equation produce oscillating trajectories and under what conditions will it produce monotonic trajectories?
4. State the necessary and sufficient conditions for convergence to a steady state.
5. When does one use the method of undetermined coefficients to find the particular solution?

Review Exercises

1. Solve the following linear, second-order differential equations:
   (a) $\ddot{y} - \dot{y} - 2y = 10$
   (b) $\ddot{y} + 6\dot{y} + 9y = 27$
   (c) $\ddot{y} + 4\dot{y} + 5y = 10$, $y(0) = 2$, $\dot{y}(0) = 1$
2. Solve the following linear, second-order differential equations:
   (a) $4\ddot{y} + 12\dot{y} - 7y = 28$, $y(0) = 2$, $\dot{y}(0) = 1$
   (b) $\ddot{y} + 6\dot{y} + 9y = 27t$
   (c) $\ddot{y} + 4\dot{y} + 5y = 10t^2$, $y(0) = 2$, $\dot{y}(0) = 1$
3. Consider the following linear, first-order differential equations:

$$\dot{y} = a_{11}y + a_{12}x$$
$$\dot{x} = a_{21}y$$

Use these equations to derive a linear second-order differential equation for y. Solve for $y(t)$ and derive the conditions on the parameters that must be satisfied for $y(t)$ to converge to its steady-state value of $y = 0$.

4. Consider a model of market equilibrium in which the current supply of firms is a function of the price that is *expected* to prevail when the product is sold. Assume that the market supply equation is

$$q^S(t) = F + Gp^e$$

where p^e is the expected price and F and G are constant parameters of the supply equation. Assume further that suppliers use information about the actual current price and its first and second derivatives with respect to time to form their prediction of the price that will prevail when their product reaches the market. In particular, assume that

$$p^e = p + b\dot{p} + c\ddot{p}$$

where $b > c > 0$. If the current price is constant, so that $\dot{p} = \ddot{p} = 0$, then suppliers expect the prevailing price to equal the current price. If the current price is rising, so that $\dot{p} > 0$, then suppliers expect the prevailing price to be higher than current price. How much higher depends on whether the current price is rising at an increasing rate, $\ddot{p} > 0$; or at a decreasing rate, $\ddot{p} < 0$. Making the necessary substitutions, the supply equation then becomes

$$q^S = F + Gp + Gb\dot{p} + Gc\ddot{p}$$

The remainder of the market equilibrium model is a linear demand equation

$$q^D = A + Bp$$

and a linear price-adjustment equation that says that price rises when there is excess demand and falls when there is excess supply:

$$\dot{p} = \alpha(q^D - q^S)$$

where $\alpha > 0$ is a constant which determines how rapidly price adjusts when the market is out of equilibrium, and A and B are constant parameters of the demand equation.

Derive the linear, second-order differential equation implied by this model. Solve this differential equation explicitly. What restrictions must be imposed on the parameters of the demand and supply functions to ensure stability of the equilibrium?

5. Solve the linear, second-order differential equation in exercise 4 for the following parameter values:

   (a) $\alpha = 0.1$, $G = 25$, $B = -20$, $b = 0.5$, $c = 0.1$

   (b) $\alpha = 0.4$, $G = 25$, $B = -20$, $b = 0.5$, $c = 0.1$

6. Suppose that rather than responding to the gap between demand and supply, price only falls when accumulated inventories of unsold merchandise exceed some critical (and constant) value, $Z > 0$. We represent this algebraically as

$$\dot{p} = \beta \left\{ Z - \int_0^t [q^S(s) - q^D(s)]\, ds,\ \beta > 0 \right\}$$

   Assume that $q^D = A + Bp$ and $q^S = F + Gp$. Derive and solve the second-order differential equation implied by this model. Show that the stability condition implies that price neither converges to nor diverges from its equilibrium value but oscillates forever around the equilibrium.

Chapter 24 Simultaneous Systems of Differential and Difference Equations

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Walrusian Price Adjustment Model with Entry
- A Markov Model of Layoffs
- Practice Exercises

It is common in economic models for two or more variables to be determined simultaneously. When the model is dynamic and involves two or more variables, a *system* of differential or difference equations arises. The purpose of this chapter is to extend our single equation techniques to solve systems of autonomous differential and difference equations.

24.1 Linear Differential Equation Systems

We begin with the simplest case—a system of two linear differential equations—and solve it using the **substitution method.** We then proceed to a more general method, known as the **direct method,** that can be used to solve a system of linear differential equations with more than two equations.

The Substitution Method

This method is suited to solving a differential equation system consisting of exactly two linear differential equations.

Definition 24.1 A *linear system* of two autonomous differential equations is expressed as

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 + b_1 \tag{24.1}$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 + b_2 \tag{24.2}$$

The system is linear because it contains only linear differential equations which, as usual, means that y_i and $\dot{y}_i$ are not raised to any power other than one. The system is autonomous because the coefficients, a_{ij}, and the terms, b_i, are constants. The equations must be solved simultaneously because $\dot{y}_1$ depends on the solution for y_2 and $\dot{y}_2$ depends on the solution for y_1.

As in previous chapters on linear differential and difference equations, we separate the problem of finding the **complete solutions** into two parts. We first find the **homogeneous solutions** and then find **particular solutions.** The complete solutions are the *sum* of the homogeneous and particular solutions. In symbols,

$$y_1 = y_1^h + y_1^p$$
$$y_2 = y_2^h + y_2^p$$

where y_i is the complete solution, y_i^h is the general homogeneous solution for y_i, and y_i^p is the particular solution for y_i.

The General Solution to the Homogeneous Forms

The first step in obtaining a complete solution is to put the differential equation system into its homogeneous form. This is done by setting the terms in each equation equal to zero.

Definition 24.2 The *homogeneous form* of the system of two linear, first-order differential equations (24.1) and (24.2) is

$$\dot{y}_1 = a_{11}y_1 + a_{12}y_2 \tag{24.3}$$

$$\dot{y}_2 = a_{21}y_1 + a_{22}y_2 \tag{24.4}$$

It is possible to convert this system of two first-order differential equations into a single second-order differential equation using a combination of differentiation and substitution. Since we already know how to solve a linear, second-order differential equation, this procedure provides an easy way to find the solution.

To transform the system into a second-order differential equation, differentiate equation (24.3) to get

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\dot{y}_2$$

This gives a second-order equation but one that still depends on $\dot{y}_2$. Therefore use equation (24.4) to substitute for $\dot{y}_2$. This gives

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}(a_{21}y_1 + a_{22}y_2)$$

Now use equation (24.3) to get the following expression that can be used to substitute for y_2

$$y_2 = \frac{\dot{y}_1 - a_{11}y_1}{a_{12}} \tag{24.5}$$

We assume that $a_{12} \neq 0$. (In cases where this assumption does not hold, we can solve equation 24.3 as a single differential equation because it would no longer depend on y_2.) Substitute this expression for y_2 to get

$$\ddot{y}_1 = a_{11}\dot{y}_1 + a_{12}\left(a_{21}y_1 + a_{22}\frac{\dot{y}_1 - a_{11}y_1}{a_{12}}\right)$$

Simplifying and rearranging gives

$$\ddot{y}_1 - (a_{11} + a_{22})\dot{y}_1 + (a_{11}a_{22} - a_{12}a_{21})y_1 = 0 \tag{24.6}$$

which is a linear, homogeneous, second-order differential equation with constant coefficients. In chapter 23, equations like these were written in the form

$$\ddot{y} + a_1\dot{y} + a_2 y = 0$$

and the solution was given in theorem 23.2 for the case of real-valued roots and in theorem 23.3 for the case of complex-valued roots. Applying those theorems to equation (24.6) gives the solution for $y_1(t)$. The solution for $y_2(t)$ is then obtained by substitution. Let's do this for each of the three types of characteristic roots that can occur.

1. *Real and distinct roots.* Theorem 23.2 gives the following solution to equation (24.6):

$$y_1^h(t) = C_1e^{r_1t} + C_2e^{r_2t} \tag{24.7}$$

where the roots r_1 and r_2 are the solution to the characteristic equation

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

and are given by

$$r_1,\ r_2 = \frac{a_{11} + a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \tag{24.8}$$

The solution for $y_2^h(t)$ is given by equation (24.5), which shows y_2 as a function of the solution for y_1 and the derivative of the solution for y_1. The solution for y_1 is given in equation (24.7). Take its derivative

$$\dot{y}_1 = r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}$$

and substitute them both into equation (24.5) to get

$$y_2^h(t) = \frac{r_1 C_1 e^{r_1 t} + r_2 C_2 e^{r_2 t}}{a_{12}} - \frac{a_{11}}{a_{12}}(C_1 e^{r_1 t} + C_2 e^{r_2 t})$$

Simplifying gives the solution

$$y_2^h(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} \tag{24.9}$$

Together, equations (24.7) and (24.9) are the solutions to the homogeneous form of the system of two linear, first-order differential equations in definition 24.2 when the roots are real and distinct.

2. *Real and equal roots.* This case occurs when

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = 0$$

Theorem 23.2 gives the solution

$$y_1^h(t) = C_1 e^{rt} + C_2 t e^{rt} \tag{24.10}$$

The solution for $y_2(t)$ is found using equation (24.5) as before. Take the derivative of equation (24.10),

$$\dot{y}_1(t) = rC_1 e^{rt} + rC_2 t e^{rt} + C_2 e^{rt}$$

then substitute it and the solution for y_2 into equation (24.5). This gives

$$y_2^h(t) = \frac{(rC_1 + rC_2 t + C_2)e^{rt} - a_{11}(C_1 + C_2 t)e^{rt}}{a_{12}}$$

Simplifying gives the solution for y_2

$$y_2^h(t) = \frac{(r - a_{11})C_1 + C_2 + (r - a_{11})C_2 t}{a_{12}} e^{rt}$$

3. *Complex-valued roots.* Theorem 23.3 gives the solution

$$y_1^h(t) = e^{ht}[A_1 \cos(vt) + A_2 \sin(vt)] \tag{24.11}$$

where $h = (a_{11} + a_{22})/2$ and $v = \sqrt{4(a_{11}a_{22} - a_{12}a_{21}) - (a_{11} + a_{22})^2}/2$. Again, the solution for y_2 is found using equation (24.5). Take the derivative of equation (24.11),

$$\dot{y}_1 = he^{ht}[A_1 \cos(vt) + A_2 \sin(vt)] + e^{ht}[-vA_1 \sin(vt) + vA_2 \cos(vt)]$$

then substitute it and the solution for y_1 into equation (24.5). This gives

$$y_2^h(t) = e^{ht}\left[\frac{(h - a_{11})A_1 + vA_2}{a_{12}} \cos(vt) + \frac{(h - a_{11})A_2 - vA_1}{a_{12}} \sin(vt)\right]$$

Putting these results together gives

Theorem 24.1 The solutions to the homogeneous form of the system of two linear, first-order differential equations (24.3) and (24.4) are

Real and distinct roots:

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$
$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t}$$

Real and equal roots:

$$y_1(t) = C_1 e^{rt} + C_2 t e^{rt}$$
$$y_2(t) = \left[\frac{r - a_{11}}{a_{12}}(C_1 + C_2 t) + \frac{C_2}{a_{12}}\right] e^{rt}$$

where

$$r_1,\ r_2 = \frac{a_{11} + a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}$$

Complex roots:

$$y_1(t) = e^{ht}[A_1 \cos(vt) + A_2 \sin(vt)]$$
$$y_2(t) = e^{ht}\left[\frac{(h - a_{11})A_1 + vA_2}{a_{12}} \cos(vt) + \frac{(h - a_{11})A_2 - vA_1}{a_{12}} \sin(vt)\right]$$

where

$$h = \frac{1}{2}(a_{11} + a_{22})$$

and

$$v = \frac{1}{2}\sqrt{4(a_{11}a_{22} - a_{12}a_{21})^2 - (a_{11} + a_{22})^2}$$

Example 24.1 Solve the following system of homogeneous differential equations:

$$\dot{y}_1 = y_1 - 3y_2$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2$$

Solution

Differentiate the first equation to get

$$\ddot{y}_1 = \dot{y}_1 - 3\dot{y}_2$$

Use the second equation to substitute for $\dot{y}_2$. This gives

$$\ddot{y}_1 = \dot{y}_1 - 3(0.25y_1 + 3y_2)$$

Finally use the first equation again to obtain an expression for y_2:

$$y_2 = \frac{-\dot{y}_1 + y_1}{3} \tag{24.12}$$

Use this to substitute for y_2. This gives

$$\ddot{y}_1 = \dot{y}_1 - 3\left(0.25y_1 + 3\frac{-\dot{y}_1 + y_1}{3}\right)$$

Simplify and rearrange to get

$$\ddot{y}_1 - 4\dot{y}_1 + \frac{15}{4}y_1 = 0$$

The characteristic equation is

$$r^2 - 4r + \frac{15}{4} = 0$$

with roots

$$r_1,\ r_2 = \frac{4}{2} \pm \frac{1}{2}\sqrt{16 - 4\left(\frac{15}{4}\right)}$$

which gives $r_1,\ r_2 = 3/2,\ 5/2$. The roots are real and distinct. The solution for y_1 then is

$$y_1(t) = C_1 e^{3t/2} + C_2 e^{5t/2}$$

The solution for y_2 can now be found using equation (24.12). First, differentiate the solution for y_1 to get

$$\dot{y}_1(t) = \frac{3}{2} C_1 e^{3t/2} + \frac{5}{2} C_2 e^{5t/2}$$

then substitute it and the solution for y_1 into equation (24.12). After simplifying, this gives

$$y_2(t) = -\frac{1}{6} C_1 e^{3t/2} - \frac{1}{2} C_2 e^{5t/2}$$

■

Example 24.2 Solve

$$\dot{y}_1 = -6y_1 - 8y_2$$
$$\dot{y}_2 = 2y_1 + 2y_2$$

Solution

Differentiate the first equation to get

$$\ddot{y}_1 = -6\dot{y}_1 - 8\dot{y}_2$$

Substitute for $\dot{y}_2$ to get

$$\ddot{y}_1 = -6\dot{y}_1 - 8(2y_1 + 2y_2)$$

Use the first equation to get

$$y_2 = -\frac{6}{8}y_1 - \frac{1}{8}\dot{y}_1$$

then substitute it into the equation for $\ddot{y}_1$ to get

$$\ddot{y}_1 = -6\dot{y}_1 - 8\left[2y_1 + 2\left(-\frac{6}{8}y_1 - \frac{1}{8}\dot{y}_1\right)\right]$$

After simplifying and rearranging, this becomes

$$\ddot{y}_1 + 4\dot{y}_1 + 4y_1 = 0$$

The roots of the characteristic equation for this differential equation are

$$r_1,\ r_2 = -\frac{4}{2} \pm \frac{\sqrt{16 - 4(4)}}{2}$$

The roots are real and equal

$$r_1 = r_2 = -2$$

Theorem 23.2 gives the solution as

$$y_1(t) = [C_1 + C_2 t]e^{-2t}$$

This solution and its derivative in the expression for y_2 lead to the following solution for y_2:

$$y_2(t) = \left(\frac{-1}{2}C_1 - \frac{1}{2}C_2 t + \frac{C_2}{-8}\right)e^{-2t}$$

■

The Particular Solutions

We found the solution to the homogeneous form of the system of two linear, first-order differential equations in definition 24.1. We turn our attention now to the task of finding particular solutions to that system so that we can add together the homogeneous and particular solutions to obtain the complete solutions.

The particular solution we always look for in the case of autonomous differential equations is the steady-state solution.

Definition 24.3

The **steady-state** solution to a system of two differential equations is the pair of values $\bar{y}_1$ and $\bar{y}_2$ at which $\dot{y}_1$ and $\dot{y}_2$ both equal zero.

Set $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$ in the complete differential equation system (24.1) and (24.2) to solve for $\bar{y}_1$ and $\bar{y}_2$. This gives

$$a_{11}\bar{y}_1 + a_{12}\bar{y}_2 + b_1 = 0$$
$$a_{21}\bar{y}_1 + a_{22}\bar{y}_2 + b_2 = 0$$

This is a linear system of two equations in two unknowns. Solve the first equation for $\bar{y}_1$ and substitute it into the second equation to get the solution for $\bar{y}_2$. The first equation implies that

$$\bar{y}_1 = -\frac{a_{12}}{a_{11}}\bar{y}_2 - \frac{b_1}{a_{11}}$$

The second equation implies that

$$\bar{y}_2 = -\frac{a_{21}}{a_{22}}\bar{y}_1 - \frac{b_2}{a_{22}}$$

Substitute the expression for $\bar{y}_1$ into the expression for $\bar{y}_2$. This gives

$$\bar{y}_2 = -\frac{a_{21}}{a_{22}}\left(-\frac{a_{12}}{a_{11}}\bar{y}_2 - \frac{b_1}{a_{11}}\right) - \frac{b_2}{a_{22}}$$

After simplifying and rearranging, this becomes

$$\bar{y}_2 = \frac{a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}} \tag{24.13}$$

After substituting this into the expression for $\bar{y}_1$ and simplifying, we get

$$\bar{y}_1 = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}} \tag{24.14}$$

The steady-state solutions are given by equations (24.13) and (24.14). They exist if and only if $(a_{11}a_{22} - a_{12}a_{21}) \neq 0$, an assumption we make throughout this chapter. Note that this is analogous to assuming $a_2 \neq 0$ in chapter 23 on second-order differential equations. If the assumption is violated, it is still possible to find a particular solution, but it has to be an alternative to the steady-state solutions.

Provided that $a_{11}a_{22} - a_{12}a_{21} \neq 0$, the particular solutions to the complete system of two linear differential equations are given by the steady-state solutions. That is,

$$y_1^p = \bar{y}_1$$
$$y_2^p = \bar{y}_2$$

Example 24.3 Find the particular solutions for

$$\dot{y}_1 = y_1 - 3y_2 + 1$$
$$\dot{y}_2 = \frac{1}{4}y_1 - y_2 + 2$$

Solution

Set $\dot{y}_1 = 0$ and solve for $\bar{y}_1$. This gives

$$\bar{y}_1 = 3y_2 - 1$$

Set $\dot{y}_2 = 0$ and solve for $\bar{y}_2$. This gives

$$\bar{y}_2 = \frac{1}{4}y_1 + 2$$

Substitute the expression for $\bar{y}_1$ to get

$$\bar{y}_2 = \frac{1}{4}(3y_2 - 1) + 2$$

Solving gives $\bar{y}_2 = 7$. Substitute this value back into the expression for $\bar{y}_1$to get $\bar{y}_1 = 20$. ■

The Complete Solutions

The complete solution to the system of two linear equations in definition 24.1 is the *sum* of the homogeneous solutions and the particular solutions. Theorem 24.2 provides the complete solutions for the three types of characteristic roots that can occur.

Theorem 24.2 The complete solutions to the system of two linear first-order differential equations (24.1) and (24.2) are

Real and distinct roots:

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{y}_1$$
$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

Real and equal roots:

$$y_1(t) = C_1 e^{rt} + C_2 t e^{rt} + \bar{y}_1$$
$$y_2(t) = \left[\frac{r - a_{11}}{a_{12}}(C_1 + C_2 t) + \frac{C_2}{a_{12}}\right] e^{rt} + \bar{y}_2$$

where

$$r_1, r_2 = \frac{a_{11} + a_{22}}{2} \pm \frac{1}{2}\sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}$$

Complex roots:

$$y_1(t) = e^{ht}[A_1 \cos(vt) + A_2 \sin(vt)] + \bar{y}_1$$
$$y_2(t) = e^{ht}\left[\frac{(h - a_{11})A_1 + vA_2}{a_{12}} \cos(vt) + \frac{(h - a_{11})A_2 - vA_1}{a_{12}} \sin(vt)\right] + \bar{y}_2$$

where

$$h = \frac{1}{2}(a_{11} + a_{22})$$

and

$$v = \frac{1}{2}\sqrt{4(a_{11}a_{22} - a_{12}a_{21})^2 - (a_{11} + a_{22})^2}$$

and

$$\bar{y}_1 = \frac{a_{12}b_2 - a_{22}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$
$$\bar{y}_2 = \frac{a_{21}b_1 - a_{11}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

Example 24.4 Find the complete solution to the following system of differential equations:

$$\dot{y}_1 = y_1 - 3y_2 - 5$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2 - 5$$

Solution

First, put the system into its homogeneous form:

$$\dot{y}_1 = y_1 - 3y_2$$
$$\dot{y}_2 = \frac{1}{4}y_1 + 3y_2$$

This is the homogeneous system solved in example 24.3. The solutions obtained were

$$y_1^h(t) = C_1e^{3t/2} + C_2e^{5t/2}$$
$$y_2^h(t) = -\frac{1}{6}C_1e^{3t/2} - \frac{1}{2}C_2e^{5t/2}$$

Next, find the steady-state solutions to use as the particular solutions. Set $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$ in the complete equations. This gives

$$\bar{y}_1 - 3\bar{y}_2 - 5 = 0$$
$$\frac{1}{4}\bar{y}_1 + 3\bar{y}_2 - 5 = 0$$

Solving these gives

$$\bar{y}_1 = 8 \quad \text{and} \quad \bar{y}_2 = 1$$

Now add the homogeneous and particular solutions together to get the complete solutions

$$y_1(t) = C_1e^{3t/2} + C_2e^{5t/2} + 8$$
$$y_2(t) = -\frac{1}{6}C_1e^{3t/2} - \frac{1}{2}C_2e^{5t/2} + 1$$

■

Initial Conditions

When the complete solutions must also satisfy given initial conditions, the values of the constants of integration must be set appropriately. Normally initial conditions are given for $y_1$and y_2 at $t = 0$.

Example 24.5 Find the constants of integration that make the solutions in example 24.4 satisfy the initial conditions $y_1(0) = 1$ and $y_2(0) = 3$.

Solution

Evaluating the solutions at $t = 0$ gives

$$\begin{aligned} y_1(0) &= C_1 + C_2 + 8 \\ y_2(0) &= -\frac{1}{6}C_1 - \frac{1}{2}C_2 + 1 \end{aligned}$$

Setting $y_1(0) = 1$ and solving the first equation for C_1 gives

$$C_1 = 1 - C_2 - 8$$

Setting $y_2(0) = 3$ and solving the second equation for C_2 gives

$$C_2 = -\frac{2}{6}C_1 - 2 \cdot 3 + 2$$

Substituting for C_1 and solving for C_2 gives $C_2 = -5/2$. Using this in the expression for C_1 gives $C_1 = -9/2$. The complete solutions then become

$$\begin{aligned} y_1(t) &= -\frac{9}{2}e^{3t/2} - \frac{5}{2}e^{5t/2} + 8 \\ y_2(t) &= \frac{3}{4}e^{3t/2} + \frac{5}{4}e^{5t/2} + 1 \end{aligned}$$

■

Example 24.6 Find the complete solution to the following system of differential equations with $y_1(0) = 4$ and $y_2(0) = 5$:

$$\begin{aligned} \dot{y}_1 &= 2y_1 + 5y_2 + 2 \\ \dot{y}_2 &= -\frac{1}{2}y_1 - y_2 - 5 \end{aligned}$$

Solution

First, put the system into its homogeneous form

$$\dot{y}_1 = 2y_1 + 5y_2$$
$$\dot{y}_2 = -\frac{1}{2}y_1 - y_2$$

Differentiate the first equation to get

$$\ddot{y}_1 = 2\dot{y}_1 + 5\dot{y}_2$$

and use the second equation to substitute for $\dot{y}_2$. This gives

$$\ddot{y}_1 = 2\dot{y}_1 + 5\left(-\frac{1}{2}y_1 - y_2\right)$$

Now use the first equation to substitute for y_2 to get

$$\ddot{y}_1 = 2\dot{y}_1 + 5\left[-\frac{1}{2}y_1 - \left(\frac{\dot{y}_1 - 2y_1}{5}\right)\right]$$

After simplifying and rearranging, this becomes

$$\ddot{y}_1 - \dot{y}_1 + \frac{1}{2}y_1 = 0$$

The characteristic roots are

$$r_1, r_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4 \cdot (1/2)} = \frac{1}{2} \pm \frac{1}{2}i$$

where i is the imaginary number $\sqrt{-1}$. This is an example of complex-valued roots.

The solutions to the homogeneous form of the system of differential equations are

$$y_1(t) = e^{t/2}\left[A_1 \cos\left(\frac{t}{2}\right) + A_2 \sin\left(\frac{t}{2}\right)\right]$$
$$y_2(t) = e^{t/2}\left[\frac{-3A_1/2 + A_2/2}{5}\cos\left(\frac{t}{2}\right) + \frac{-3A_2/2 - A_1/2}{5}\sin\left(\frac{t}{2}\right)\right]$$

Next, find the steady-state solutions that will provide the *particular* solutions we need. Set $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$ in the complete differential equations. This gives

$$2\bar{y}_1 + 5\bar{y}_2 + 2 = 0$$
$$-\frac{1}{2}\bar{y}_1 - \bar{y}_2 - 5 = 0$$

Solving these two equations for $\bar{y}_1$ and $\bar{y}_2$ gives

$$\bar{y}_1 = -46, \quad \bar{y}_2 = 18$$

Finally, add the *homogeneous* and *particular* solutions together to get the complete solutions

$$y_1(t) = e^{t/2}\left[A_1 \cos\left(\frac{t}{2}\right) + A_2 \sin\left(\frac{t}{2}\right)\right] - 46$$
$$y_2(t) = e^{t/2}\left[\frac{-3A_1/2 + A_2/2}{5} \cos\left(\frac{t}{2}\right) + \frac{-3A_2/2 - A_1/2}{5} \sin\left(\frac{t}{2}\right)\right] + 18$$

To make the solutions also satisfy the initial conditions, we must set the constants appropriately. At $t = 0$, since $\cos(0) = 1$ and $\sin(0) = 0$, the solutions are

$$y_1(0) = A_1 - 46$$
$$y_2(0) = \frac{-3A_1/2 + A_2/2}{5} + 18$$

and these must satisfy the given conditions: $y_1(0) = 4$ and $y_2(0) = 5$. Thus

$$4 = A_1 - 46$$
$$5 = \frac{-3A_1/2 + A_2/2}{5} + 18$$

Solving these two equations for A_1 and A_2 gives $A_1 = 50$ and $A_2 = 20$. The final complete solutions then are

$$y_1(t) = e^{t/2}\left[50 \cos\left(\frac{t}{2}\right) + 20 \sin\left(\frac{t}{2}\right)\right] - 46$$
$$y_2(t) = e^{t/2}\left[-13 \cos\left(\frac{t}{2}\right) - 11 \sin\left(\frac{t}{2}\right)\right] + 18$$

■

The Direct Method

Although the substitution method works well for systems of two differential equations, it can become cumbersome for larger systems. The following direct approach to solving a system of linear differential equations circumvents this limitation.

Definition 24.4 A **linear system** of n autonomous differential equations is expressed in matrix form as

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{b}$$

where A is an $n \times n$ matrix of constant coefficients, $\mathbf{b}$ is a vector of n constant terms, $\mathbf{y}$ is a vector of n variables, and $\dot{\mathbf{y}}$ is a vector of n derivatives.

Example 24.7 Write the 2×2 matrix of coefficients and the vector of two terms in the case of a system of $n = 2$ linear, autonomous differential equations.

Solution

The matrix of coefficients is

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and the vector of terms is

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

■

The solution to the complete system of equations is obtained by adding together the *homogeneous solutions* and the *particular solutions*. Begin by putting the complete system into its homogeneous form

$$\dot{\mathbf{y}} = A\mathbf{y} \tag{24.15}$$

We proceed by “guessing" that the homogeneous solutions are of the form

$$\mathbf{y} = \mathbf{k}e^{rt}$$

where $\mathbf{k}$ is an n-dimensional vector of constants and r is a scalar. To see if this guess is correct, check that the guessed solution and its first derivative satisfy the differential equation system. The derivative of the proposed solution is

$$\dot{\mathbf{y}} = r\mathbf{k}e^{rt}$$

Substitution of these derivatives and the proposed solutions into the original system of equations gives

$$r\mathbf{k}e^{rt} = A\mathbf{k}e^{rt}$$

Simplifying gives

$$[A - rI]\mathbf{k} = \mathbf{0} \tag{24.16}$$

where I is the identity matrix, and $\mathbf{0}$ is the zero-vector. We wish to find the values of r that solve this equation. These values of r make our guessed solution correct.

In chapter 8, we learned that a system of linear *homogeneous* equations such as equation (24.16) has a nontrivial solution if and only if the determinant of $[A - rI]$ is identically equal to zero. Thus, the solution values for r are found by solving

$$|A - rI| = 0 \tag{24.17}$$

which is a polynomial equation of degree n in the unknown number r. This is known as the **characteristic equation** of matrix A and its solutions are called the characteristic roots or the **eigenvalues** of the A matrix. A nonzero vector $\mathbf{k}_1$, which is a solution of equation (24.16) for a particular eigenvalue, r_1, is called the **eigenvector** of the matrix A corresponding to the eigenvalue r_1.

In the case $n = 2$, equation (24.17) becomes

$$\begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$$

which, after simplifying, gives

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

which is, of course, the same characteristic equation obtained using the substitution method.

Before proceeding, it is useful to pause and make note of the fact that in the case of $n = 2$, the characteristic equation can be written as

$$r^2 - \text{tr}(A)r + |A| = 0$$

where $\text{tr}(A) = a_{11} + a_{22}$ is the sum of the diagonal elements of the coefficient matrix and $|A| = a_{11}a_{22} - a_{12}a_{21}$ is the determinant of the coefficient matrix. The solution for the characteristic roots can then be expressed as

$$r_1, r_2 = \frac{\text{tr}(A)}{2} \pm \frac{1}{2}\sqrt{\text{tr}(A)^2 - 4|A|}$$

This expression provides a fast way of calculating the characteristic roots directly from the coefficient matrix.

In general, there are n equations and n characteristic roots; therefore, there are n solutions to the system of differential equations. As in the case of a two-dimensional system of linear equations, the general solution to the homogeneous form is a linear combination of n distinct solutions.

Theorem 24.3 If $\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n$ are linearly independent solutions of the homogeneous system in equation (24.15), then the general solution of the system is the linear combination

$$\mathbf{y}(t) = c_1\mathbf{y}^1(t) + c_2\mathbf{y}^2(t) + \cdots + c_n\mathbf{y}^n(t)$$

for a unique choice of the constants.

Example 24.8 Solve the following 2×2 system of differential equations using the direct method:

$$\dot{\mathbf{y}} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}\mathbf{y}$$

Solution

In this example, $\dot{\mathbf{y}}$ and $\mathbf{y}$ are two-dimensional vectors. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 4-r & -1 \\ -4 & 4-r \end{vmatrix} = 0$$

which becomes $r^2 - 8r + 12 = 0$. The solutions are $r_1 = 2$ and $r_2 = 6$. For $r_1 = 2$ we want to compute nontrivial solutions for the eigenvectors

$$\begin{bmatrix} 4-2 & -1 \\ -4 & 4-2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives $2k_1 - k_2 = 0$. As we often do with eigenvectors, we set $k_1 = 1$ which gives $k_2 = 2$. Thus the first set of solutions is

$$\mathbf{y}^1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

For $r_2 = 6$ the eigenvectors are the solution to

$$\begin{bmatrix} 4-6 & -1 \\ -4 & 4-6 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives $-2k_1 - k_2 = 0$. With $k_1 = 1$, we get $k_2 = -2$. Thus the second set of solutions is

$$\mathbf{y}^2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

Since these two solutions are linearly independent, the general solution is

$$\mathbf{y}(t) = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{6t}$$

■

Example 24.9 Solve the differential equation system in example 24.1 using the direct method.

Solution

The coefficient matrix for that system is

$$A = \begin{bmatrix} 1 & -3 \\ 1/4 & 3 \end{bmatrix}$$

The characteristic equation then is

$$\begin{vmatrix} 1-r & -3 \\ 1/4 & 3-r \end{vmatrix} = 0$$

which gives $r^2 - 4r + 15/4 = 0$. The solutions are $r_1 = 3/2$ and $r_2 = 5/2$. For $r_1 = 3/2$, the eigenvector is the solution to

$$\begin{bmatrix} 1-3/2 & -3 \\ 1/4 & 3-3/2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives $-k_1/2 - 3k_2 = 0$. With $k_1 = 1$, we get $k_2 = -1/6$. The first set of solutions then is

$$\mathbf{y}^1(t) = \begin{bmatrix} 1 \\ -1/6 \end{bmatrix} e^{3t/2}$$

For $r_2 = 5/2$, the eigenvector is the solution to

$$\begin{bmatrix} 1 - 5/2 & -3 \\ 1/4 & 3 - 5/2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives $-3k_1/2 - 3k_2 = 0$. With $k_1 = 1$, this gives $k_2 = -1/2$. The second set of solutions then is

$$\mathbf{y}^2(t) = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} e^{5t/2}$$

Since these two solutions are linearly independent, the general solutions are

$$\mathbf{y}(t) = C_1 \begin{bmatrix} 1 \\ -1/6 \end{bmatrix} e^{3t/2} + C_2 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} e^{5t/2}$$ ■

Examples 24.8 and 24.9 demonstrate the direct method when the characteristic roots (eigenvalues) are real and distinct. The next example demonstrates the direct method when the characteristic roots are complex.

Example 24.10 Solve the homogeneous differential equation system

$$\dot{\mathbf{y}} = A\mathbf{y}, \quad \text{where } A = \begin{bmatrix} 2 & -5 \\ 2 & -4 \end{bmatrix}$$

Solution

The characteristic roots are $r_1 = -1 + i$ and $r_2 = -1 - i$. For $r_1 = -1 + i$, the eigenvectors are the solutions to

$$\begin{bmatrix} 3 - i & -5 \\ 2 & -3 - i \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This result gives $(3 - i)k_1 - 5k_2 = 0$ or $k_2 = (3 - i)k_1/5$. It is easier to set $k_1 = 5$ this time as this value gives $k_2 = 3 - i$. Since the eigenvectors end up being multiplied by constants anyway in the general solution, this rescaling has no effect on the general solution. The first solution then is

$$\mathbf{y}^1(t)\begin{bmatrix} 5 \\ 3-i \end{bmatrix} e^{(-1+i)t}$$

For $r_2 = -1 - i$, the eigenvectors are the solutions to

$$\begin{bmatrix} 3+i & -5 \\ 2 & -3+i \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

This gives $(3+i)k_1 - 5k_2 = 0$ or $k_2 = (3+i)k_1/5$. With $k_1 = 5$, we get $k_2 = 3+i$. The second solution then is

$$\mathbf{y}^2(t) = \begin{bmatrix} 5 \\ 3+i \end{bmatrix} e^{(-1-i)t}$$

The two combine to give the general solution

$$\mathbf{y}(t) = C_1\begin{bmatrix} 5 \\ 3-i \end{bmatrix} e^{(-1+i)t} + C_2\begin{bmatrix} 5 \\ 3+i \end{bmatrix} e^{(-1-i)t}$$

By Euler's formula, this transforms into

$$\mathbf{y}(t) = C_1\begin{bmatrix} 5 \\ 3-i \end{bmatrix} e^{-t}[\cos(t) + i\sin(t)] + C_2\begin{bmatrix} 5 \\ 3+i \end{bmatrix} e^{-t}[\cos(t) - i\sin(t)]$$

The right-hand side of the second equation here is

$$\{C_1(3-i)[\cos(t) + i\sin(t)] + C_2(3+i)[\cos(t) - i\sin(t)]\}\, e^{-t}$$

After combining the real and imaginary parts, and using the fact that $i^2 = -1$, this becomes

$$\{[(C_1 + C_2)[3\cos(t) + \sin(t)] + i(C_1 - C_2)[-\cos(t) + \sin(t)]\}\, e^{-t}$$

After defining new constants $A_1 = C_1 + C_2$ and $A_2 = i(C_1 - C_2)$, the general solution becomes

$$\mathbf{y}(t) = \left\{A_1\begin{bmatrix} 5\cos(t) \\ 3\cos(t) + \sin(t) \end{bmatrix} + A_2\begin{bmatrix} 5\sin(t) \\ -\cos(t) + 3\sin(t) \end{bmatrix}\right\} e^{-t}$$

■

The Particular Solutions

The steady-state solutions provide the particular solutions we require. Set $\dot{\mathbf{y}} = \mathbf{0}$ in the complete system of differential equations. This gives

$$A\bar{\mathbf{y}} + \mathbf{b} = \mathbf{0}$$

for which the solution is

$$\bar{\mathbf{y}} = -A^{-1}\mathbf{b}$$

provided the inverse matrix A^{-1} exists. This matrix exists if and only if A is nonsingular (i.e., the determinant must be nonzero). We have already assumed this for the case of $n = 2$. We now assume that $|A| \neq 0$ for any n.

Before concluding this section, we show that any single second-order differential equation can be transformed into a system of two first-order differential equations. The important implication of this is that systems containing higher-order equations can always be transformed into a system containing only first-order equations using the technique explained next. It is sufficient therefore to study first-order systems of differential equations.

Example 24.11 Consider the differential equation

$$\ddot{y} - a_1\dot{y} - a_2 y = b$$

To transform this second-order, linear differential equation into a system of two first-order, linear differential equations, define a new variable x as

$$x = \dot{y}$$

The definition implies that $\dot{x} = \ddot{y}$, which, after substitution into the original second-order differential equation, gives an equivalent system of two first-order differential equations

$$\dot{x} = a_1 x + a_2 y + b$$
$$\dot{y} = x$$

Comparing this system of equations to the general form in definition 24.1, we see that $x = y_1$, $y = y_2$, $a_1 = a_{11}$, $a_2 = a_{12}$, $a_{21} = 1$, and $a_{22} = 0$. This system can be solved for $y(t)$ and $x(t)$ simultaneously using the methods outlined in this chapter; of course, in this particular case, we would only be interested in the solution for $y(t)$. ■

EXERCISES

1. Transform the following second-order differential equations into systems of two first-order differential equations and solve:

(a) $2\ddot{y} - 5\dot{y} + y - 10 = 0$

(b) $\ddot{y} - 2y = 1$

(c) $\ddot{y} + 10\dot{y} + y = 1$

2. For each of the following systems of linear differential equation systems, solve using:

(a) the substitution method

(b) the direct method

Ensure the solutions satisfy any initial conditions that are given:

(i) $\dot{y}_1 = y_1 + 5y_2$, $\dot{y}_2 = \frac{1}{4}y_1 - y_2$

(ii) $\dot{y}_1 = y_1 + 5y_2 + 18$, $\dot{y}_2 = \frac{1}{4}y_1 - y_2 + 9$, and $y_1(0) = 6$, $y_2(0) = 0$

(iii) $\dot{y}_1 = 2y_1 + y_2/2$, $\dot{y}_2 = 7y_1/2 - y_2 + 15$, and $y_1(0) = 2$, $y_2(0) = 4$

(iv) $\dot{y}_1 = 3y_1 + y_2 + 4$, $\dot{y}_2 = 2y_1 + 2y_2 - 12$, and $y_1(0) = -2$, $y_2(0) = 5$

3. For each of the following systems of linear differential equation systems, solve using:

(a) the substitution method

(b) the direct method

Ensure that the solutions satisfy any initial conditions that are given:

(i) $\dot{y}_1 = -2y_1 + 2y_2 + 12$, $\dot{y}_2 = y_1 - 3y_2 - 12$, and $y_1(0) = -2$, $y_2(0) = 5$

(ii) $\dot{y}_1 = -y_1 - 9y_2/4 + 2$, $\dot{y}_2 = -3y_1 + 2y_2 - 1$, and $y_1(0) = 20$, $y_2(0) = 2$

(iii) $\dot{y}_1 = 2y_1 - 2y_2 + 5$, $\dot{y}_2 = 2y_1 + 2y_2 + 1$, and $y_1(0) = 2.5$, $y_2(0) = -1$

24.2 Stability Analysis and Linear Phase Diagrams

The steady-state solutions to an autonomous system of differential equations are said to be *stable* if the system *converges* to the steady state solutions and *unstable* otherwise. As in previous chapters we emphasize the issue of stability here because of its importance in economic applications.

In chapters 21 to 23 we found that the stability characteristics of differential equations depend on the *signs* of the characteristic roots. Roots with negative real parts are associated with differential equations that converge to the steady state (stable); roots with positive real parts are associated with differential equations that diverge from the steady state (unstable).

In this section we will see that the stability of a *system* of differential equations also depends on the signs of the roots of the characteristic equation. Theorem 24.4 states the conditions for convergence.

Theorem 24.4 The steady-state solution of a system of linear, autonomous differential equations is asymptotically stable if and only if the characteristic roots are negative (the real part is negative in the case of complex-valued roots).

Proof

Theorem 24.4 applies regardless of the number of equations in the system; however, we prove it only for the case of two equations. We consider the three possible types of roots that can occur:

Roots real and distinct. The solutions to the system of two autonomous, linear differential equations are

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{y}_1$$
$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

Under what conditions does $y_1(t)$ *converge* to $\bar{y}_1$ and $y_2(t)$ *converge* to $\bar{y}_2$? Inspection of the solutions reveals that if r_1 and r_2 are *negative*, the exponential terms, $e^{r_1 t}$ and $e^{r_2 t}$, tend to zero in the limit as t goes to infinity. Therefore both solutions *converge* to their steady states as $t \to \infty$ for all values of the constants of integrations. Because $y_i(t)$ converges to $\bar{y}_i$ $(i = 1, 2)$ only as $t \to \infty$, the path $y_i(t)$ is asymptotic to the value $\bar{y}_i$. For this reason the steady state is said to be *asymptotically* stable if both roots are negative.

If r_1 and r_2 are *positive*, the exponential terms tend to infinity as $t \to \infty$. Hence both solutions diverge from the steady state, except when the constants of integration are both equal to zero. Therefore the steady state is unstable when both roots are positive.

If one root is *negative* and the other is *positive*, the exponential term containing the negative root goes to zero as $t \to \infty$ but the exponential term containing the positive root diverges to infinity as $t \to \infty$. Hence both solutions diverge from the steady state except when the constant of integration on the divergent exponential term is equal to zero. (This special case actually plays an important role in economic applications so we will have much more to say about it.) Therefore the steady state is unstable when even one of the roots is positive.

Roots real and equal. The solutions to the system of two autonomous, linear differential equations are

$$y_1(t) = C_1 e^{rt} + C_2 t e^{rt} + \bar{y}_1$$
$$y_2(t) = \left[\frac{r - a_{11}}{a_{12}}(C_1 + C_2 t) + \frac{C_2}{a_{12}}\right] e^{rt} + \bar{y}_2$$

The equal roots are either positive or negative. If positive, the exponential terms tend to infinity as $t \to \infty$, so both solutions diverge from the steady state. If negative, both solutions converge to their steady-state values. The proof of this is identical to the proof of convergence in the case of equal roots for a second-order differential equation in chapter 23.

Complex roots. The solution in this case is

$$y_1(t) = e^{ht}[A_1 \cos(vt) + A_2 \sin(vt)] + \bar{y}_1$$
$$y_2(t) = e^{ht}\left[\frac{(h - a_{11})A_1 + vA_2}{a_{12}} \cos(vt) + \frac{(h - a_{11})A_2 - vA_1}{a_{12}} \sin(vt)\right] + \bar{y}_2$$

Since the sine and cosine functions are bounded even as $t \to \infty$, the solutions diverge if the real part of the roots is positive, $h > 0$, and converge if the real part is negative, $h < 0$. ■

Theorem 24.4 says that y_1 and y_2 converge to $\bar{y}_1$ and $\bar{y}_2$, respectively, if the roots are negative, no matter what values the constants of integration take. Since the constants are determined by initial conditions, we can interpret this result as saying that no matter what the initial conditions, $y_1(t)$ and $y_2(t)$ will always converge towards the values $\bar{y}_1$ and $\bar{y}_2$ if the roots are negative.

In economic models of dynamic optimization, it is common to obtain a system of differential equations in which one of the characteristic roots is positive and the other is negative. We examine this important case next.

Theorem 24.5 If one of the characteristic roots is positive and the other is negative, the steady-state equilibrium is called a **saddle-point** equilibrium. It is unstable. However, $y_1(t)$ and $y_2(t)$ converge to their steady-state solutions if the initial conditions for $y_1$and y_2, satisfy the following equation:

$$y_2 = \frac{r_1 - a_{11}}{a_{12}}(y_1 - \bar{y}_1) + \bar{y}_2$$

where r_1 is the negative root and r_2 is the positive root. The locus of points (y_1, y_2) defined by this equation is known as the **saddle path.**

Proof

The characteristic roots are real valued if they are of opposite sign. The solutions are

$$y_1(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{y}_1$$
$$y_2(t) = \frac{r_1 - a_{11}}{a_{12}} C_1 e^{r_1 t} + \frac{r_2 - a_{11}}{a_{12}} C_2 e^{r_2 t} + \bar{y}_2$$

Without loss of generality, assume that r_1 is the negative root and r_2 is the positive root. Then, $y_1(t)$ and $y_2(t)$ converge to their steady-state solutions if and only if $C_2 = 0$. Solving for C_1 and C_2 in the solutions above gives

$$C_1 = \frac{(y_1 - \bar{y}_1)(r_2 - a_{11}) - a_{12}(y_2 - \bar{y}_2)}{r_2 - r_1} e^{-r_1 t}$$
$$C_2 = \frac{a_{12}(y_2 - \bar{y}_2) - (y_1 - \bar{y}_1)(r_1 - a_{11})}{r_2 - r_1} e^{-r_2 t}$$

where the t arguments for y_1 and y_2 are suppressed to shorten the expressions. Setting $C_2 = 0$ and simplifying implies that

$$y_2 = (y_1 - \bar{y}_1)\frac{r_1 - a_{11}}{a_{12}} + \bar{y}_2$$ ■

Theorem 24.5 tells us that if even one of the characteristic roots is positive, the solutions will not converge to the steady state from *arbitrarily* chosen initial conditions; on the other hand, if the initial conditions for y_1 and y_2 happen to satisfy the equation given in theorem 24.5, the solutions will converge. This is called a *saddle-point equilibrium* and can only occur for the case of real and distinct roots. It plays quite an important role in economic dynamics so we will have more to say about it throughout this chapter and the next.

Linear Phase Diagrams

The phase diagram proved to be a useful tool for conducting a qualitative analysis of a single nonlinear differential equation in chapter 22. It will prove to be equally valuable in the analysis of a system of two differential equations. We explain the construction of a phase diagram, beginning with a system of two *linear* differential equations. The method, and the interesting variety of trajectory systems that can arise, carry over to the analysis of systems of two *nonlinear* differential equations.

In chapter 22 we constructed phase diagrams for single nonlinear differential equations by plotting $\dot{y}$ against y so that we could see the *range* of y values for which $\dot{y}$ is positive and those for which it is negative. That allowed us to *see* whether y converged and, if so, to what value. The objective is the same in the case of a phase diagram for a *system* of two differential equations but the method is slightly different. We explain the method by example.

Example 24.12 **Phase Diagram for Both Roots Negative (Stable Node)**

Solve the following differential equation system, and draw its phase diagram:

$$\dot{y}_1 = -2y_1 + 2$$
$$\dot{y}_2 = -3y_2 + 6$$

Solution

Since $a_{12} = 0 = a_{21}$, these differential equations are actually independent of one another and so can be solved separately as single equations. From chapter 21 the solutions are

$$y_1(t) = C_1 e^{-2t} + 1$$
$$y_2(t) = C_2 e^{-3t} + 2$$

where C_1 and C_2 are arbitrary constants of integration. In this system, it is clear that $y_1(t)$ converges to its steady-state solution $\bar{y}_1 = 1$ and $y_2(t)$ converges to its steady-state solution $\bar{y}_2 = 2$ because the exponential terms go to zero as $t \to \infty$. The steady-state point $(1, 2)$ is therefore a stable equilibrium.

Suppose that we choose initial values of $y_1^0 = 3$ and $y_2^0 = 1/2$. Figure 24.1 shows the trajectories for $y_1(t)$ and $y_2(t)$ that emanate from these initial conditions.

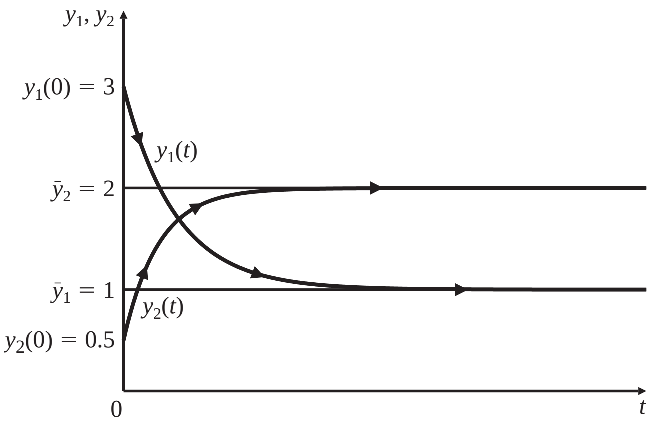


Figure 24.1 Solutions for y_1 and y_2 plotted as an explicit function of time

The $y_1(t)$ trajectory falls (since it starts above its steady-state solution) as it converges and the $y_2(t)$ trajectory rises (since it starts below its steady-state solution) as it converges.

Figure 24.1 provides a useful way to “see" the solutions for $y_1(t)$ and $y_2(t)$. However, it is not always possible to construct a diagram like figure 24.1 with y_1 and y_2 plotted as explicit functions of t because explicit solutions cannot always be obtained (i.e., for nonlinear differential equations). A phase diagram circumvents this problem and, at the same time, provides a different way to “see" the solution.

A phase diagram for a system of two differential equations is drawn with y_2 on the vertical axis and y_1 on the horizontal axis. The y_1, y_2 plane is referred to as the **phase plane.** We construct the phase diagram for this example in two steps:

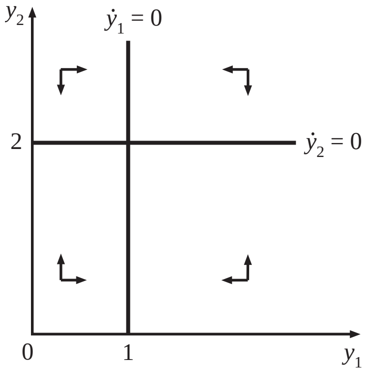


Figure 24.2 Phase diagram for example 24.12

Step 1 *Determine the motion of* y_1 *in the phase plane.* Begin by graphing the locus of points for which $\dot{y}_1 = 0$. To find these points, set $\dot{y}_1 = 0$. This gives

$$y_1 = 1$$

In figure 24.2, a vertical line is drawn at $y_1 = 1$ to show this locus of points. This line is called the y_1 **isocline;** it divides the phase plane into two regions or **isosectors.** In the region to the *right of* the isocline ($y_1 > 1$), the differential equation for y_1 shows that $\dot{y}_1$ is negative. In the region to the *left* of the isocline ($y_1 < 1$), it shows that $\dot{y}_1$ is positive.

We have established the motion of y_1 in the two regions separated by the y_1 isocline: y_1 is decreasing to the right (because $\dot{y}_1 < 0$) of the isocline and increasing to the left (because $\dot{y}_1 > 0$) of the isocline. To indicate this motion in the diagram, we draw horizontal arrows pointing in the appropriate directions.

Step 2 *Determine the motion of* y_2. Begin by graphing the y_2 isocline. Setting $\dot{y}_2 = 0$ gives

$$y_2 = 2$$

which is a horizontal line at $y_2 = 2$ in figure 24.2. It too divides the plane into two regions. Inspection of the differential equation for y_2 shows that $\dot{y}_2$ is negative above the isocline (for $y_2 > 2$) and positive below the isocline (for $y_2 < 2$). To indicate the motion of y_2 in the diagram, we draw vertical arrows pointing in the appropriate directions.

The two isoclines intersect where both $\dot{y}_1$ and $\dot{y}_2$ equal zero. By definition 24.3, this is the steady-state point. In figure 24.2, this occurs at the point (1, 2).

The *arrows of motion* give a rough picture of what the trajectories in the phase plane look like and whether they move towards the steady state. For example, to the southeast of the steady-state solution, the arrows of motion indicate that trajectories in this sector of the phase plane move in a northwesterly direction, meaning that $y_1(t)$ decreases and $y_2(t)$ increases over time. Trajectories in the southwestern sector of the phase plane move in a northeasterly direction. Trajectories in the northwest move southeast and trajectories in the northeast move southwest. Overall, the arrows of motion tell us that no matter what sector of the phase plane we start in, trajectories always move towards the steady state. The phase diagram then provides a good indication that the steady state is globally stable.

To get a precise picture of the trajectories requires that we actually plot them using the solutions to the two differential equations. We do this in figure 24.3.

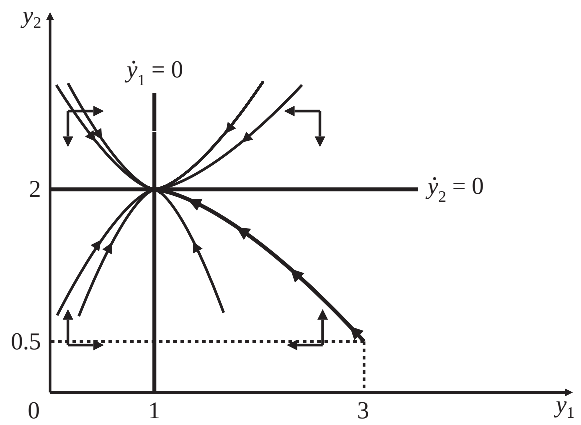


Figure 24.3 Phase diagram for example 24.12 showing representative trajectories: a stable node

A **trajectory** in the y_1, y_2 phase plane is the *path* followed by the pair y_1, y_2. For example, suppose we start at the point $(3, 0.5)$, the same initial point that was used in figure 24.1. The trajectory emanating from this initial point travels northwest over time. Compare this trajectory of (y_1, y_2) to the trajectories of y_1 and y_2 plotted against t in figure 24.1. Although time is not explicitly shown in the phase diagram, it is definitely implicit in the trajectories. For example, at $t = 0$, the trajectory is at the point $(3, 1/2)$. After some time has passed, the pair $y_1(t)$ and $y_2(t)$ have moved along the trajectory to the northwest. This means $y_1(t)$ has decreased while $y_2(t)$ has increased, as shown explicitly in figure 24.1. After more time has passed, the pair is further along the trajectory which means $y_1(t)$ has decreased further while $y_2(t)$ has increased further, as shown in figure 24.1. As $t \to \infty$, the pair $y_1(t)$ and $y_2(t)$ *converges* to the point $(1, 2)$ in figure 24.3 just as $y_1(t)$ and $y_2(t)$ converge to $\bar{y}_1$ and $\bar{y}_2$ in figure 24.1.

An important characteristic of a dynamic system in which both roots are negative is that no matter what the initial values of y_1 and y_2, their paths converge to the steady state. This is demonstrated in figure 24.3 where a number of representative trajectories are drawn. All trajectories in the phase plane converge asymptotically to the steady state. This kind of equilibrium is called a **stable node.** ■

Example 24.13 Phase Diagram for Both Roots Positive (Unstable Node)

Solve and graph the phase diagram for the differential equation system

$$\dot{y}_1 = 2y_1 - 2$$
$$\dot{y}_2 = 3y_2 - 6$$

Solution

The only differences between these equations and those in example 24.12 are the signs of the coefficients and terms. The solutions are

$$y_1(t) = C_1 e^{2t} + 1$$
$$y_2(t) = C_2 e^{3t} + 2$$

The phase diagram is constructed in the same way as for the system in the previous example. It has the same isoclines as that system but the motion is exactly opposite. Thus, the trajectories in the phase diagram, drawn in figure 24.4 have exactly the same shape but go in the opposite direction to the trajectories for the previous example. As a result, we see clearly that the system diverges from the steady state from all points in the phase plane except the steady state itself. The steady state in this case is called an **unstable node.**

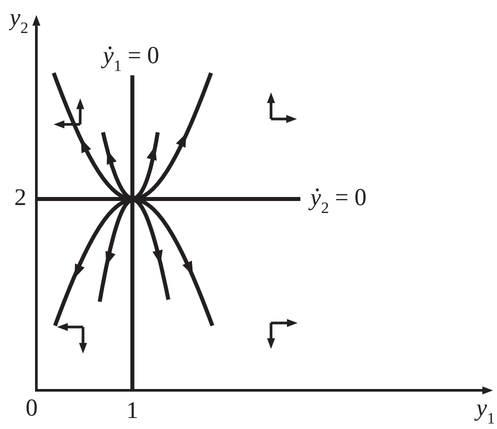


Figure 24.4 Phase diagram for example 24.13: An unstable node

In the case of real and equal roots, the dynamic system is called an **improper node.** We do not show a phase diagram here for this case; it is sufficient for our purposes to note that, if the repeated root is negative, all trajectories converge to the steady state and if the repeated root is positive, all trajectories diverge from the steady state. ■

Example 24.14 **Phase Diagram for Roots of Opposite Sign (Saddle Point)**

Draw the phase diagram for

$$\dot{y}_1 = y_2 - 2$$
$$\dot{y}_2 = \frac{y_1}{4} - \frac{1}{2}$$

Solution

The characteristic equation is

$$|A - rI| = \begin{vmatrix} 0 - r & 1 \\ 1/4 & 0 - r \end{vmatrix} = 0$$

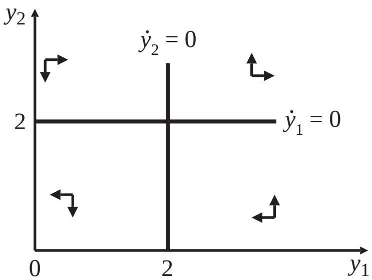


Figure 24.5 Phase diagram for example 24.14

for which the solutions are $r_1 = -1/2$ and $r_2 = 1/2$. Since the roots are of opposite sign, the steady-state solution is a saddle-point equilibrium.

Construct the phase diagram:

Step 1 *Determine the motion of* y_1. Begin by graphing the y_1 isocline: setting $\dot{y}_1 = 0$ to find the isocline gives the horizontal line $y_2 = 2$. Next, we note that $\dot{y}_1 < 0$ below this isocline (when $y_2 < 2$) and $\dot{y}_1 > 0$ above the isocline (when $y_2 > 2$). The appropriate horizontal arrows of motion are shown in figure 24.5.

Step 2 *Determine the motion of* y_2. Begin by graphing the y_2 isocline: setting $\dot{y}_2 = 0$ gives the vertical line $y_1 = 2$. To the right of this line $(y_1 > 2)$, $\dot{y}_2 > 0$ and to the left of it $(y_1 < 2)$, $\dot{y}_2 < 0$. The appropriate vertical arrows of motion are shown in the phase diagram.

The arrows of motion in figure 24.5 indicate that trajectories in the southwest and northeast sectors of the phase plane definitely move away from the steady state. But the arrows of motion in the northwest and southeast sectors show that trajectories move toward the steady state. What do the trajectories actually look like?

Figure 24.6 shows some representative trajectories. Consider an arbitrary starting point in figure 24.6 such as point a. At this point, the arrows of motion indicate that y_1 is decreasing and y_2 is increasing. The motion is northwesterly, therefore. Follow this trajectory along its path. As it gets close to the y_1 isocline where $\dot{y}_1 = 0$, the motion of y_1 slows down but y_2 continues to increase. As a result the

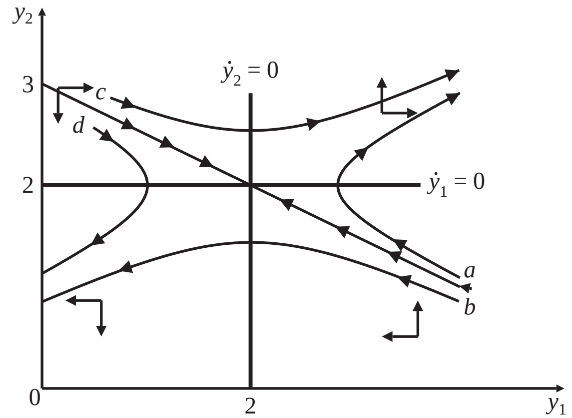


Figure 24.6 Phase diagram for example 24.14 showing representative trajectories: A saddle point

trajectory bends upward. As it crosses the y_1 isocline, y_1 is stationary for an instant even though y_2 keeps increasing. As a result the trajectory must be vertical at the crossing. From there it proceeds into a new isosector in which both y_1 and y_2 are increasing. Thus the trajectory bends back and goes in a northeasterly direction. It stays in that isosector, traveling ever farther and farther away from the steady state.

Consider a trajectory starting at point b. Like the trajectory that started at a, the motion is northwesterly; however, this time the trajectory gets close to the y_2 isocline where $\dot{y}_2 = 0$, so the motion of y_2 slows down while y_1 continues to decrease. As a result the trajectory bends to the left. As it crosses the y_2 isocline, it is horizontal because y_2 is stationary at that point, even though y_1 keeps decreasing. In the new isosector, the trajectory turns southwesterly and continues in that direction, traveling away from the steady state.

Trajectories starting from points c and d are also shown. These have the mirror-image properties of the trajectories starting from points a and b. These four arbitrarily chosen trajectories verify that most trajectories end up diverging from a steady state which is a **saddle-point equilibrium.** These trajectories also demonstrate the very important property that trajectories must obey the arrows of motion and must be horizontal when they cross the y_2 isocline and vertical when they cross the y_1 isocline.

Since the steady state is a saddle-point equilibrium, we know that some trajectories do converge to the steady state, provided they start from initial conditions satisfying theorem 24.5. By theorem 24.5, the saddle path is given by

$$y_2 = 3 - \frac{1}{2}y_1$$

This equation is graphed in figure 24.6; it is a straight line with intercept 3 and slope $-1/2$. If the pair of initial conditions for y_1 and y_2 lie anywhere on this line, the pair $y_1(t)$ and $y_2(t)$ converge to the steady state. Since trajectories beginning anywhere along this line do converge, we can think of the line itself as a trajectory, but one with the special property that it is the only path that converges to the saddle-point equilibrium. This is why it is called the saddle path. ■

Example 24.15 **Phase Diagram for Complex Roots with Negative Real Parts (Stable Focus)**

Solve and draw the phase diagram for

$$\dot{y}_1 = -y_2 + 2$$
$$\dot{y}_2 = y_1 - y_2 + 1$$

Solution

In matrix form, the homogeneous form of this system is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The roots are

$$r_1, r_2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

where i is the imaginary number. The steady-state solutions are: $\bar{y}_1 = 1$ and $\bar{y}_2 = 2$. The complete solutions then are

$$y_1(t) = e^{-t/2}\left[A_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + A_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] + 1$$

$$y_2(t) = -e^{-t/2}\left[\left(-\frac{A_1}{2} + \frac{\sqrt{3}}{2}A_2\right)\cos\left(\frac{\sqrt{3}}{2}t\right) + \left(-\frac{A_2}{2} - \frac{\sqrt{3}}{2}A_1\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right] + 2$$

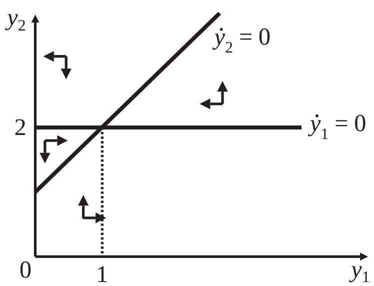


Figure 24.7 Phase diagram for example 24.15

Construct the phase diagram for this system.

First, determine the motion for y_1. Setting $\dot{y}_1 = 0$ gives the y_1 isocline as the horizontal line $y_2 = 2$ in figure 24.7. If $y_2 > 2$, then $\dot{y}_1 < 0$, so above the isocline, y_1 is decreasing; below the isocline, y_1 is increasing.

Second, determine the motion for y_2. The y_2 isocline occurs along the line $y_2 = y_1 + 1$. This is a straight line in figure 24.7 with intercept 1 and slope 1. To determine the motion of y_2 in the two isosectors created by this line is a bit more difficult than before because $\dot{y}_2$ depends on y_1 as well as y_2. Begin with an arbitrary point on the isocline; then, holding y_2 constant, move to a point to the right. Since moving to the right increases y_1 but leaves y_2 unchanged, the differential equation indicates that $\dot{y}_2$ becomes positive. This establishes that $\dot{y}_2 > 0$ to the right of the isocline. Similarly moving from any point on the isocline to the left decreases y_1 but leaves y_2 unchanged. The differential equation indicates that $\dot{y}_2$ becomes negative to the left of the isocline.

It is worthwhile reviewing the technique used above to determine the motion of y_2 on either side of the isocline because of the central role it plays in constructing phase diagrams. Effectively this technique is equivalent to taking the partial derivative of the $\dot{y}_2$ equation with respect to y_1. We see that $\partial \dot{y}_2 / \partial y_1 = 1 > 0$. This implies that a horizontal movement to the right (left) in the phase plane increases (decreases) $\dot{y}_2$. Thus $\dot{y}_2 > 0$ to the right of the isocline (on which $\dot{y}_2 = 0$) and $\dot{y}_2 < 0$ to the left of the isocline.

The arrows of motion suggest that trajectories might be spirals. This is indeed the case but there is nothing about the phase diagram itself that indicates whether the spirals converge to or diverge from the steady state. However, because the real part of the characteristic roots is negative, theorem 24.4 tells us that the trajectories converge. Figure 24.8 shows representative trajectories.

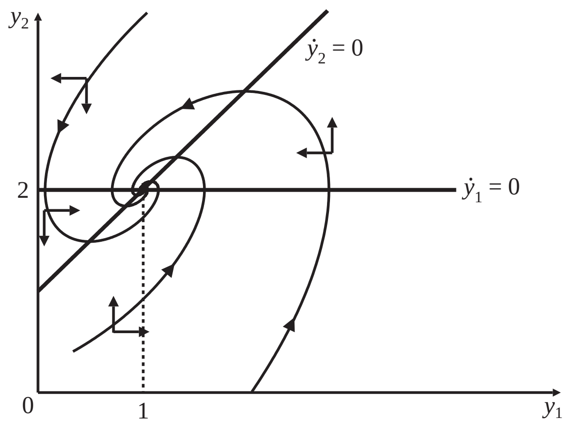


Figure 24.8 Phase diagram for example 24.15 showing representative trajectories: A stable focus

The trajectories for this system are spirals centered on and converging to the point $(1, 2)$. The steady state in this case is called a **stable focus.** The spirals converge to the steady state because the real part of the complex roots is negative.

If it had been positive, the trajectories would go in the opposite direction, diverging from the steady state. This case is called an **unstable focus.** If the real part of the complex roots turned out to be zero, trajectories would neither converge nor diverge. Instead, they would orbit the steady state endlessly. This is called a **center.** ■

Determining Stability from the Coefficient Matrix

Calculating the values of the characteristic roots allows us to determine if a steady state is stable or unstable; however, it would be helpful if there were a faster and more direct way of determining stability. It turns out there is: stability can be determined directly from the coefficients of the differential equations. Theorem 24.6 states the result.

Theorem 24.6 Let $|A| = a_{11}a_{22} - a_{12}a_{21}$ be the determinant of the coefficient matrix A in a system of two linear differential equations, and assume that $|A| \neq 0$. Let $\text{tr}(A) = a_{11} + a_{22}$ be the trace of A. The stability properties of the steady-state equilibrium of the system are determined as follows:

(i) If $|A| < 0$, the characteristic roots are real and of opposite sign. In this case the steady state is a saddle-point equilibrium.
(ii) If $|A| > 0$, the characteristic roots are of equal sign if real-valued but could have complex values. If $\text{tr}(A) < 0$, the real parts of both roots are negative, giving an asymptotically stable steady state. If $\text{tr}(A) > 0$, the real parts of both roots are positive, giving an unstable steady state.

Proof

The proof uses the following two properties of characteristic roots (eigenvalues) developed in theorems 10.6 and 10.7: the sum of the two roots of matrix A is equal to the trace of A,

$$r_1 + r_2 = \text{tr}(A)$$

and the product of the two roots of A is equal to the determinant A,

$$r_1 r_2 = |A|$$

(i) Since $|A| = r_1 r_2$, then if $|A| < 0$, r_1 and r_2 must be of opposite sign.
(ii) If $|A| > 0$, then r_1 and r_2 must be either both negative or both positive, if real valued. Since $\text{tr}(A) = r_1 + r_2$, they are both negative if $\text{tr}(A) < 0$ and both positive if $\text{tr}(A) > 0$. If complex valued, the real part of the roots equals $\text{tr}(A)/2$. Hence, the real part is negative if and only if $\text{tr}(A) < 0$. ■

Table 24.1 Types of steady states in systems of two linear differential equations

1. If $\|A\| < 0$, then r_1, r_2 opposite signs	Saddle point
2. If $\|A\| > 0$, then:	
r_1 and $r_2 < 0$	Stable node
r_1 and $r_2 > 0$	Unstable node
$r_1 = r_2 = r < 0$	Improper stable node
$r_1 = r_2 = r > 0$	Improper unstable node
r_1, r_2 complex, $a_{11} + a_{22} < 0$	Stable focus
r_1, r_2 complex, $a_{11} + a_{22} > 0$	Unstable focus
r_1, r_2 complex, $a_{11} + a_{22} = 0$	Center

Theorem 24.6 provides a quick method by which we can determine the stability property of a system of differential equations. We need only calculate the *sign* of the determinant and the trace of the coefficient matrix. Table 24.1 summarizes the stability properties of systems of two linear, autonomous differential equations.

The Dornbusch Model of Exchange-Rate Overshooting

How do the price level and exchange rate in an economy respond to a change in the money supply? The Dornbusch overshooting model provides a framework for conducting an interesting analysis of this and related questions. In addition, it provides a classic example of a system of two linear, first-order differential equations for which the steady state is a saddlepoint equilibrium.

There are two markets in the model: an asset market in which the exchange rate, e, is determined, and a goods market in which the domestic price level, p, is determined. We begin with the asset market.

The demand for money, m^D, is given by

$$m^D = -ar + b\bar{y}$$

where r is the domestic interest rate, $\bar{y}$ is the domestic level of output, which is assumed to be constant, and a and b are constant positive coefficients of the money demand function.

All lowercase variables are in logarithms in the Dornbusch model. As a result the real supply of money is given by $m - p$, where m is the exogenous nominal supply of money. Equilibrium in the asset market requires that supply equals demand

$$m - p = -ar + b\bar{y} \tag{24.18}$$

If this economy were not open to trade with other countries, this equation would determine the equilibrium interest rate, r. However, because the economy is assumed to be open to trade, international capital movements will force the interest rate to equal the world interest rate, r^*, in the absence of exchange rate fluctuations. However, when the exchange rate can change, an interest rate differential equal to the expected rate of depreciation of the home currency can persist. Symbolically this means that

$$r = r^* + E(\dot{e}) \tag{24.19}$$

where $E(\dot{e})$ is the expected rate of depreciation of the exchange rate. The exchange rate, e, is defined as the domestic price of foreign currency. Note that it is correct to interpret $\dot{e}$ as a *rate* of change or depreciation because e is the logarithm of the actual exchange rate. An example illustrates this relationship: if the U.S. dollar is expected to depreciate by 1% against other currencies, then the U.S. interest rate will be 1% higher than the world rate in equilibrium. Investors are not willing to exploit the interest differential any further because their earnings are denominated in dollars, which are expected to depreciate by 1%, thereby canceling the higher interest rate.

Dornbusch assumes that economic agents have perfect foresight when forecasting exchange rate movements. This means that the *expected* and *actual* rates of depreciation of the exchange rate are assumed to be equal. This implies that

$$\dot{e} = E(\dot{e}) \tag{24.20}$$

Substituting this equation into equation (24.19) and then equation (24.19) into equation (24.18) gives

$$\dot{e} = \frac{p}{a} + \frac{b\bar{y} - m}{a} - r^*$$

after some rearranging. This is one of the two linear differential equations in the model. The second one, which we derive next, describes the dynamics of the domestic price level.

Whereas we have assumed that the asset market is always in equilibrium (exchange rate and interest rates adjust continuously), we assume that domestic prices adjust sluggishly in response to excess demand. Specifically, we assume that

$$\dot{p} = \alpha(y^D - y^S), \qquad \alpha > 0$$

where y^D and y^S are (the logarithms of) aggregate quantity demanded and supplied respectively, and α is a speed-of-adjustment coefficient. Under the assumption that supply is fixed at $\bar{y}$ and demand is given by

$$y^D = u + v(e - p)$$

where $e - p$ is the relative price of domestic goods and u and v are positive coefficients of the demand function, the differential equation for price becomes

$$\dot{p} = -\alpha v p + \alpha v e + \alpha(u - \bar{y})$$

The model is now complete. The system of first-order linear differential equations is rewritten as

$$\begin{bmatrix} \dot{p} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} -\alpha v & \alpha v \\ 1/a & 0 \end{bmatrix} \begin{bmatrix} p \\ e \end{bmatrix} + \begin{bmatrix} \alpha(u - \bar{y}) \\ (b\bar{y} - m)/a - r^* \end{bmatrix} \tag{24.21}$$

Using theorem 24.2, we have the solutions

$$p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{p} \tag{24.22}$$

$$e(t) = \frac{r_1 + \alpha v}{\alpha v} C_1 e^{r_1 t} + \frac{r_2 + \alpha v}{\alpha v} C_2 e^{r_2 t} + \bar{e} \tag{24.23}$$

where

$$r_1, r_2 = \frac{-\alpha v}{2} \pm \frac{1}{2}\sqrt{\alpha^2 v^2 + \frac{4\alpha v}{a}}$$

and where $\bar{p}$ and $\bar{e}$, the steady-state price and exchange rate respectively, are found by setting $\dot{p} = \dot{e} = 0$. This gives

$$\bar{p} = ar^* - b\bar{y} + m$$

$$\bar{e} = \bar{p} - \frac{u - \bar{y}}{v}$$

The determinant of the coefficient matrix is negative ($|A| = -\alpha v/a < 0$). As a result the roots are real valued and of opposite sign. Therefore the steady state is a saddle-point equilibrium.

The phase plane for the linear system of differential equations in equation (24.21) is drawn in figure 24.9. The p isocline is obtained by setting $\dot{p} = 0$, which gives the line

$$p = e + \frac{u - \bar{y}}{v}$$

The e isocline is obtained by setting $\dot{e} = 0$, which gives the line

$$p = \bar{p}$$

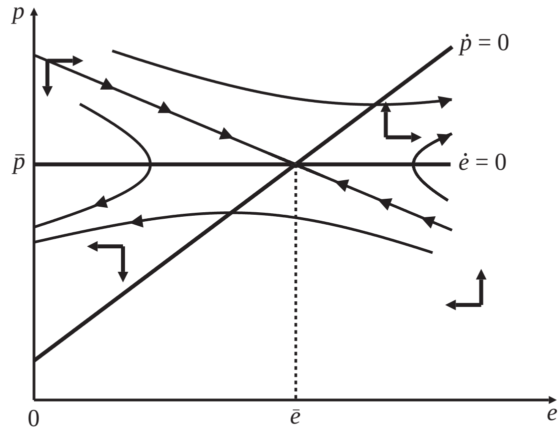


Figure 24.9 Phase diagram in the exchange rate, e, and price, p, phase plane for the Dornbusch overshooting model

The *motion* of p in the two isosectors separated by the p isocline is determined as follows: p is declining in the region *above* the $\dot{p} = 0$ line, and is increasing in the region *below*. This is confirmed by calculating that $\partial \dot{p}/\partial p = -\alpha v < 0$. (Alternatively, $\partial \dot{p}/\partial e = \alpha v > 0$ gives the same result.)

The motion of e is determined as follows: e is increasing *above* the $\dot{e} = 0$ line and decreasing *below* it. This is confirmed by calculating $\partial \dot{e}/\partial p = 1/a > 0$. The arrows of motion, some representative trajectories, and the saddle path are shown in figure 24.9.

Why is this an "overshooting" model? In the phase diagram in figure 24.10, the economy is initially in long-run equilibrium at point 1 and the relevant isoclines are those labeled with subscript 1. Now suppose there is an exogenous increase in the nominal supply of money m. This causes the e isocline to shift up to the one labeled with subscript 2, but it does not affect the p isocline. As a result $\bar{p}$ and $\bar{e}$ rise by the same amount (which is clear from the diagram because the slope of the p isocline is unity). The new long-run equilibrium is at point 2.

The economy cannot jump to the new steady state instantly, however, because domestic price, p, changes sluggishly. On the other hand, e can adjust instantly. Given a strong assumption of perfect foresight on the part of economic agents in the model, the exchange rate will *jump* to the point e_2 to reach the new saddle path. (Only this jump is consistent with the economy reaching the new long-run equilibrium at the new steady state.) The economy then follows the saddle path toward the new steady state. In this model the short-run increase in the exchange rate therefore overshoots the long-run increase.

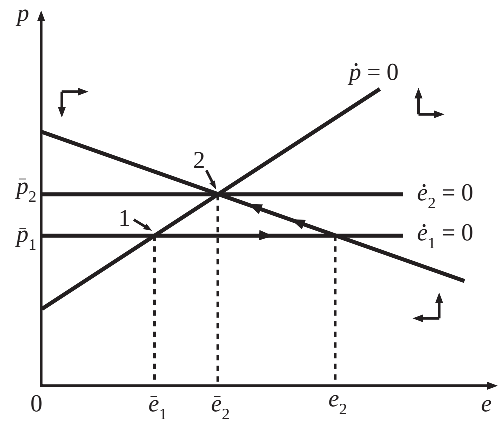


Figure 24.10 Short-run increase in the exchange rate, $\bar{e}_1$ to e_2, overshooting the long-run increase, $\bar{e}_1$ to $\bar{e}_2$

Stability Analysis of Nonlinear Differential Equation Systems

Differential equation systems often contain *nonlinear* equations in economic applications. When this happens in a system of two differential equations, phase diagrams are useful in analyzing the dynamics of the system. In addition, the behavior of the nonlinear differential equation system around the steady state (if it exists) is given by a linear approximation of the nonlinear model. This is an extremely useful result because it allows us to ascertain whether a nonlinear system is a stable or unstable node, a saddle point, a focus, or a center, simply by determining the signs of the characteristic roots of the linearized system at the equilibrium point.

Definition 24.5 A nonlinear system of two autonomous differential equations is expressed in general as

$$\dot{y}_1 = F(y_1, y_2)$$
$$\dot{y}_2 = G(y_1, y_2)$$

Assume that a steady state exists at the point $(\bar{y}_1, \bar{y}_2)$. Thus $F(\bar{y}_1, \bar{y}_2) = G(\bar{y}_1, \bar{y}_2) = 0$. Assuming that F and G have continuous second-order derivatives, we can take a first-order linear approximation to these differential equations for y_1 and y_2 close to $(\bar{y}_1, \bar{y}_2)$. This gives

$$\dot{y}_1 = F(\bar{y}_1, \bar{y}_2) + \frac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_1}(y_1 - \bar{y}_1) + \frac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_2}(y_2 - \bar{y}_2)$$
$$\dot{y}_2 = G(\bar{y}_1, \bar{y}_2) + \frac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_1}(y_1 - \bar{y}_1) + \frac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_2}(y_2 - \bar{y}_2)$$

But since $F(\bar{y}_1, \bar{y}_2) = G(\bar{y}_1, \bar{y}_2) = 0$, this reduces to

$$\dot{y}_1 = \frac{\partial F}{\partial y_1} y_1 + \frac{\partial F}{\partial y_2} y_2 - \left(\frac{\partial F}{\partial y_1} \bar{y}_1 + \frac{\partial F}{\partial y_2} \bar{y}_2 \right) \tag{24.24}$$

$$\dot{y}_2 = \frac{\partial G}{\partial y_1} y_1 + \frac{\partial G}{\partial y_2} y_2 - \left(\frac{\partial G}{\partial y_1} \bar{y}_1 + \frac{\partial G}{\partial y_2} \bar{y}_2 \right) \tag{24.25}$$

Since the partial derivatives are all evaluated at a specific point, $(\bar{y}_1, \bar{y}_2)$, they are all constants. Therefore equations (24.24) and (24.25) can be re-expressed as

$$\dot{y}_1 = a_{11} y_1 + a_{12} y_2 + b_1$$
$$\dot{y}_2 = a_{21} y_1 + a_{22} y_2 + b_2$$

where $a_{11} = \partial F / \partial y_1, a_{12} = \partial F / \partial y_2$, and so on. The stability property of the steady-state point $(\bar{y}_1, \bar{y}_2)$ in a linear system of differential equations such as this is determined directly from the coefficient matrix. The coefficient matrix for equations (24.24) and (24.25) is

$$A = \begin{bmatrix} \dfrac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_1} & \dfrac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_2} \\ \dfrac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_1} & \dfrac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_2} \end{bmatrix} \tag{24.26}$$

The following theorem states the relevance of this matrix.

Theorem 24.7 If the determinant of the coefficient matrix in (24.26) is nonzero, the qualitative behavior of the trajectories of the nonlinear system in definition 24.5 in the neighborhood of its steady-state point $(\bar{y}_1, \bar{y}_2)$ is the same as that of the trajectories of the linear homogeneous system consisting of (24.24) and (24.25), except if the linear system is a center. In that case the nonlinear system could be a center or a focus.

Theorem 24.7 implies that the **local stability** property of a nonlinear differential equation system can be determined directly from the coefficient matrix of its linearized form. Thus, the stability results summarized in table 24.1 can be used to determine the qualitative behavior of the trajectories of a nonlinear system around its steady state. For example, if the determinant of A in equation (24.26) is negative, the steady state of the nonlinear system is a saddle point, so the trajectories display the properties of a saddle point, at least locally. The qualifier that the results are locally valid is a direct consequence of the fact that the linearization of the nonlinear system is only valid in the local neighborhood of the steady state.

Example 24.16 Determine the behavior of the trajectories of the following nonlinear differential equation system in the neighborhood of the steady state:

$$\dot{y}_1 = ay_1 - by_2^{b-1}$$
$$\dot{y}_2 = y_1 - cy_2$$

where a and c are positive constants and $0 < b < 1$.

Solution

The steady state is found by setting $\dot{y}_1 = 0 = \dot{y}_2$. This gives

$$y_2\left(ac - by_2^{b-2}\right) = 0$$

The solutions are $\bar{y}_2 = 0$ and

$$\bar{y}_2 = \left(\frac{ac}{b}\right)^{1/(b-2)} > 0$$

The solutions for y_1 are $\bar{y}_1 = c\bar{y}_2$. We shall focus our attention on the strictly positive values of the two steady states. The coefficient matrix of the first-order linear approximation to the nonlinear system is

$$A = \begin{bmatrix} a & -(b-1)by_2^{-b-2} \\ 1 & -c \end{bmatrix}$$

The determinant of the coefficient matrix is $-ac + (b-1)b\bar{y}_2^{-b-2}$ and is negative because $0 < b < 1$ and $a, c > 0$, and because $\bar{y}_2 > 0$. Thus we know immediately that the strictly positive steady state is a saddle-point equilibrium. We therefore conclude that the behavior of the nonlinear system is that of a saddle-point equilibrium in the neighborhood of the steady-state point.

Determining the **global** behavior of a nonlinear system can be a much more difficult task. However, for many problems encountered in economics, and all of the problems encountered in this book, the nonlinear differential equation systems are sufficiently well behaved that the global behavior of the nonlinear system can be determined using phase diagram analysis in conjunction with theorem 24.7. ■

Example 24.17

A Nonlinear Phase Diagram

Construct the phase diagram for the nonlinear differential equation system in example 24.16.

Solution

The procedure is the same as for a linear system except that qualitative graphing techniques may have to be substituted for explicit graphing. Begin by analyzing the motion of y_1.

Setting $\dot{y}_1 = 0$ gives

$$y_2 = \left(\frac{a}{b} y_1\right)^{1/(b-1)}$$

as the y_1 isocline. To determine the shape of this equation, it is helpful to determine its slope and intercepts if possible. As $y_1 \to 0$, $y_2 \to \infty$ because the exponent is negative $(0 < b < 1)$. In addition, as $y_1 \to \infty$, $y_2 \to 0$. Thus the graph is asymptotic to both axes. The slope is

$$\frac{dy_2}{dy_1} = \frac{1}{b-1}\left(\frac{a}{b} y_1\right)^{(2-b)/(b-1)} < 0$$

Figure 24.11 shows the y_1 isocline labeled $\dot{y}_1 = 0$.

The motion of y_1 at points off the isocline is determined as follows. We calculate

$$\frac{\partial \dot{y}_1}{\partial y_1} = a > 0$$

As a result y_1 is increasing at points to the right of the isocline. Since the differential equation is monotonically increasing in y_1, we know that y_1 must continue to be increasing everywhere to the right of the isocline. By similar reasoning, $\dot{y}_1 < 0$ to the left of the isocline. The appropriate horizontal arrows are marked on figure 24.11.

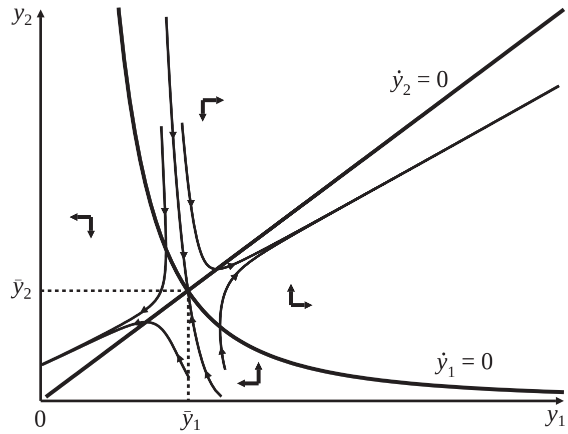


Figure 24.11 Phase diagram for example 24.16

Now determine the motion of y_2. Since $\dot{y}_2$ is a linear differential equation, this is relatively straightforward: the y_2 isocline is the line $y_2 = y_1/c$ and y_2 is decreasing above and increasing below the y_2 isocline.

Figure 24.11 shows the phase diagram. We have determined that the steady state is a saddle-point equilibrium, so we know that in the neighborhood of the steady state there is a saddle path and that all other trajectories diverge. The phase diagram analysis shows that globally, the behavior of trajectories is consistent with the presence of a saddle point. ■

EXERCISES

1. For each of the following differential equation systems, determine whether the system is a stable or unstable node, saddle point, stable or unstable focus, or center:

(a)
$$\dot{y}_1 = 10y_1 + 3y_2 + 2$$
$$\dot{y}_2 = -3y_1 + y_2 + 1$$

(b)
$$\dot{y}_1 = y_1 + 3y_2 + 10$$
$$\dot{y}_2 = -2y_1 + y_2 - 5$$

(c)
$$\dot{y}_1 = 2y_1 - 6y_2 - 1$$
$$\dot{y}_2 = -3y_1 + 5y_2 + 2$$

(d)
$$\dot{y}_1 = -2y_1 - 4y_2 + 5$$
$$\dot{y}_2 = -2y_1 - 9y_2 + 1$$

2. Solve the following linear differential equation system, draw the phase diagram, and find the equation for the saddle path:

$$\dot{y}_1 = 2y_1 - 9y_2$$
$$\dot{y}_2 = -3y_1 - 4y_2$$

3. Solve the following linear differential equation system, draw the phase diagram, and find the equation for the saddle path. If $y_1(0) = 8$, what value must be chosen for $y_2(0)$ to ensure that the system converges to the steady state?

$$\dot{y}_1 = 2y_1 - 9y_2 + 35$$
$$\dot{y}_2 = -3y_1 - 4y_2 + 70$$

Assume that $b < 0$ and all other parameters $(\alpha, \gamma, a, F, G, \bar{c})$ are greater than zero.

Show that if γ is not too large, the equilibrium is a stable node, but that if γ is large enough, the steady state could be a stable focus.

4. In the Dornbusch model, use the following parameter values: $\alpha = 1$, $v = 1$, $a = 4/3$, $b = 1/3$, $\bar{y} = 3.4$, $m = 3$, $r^* = 0.1$, and $u = 4$. Solve for the steady-state price and exchange rate, $\bar{p}$ and $\bar{e}$; solve the differential equation system. If $p(0) = 4$, to what value must $e(0)$ "jump" to put the economy on the saddle path?

24.3 Systems of Linear Difference Equations

The techniques for solving systems of linear difference equations are similar to those developed in section 24.1 for solving systems of linear differential equations. We therefore provide a briefer coverage of this topic.

Definition 24.6

The **general form** for a system of two linear difference equations with constant coefficients and terms is

$$y_{t+1} = a_{11}y_t + a_{12}x_t + b_1$$
$$x_{t+1} = a_{21}y_t + a_{22}x_t + b_2$$

As usual with linear difference or differential equations, the solutions to the complete equations are equal to the sum of the homogeneous solutions and particular solutions to the complete equations. We begin by solving the homogeneous form.

The Homogeneous Solutions

Begin by putting the difference equation system into its homogeneous form. This gives

$$y_{t+1} = a_{11}y_t + a_{12}x_t \tag{24.27}$$

$$x_{t+1} = a_{21}y_t + a_{22}x_t \tag{24.28}$$

We now show two approaches to solving this homogeneous system of difference equations. The first is the **substitution method;** the second is the **direct method.**

The Substitution Method

In this approach we use substitution to reduce the system of two first-order difference equations to a single second-order difference equation. From equation (24.27) we know that

$$y_{t+2} = a_{11}y_{t+1} + a_{12}x_{t+1}$$

Using equation (24.28) to substitute for x_{t+1}, we write

$$y_{t+2} = a_{11}y_{t+1} + a_{12}(a_{21}y_t + a_{22}x_t)$$

which is a second-order difference equation. Since it still depends on x_t, we use equation (24.27) again but this time to get the following expression:

$$x_t = \frac{y_{t+1} - a_{11}y_t}{a_{12}}, \qquad a_{12} \neq 0 \tag{24.29}$$

Note the restriction $a_{12} \neq 0$. If this were not true, then y_{t+1} would not depend on x_t, and we could then solve it directly as a single, first-order difference equation. Using this to substitute for x_t gives

$$y_{t+2} = a_{11}y_{t+1} + a_{12}a_{21}y_t + a_{12}a_{22}\left(\frac{y_{t+1} - a_{11}y_t}{a_{12}}\right)$$

After simplifying and rearranging this becomes

$$y_{t+2} - (a_{11} + a_{22})y_{t+1} + (a_{11}a_{22} - a_{12}a_{21})y_t = 0$$

which is a homogeneous, linear, second-order difference equation with constant coefficients. Equations like this were solved in chapter 20; theorem 20.2 gives the solution (for the case of *real and distinct roots*):

$$y_t = C_1 r_1^t + C_2 r_2^t \tag{24.30}$$

where C_1 and C_2 are arbitrary constants (the values of which are determined from initial conditions) and r_1 and r_2 are the roots of the *characteristic equation* for this second-order difference equation, which is

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

To find x_t, we need only substitute the solution for y_t back into equation (24.29). This gives

$$x_t = \frac{C_1 r_1^{t+1} + C_2 r_2^{t+1} - a_{11}\left(C_1 r_1^t + C_2 r_2^t\right)}{a_{12}}$$

Simplifying this expression gives

$$x_t = \frac{r_1 - a_{11}}{a_{12}} C_1 r_1^t + \frac{r_2 - a_{11}}{a_{12}} C_2 r_2^t \tag{24.31}$$

Together, equations (24.30) and (24.31) give the solutions to the homogeneous difference equation system in (24.27) and (24.28) for the case of real and distinct roots.

If the roots turn out to be real and equal or complex valued, we would use the appropriate solution to the second-order differential equation for y_t, followed by substitution to obtain the solution for x_t.

Example 24.18 Solve the following system of homogeneous difference equations:

$$y_{t+1} = 6y_t + 8x_t$$
$$x_{t+1} = y_t + x_t$$

Solution

The first difference equation implies that

$$y_{t+2} = 6y_{t+1} + 8x_{t+1}$$

Substitute for x_{t+1} to get

$$y_{t+2} = 6y_{t+1} + 8(y_t + x_t)$$

Use the first equation again to get the following expression for x_t

$$x_t = \frac{-6y_t + y_{t+1}}{8}$$

and use it to substitute for x_t. This gives

$$y_{t+2} = 6y_{t+1} + 8y_t + 8\left(\frac{-6y_t + y_{t+1}}{8}\right)$$

Rearrange this equation to get

$$y_{t+2} - 7y_{t+1} - 12y_t = 0$$

The characteristic equation is

$$r^2 - 7r - 2 = 0$$

with real and distinct roots $r_1, r_2 = 7/2 \pm \sqrt{57}/2$. The solution is

$$y_t = C_1 r_1^t + C_2 r_2^t$$

The solution for x_t is obtained by substituting the solution for y_t back into the expression for x_t. This gives

$$x_t = \frac{-6}{8}\left(C_1 r_1^t + C_2 r_2^t\right) + \frac{1}{8}\left(C_1 r_1^{t+1} + C_2 r_2^{t+1}\right)$$

After simplifying, this equations becomes

$$x_t = \frac{r_1 - 6}{8} C_1 r_1^t + \frac{r_2 - 6}{8} C_2 r_2^t$$

■

The Direct Method

The direct method offers an approach that can be extended quite easily to the case of more than two difference equations.

In matrix form the system of homogeneous difference equations is

$$\mathbf{y}_{t+1} = A\mathbf{y}_t$$

where A is an $n \times n$ matrix of constant coefficients and $\mathbf{y}_{t+1}$ and $\mathbf{y}_t$ are vectors of n variables. By analogy with the solution for a single difference equation, we postulate the solution to be

$$\mathbf{y}_t = \mathbf{k}r^t$$

where $\mathbf{k}$ is a vector of arbitrary constants and r is a scalar. If correct, this solution must satisfy the system of homogeneous difference equations. Let us substitute the proposed solution into the difference equations and see if it does indeed satisfy them:

$$\mathbf{k}r^{t+1} = A\mathbf{k}r^t$$

After simplifying and ruling out the trivial solution $r = 0$, this becomes

$$(A - rI)\mathbf{k} = \mathbf{0} \tag{24.32}$$

where I is the identity matrix and $\mathbf{0}$ is the zero-vector. If we choose r to solve this system of equations, then our proposed solution works! As in the case of differential equations, the solution values of r are found by solving

$$|A - rI| = 0$$

which is a polynomial of degree n in the unknown number r. As before, this is the characteristic equation of matrix A and its solutions are the characteristic roots (eigenvalues) of matrix A. A nonzero vector, $\mathbf{k}_1$, which is a solution of equation (24.32) for a particular eigenvalue, r_1, is called the eigenvector of matrix A corresponding to the eigenvalue r_1.

In general, there are n equations and n characteristic roots; therefore, there are n solutions to the system of difference equations. The general solution to the homogeneous form is a linear combination of n distinct solutions.

Example 24.19 Solve the following system of homogeneous difference equations:

$$\mathbf{y}_{t+1} = \begin{bmatrix} 6 & -8 \\ 1 & 0 \end{bmatrix} \mathbf{y}_t$$

Solution

The characteristic roots are the solution to

$$|A - rI| = \begin{vmatrix} 6 - r & -8 \\ 1 & 0 - r \end{vmatrix} = 0$$

which gives the characteristic equation

$$r^2 - 6r + 8 = 0$$

for which the solutions are $r_1 = 2$ and $r_2 = 4$.

For $r_1 = 2$, the eigenvector is the solution to

$$(A - rI)\mathbf{k} = \begin{bmatrix} 6-2 & -8 \\ 1 & 0-2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives $4k_1 - 8k_2 = 0$. Setting $k_1 = 1$ gives $k_2 = 1/2$. Thus the first set of solutions is

$$\mathbf{y}_t^1 = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} 2^t$$

For $r_2 = 4$, the eigenvector is the solution to

$$(A - rI)\mathbf{k} = \begin{bmatrix} 6-4 & -8 \\ 1 & 0-4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$$

which gives $2k_1 - 8k_2 = 0$. Setting $k_1 = 1$ gives $k_2 = 1/4$. Thus the second set of solutions is

$$\mathbf{y}_t^2 = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} 4^t$$

The general homogeneous solutions then are

$$\mathbf{y}_t = C_1 \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} 2^t + C_2 \begin{bmatrix} 1 \\ 1/4 \end{bmatrix} 4^t$$

■

The Particular (Steady-State) Solutions

The particular solutions to the system of complete difference equations in definition 24.6 that we use are the steady-state solutions, if they exist. Setting $y_{t+1} = y_t = \bar{y}$ and $x_{t+1} = x_t = \bar{x}$ to find the steady state gives

$$\bar{y} = a_{11}\bar{y} + a_{12}\bar{x} + b_1$$
$$\bar{x} = a_{21}\bar{y} + a_{22}\bar{x} + b_2$$

Solving gives

$$\bar{y} = \frac{(1 - a_{22})b_1 + a_{12}b_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \tag{24.33}$$

$$\bar{x} = \frac{a_{21}b_1 + (1 - a_{11})b_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \tag{24.34}$$

The steady-state solutions exist provided the denominator in equations (24.33) and (24.34) is not equal to zero, an assumption we make throughout.

The Complete Solutions

The complete solutions are obtained by adding the particular solutions given in equations (24.33) and (24.34) to the homogeneous solutions. Theorem 24.8 gives the complete solutions.

Theorem 24.8 The complete solutions to the general system of two linear difference equations with constant coefficients and terms in definition 24.6 are

Real and distinct roots:

$$y_t = C_1 r_1^t + C_2 r_2^t + \bar{y}$$
$$x_t = \frac{r_1 - a_{11}}{a_{12}} C_1 r_1^t + \frac{r_2 - a_{11}}{a_{12}} C_2 r_2^t + \bar{x}$$

Real and equal roots:

$$y_t = (C_1 + C_2 t) r^t + \bar{y}$$
$$x_t = \left[\frac{r - a_{11}}{a_{12}} (C_1 + C_2 t) + \frac{r}{a_{12}} C_2 \right] r^t + \bar{x}$$

where

$$r_1, r_2 = \frac{\text{tr}(A)}{2} \pm \frac{1}{2}\sqrt{(\text{tr}(A))^2 - 4|A|}$$

$$\text{tr}(A) = a_{11} + a_{22}, \quad |A| = a_{11}a_{22} - a_{12}a_{21}$$

Complex roots:

$$y_t = R^t[C_1 \cos(\theta t) + C_2 \sin(\theta t)] + \bar{y}$$
$$x_t = R^t\left[\frac{C_1 R\cos(\theta) + C_2 R\sin(\theta) - a_{11}C_1}{a_{12}}\right]\cos(\theta t) + R^t\left[\frac{C_2 R\cos(\theta) - C_1 R\sin(\theta) - a_{11}C_2}{a_{12}}\right]\sin(\theta t) + \bar{x}$$

where $R = \sqrt{|A|}$, $\cos(\theta) = \mathrm{tr}(A)/2R$, $\sin(\theta) = \sqrt{4|A| - \mathrm{tr}(A)^2}/(2R)$

Example 24.20 Solve the following system of difference equations:

$$y_{t+1} = 6y_t - 8x_t + 10$$
$$x_{t+1} = y_t + 1$$

Solution

The homogeneous form of this system was solved in example 24.19, so we need only find the particular solutions to complete the solution.

To determine the values of $\bar{y}$ and $\bar{x}$, set $y_{t+1} = y_t = \bar{y}$ and $x_{t+1} = x_t = \bar{x}$

$$\bar{y} = 6\bar{y} - 8\bar{x} + 10$$
$$\bar{x} = \bar{y} + 1$$

The first equation reduces to

$$\bar{y} = \frac{8}{7}\bar{x} - \frac{10}{7}$$

Substituting this into the expression for $\bar{x}$ and simplifying gives $\bar{x} = 3$. Substituting this back into the expression for $\bar{y}$ and simplifying gives $\bar{y} = 2$.

Using the homogeneous solutions obtained in example 24.19, the complete solutions are

$$y_t = C_1 2^t + C_2 4^t + 2$$
$$x_t = \frac{C_1}{2}2^t + \frac{C_2}{4}4^t + 3$$

■

Example 24.21 Solve the following system of difference equations:

$$y_{t+1} = 2y_t + \frac{1}{2}x_t + 1$$
$$x_{t+1} = \frac{-9}{2}y_t - x_t + 2$$

Solution

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 1/2 \\ -9/2 & -1 \end{bmatrix}$$

The roots of the characteristic equation (eigenvalues of A) are

$$r_1, r_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4(-2 + 9/4)} = \frac{1}{2} \pm 0 = \frac{1}{2}$$

The roots are real and equal. The solution, using the substitution method or the direct method, is given in theorem 24.8 as

$$y_t = (C_1 + C_2 t)\left(\frac{1}{2}\right)^t + \bar{y}$$
$$x_t = (-3C_1 - 3C_2 t + C_2)\left(\frac{1}{2}\right)^t + \bar{x}$$

The solutions for $\bar{y}$ and $\bar{x}$ are determined by solving

$$\bar{y} = 2\bar{y} + \frac{\bar{x}}{2} + 1$$
$$\bar{x} = \frac{-9}{2}\bar{y} - \bar{x} + 2$$

This gives $\bar{y} = 12$ and $\bar{x} = -26$. ■

Example 24.22 Solve the following system of difference equations:

$$y_{t+1} = 2y_t + 5x_t + 2$$
$$x_{t+1} = -\frac{1}{2}y_t - x_t - 1$$

Solution

The coefficient matrix is

$$A = \begin{bmatrix} 2 & 5 \\ -1/2 & -1 \end{bmatrix}$$

The roots of the characteristic equation (eigenvalues of A) are

$$r_1, r_2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4(1/2)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-1} = \frac{1}{2} \pm \frac{1}{2}i$$

where i is the imaginary number. The roots have complex values. The solution given in theorem 24.8 in this case is

$$\begin{aligned} y_t &= R^t[C_1 \cos(\theta t) + C_2 \sin(\theta t)] + \bar{y} \\ x_t &= R^t\left[\frac{C_1 R \cos(\theta) + C_2 R \sin(\theta) - a_{11} C_1}{a_{12}}\right]\cos(\theta t) \\ &\quad + R^t\left[\frac{C_2 R \cos(\theta) - C_1 R \sin(\theta) - a_{11} C_2}{a_{12}}\right]\sin(\theta t) + \bar{x} \end{aligned}$$

where $R = \sqrt{|A|} = (1/2)^{1/2}$ and $\bar{y} = -2$ and $\bar{x} = -6$.

To determine the values of θ, we use the fact given that $\cos(\theta) = \text{tr}(A)/2R = \sqrt{2}/2$. We therefore know that $\theta = \pi/4$ (i.e., the inverse cosine of $\sqrt{2}/2$ in radians is $\pi/4$.)

The complete solution then becomes

$$\begin{aligned} y_t &= \left(\frac{1}{2}\right)^{t/2}\left[C_1 \cos\left(\frac{\pi}{4}t\right) + C_2 \sin\left(\frac{\pi}{4}t\right)\right] - 2 \\ x_t &= \left(\frac{1}{2}\right)^{t/2}\left[\frac{C_1\sqrt{1/2}\cos(\pi/4) + C_2\sqrt{1/2}\sin(\pi/4) - 2C_1}{5}\right]\cos\left(\frac{\pi}{4}t\right) \\ &\quad + \left(\frac{1}{2}\right)^{t/2}\left[\frac{C_2\sqrt{1/2}\cos(\pi/4) - C_1\sqrt{1/2}\sin(\pi/4) - 2C_2}{5}\right]\sin\left(\frac{\pi}{4}t\right) - 6 \end{aligned}$$

■

Initial Conditions

When the solution is required to satisfy given initial conditions, the constants C_1 and C_2 take on specific values.

Example 24.23 Find the values of C_1 and C_2 that make the solution to the difference equation system in example 24.20 satisfy $y_0 = 2$ and $x_0 = 1$.

Solution

Setting $t = 0$ in the solutions given in example 24.20 and setting $y_0 = 2$ and $x_0 = 1$ gives

$$2 = C_1 + C_2 + 2$$
$$1 = \frac{C_1}{2} + \frac{C_2}{4} + 3$$

Solving these two equations now for C_1 and C_2 gives, after some simplification, $C_1 = -8$ and $C_2 = 8$. ■

Steady States and Stability

In chapters 18 to 20 we learned that single difference equations converge to the steady state if the characteristic roots, r_1 and r_2, are between -1 and $+1$ and diverge otherwise. The same property holds for systems of difference equations.

Theorem 24.10 A system of two linear difference equations with constant coefficients and terms is asymptotically stable if and only if the absolute values of both characteristic roots are less than unity.

Proof

The proof of this theorem is virtually identical to the proof of theorem 20.5 regarding convergence for a second-order difference equation so we do not repeat it here. The important point is that the terms containing r^t converge to zero as $t \to \infty$ if and only if the absolute value of r is less than one. ■

Example 24.24 Determine whether the system of difference equations in example 24.19 is stable or unstable.

Solution

We found the roots in this system to be 2 and 4. The system is therefore not stable. The sequence of points (y_t, x_t) generated by this system of equations diverges from the steady state. ■

Example 24.25 Determine whether the system of difference equations in example 24.21 is stable or unstable.

Solution

We found the roots in this system to be equal, and equal to $1/2$. Therefore the system is asymptotically stable. No matter what the starting values, the sequence of points (y_t, x_t) will always converge to the same steady-state point $(\bar{y}, \bar{x})$. ■

Example 24.26 Determine whether the system of difference equations in example 24.22 is stable or unstable.

Solution

We found complex roots in this system. The absolute value of the complex roots is $1/2$, however, so the system is asymptotically stable. ■

Price Wars

In this application, we consider the possible implications of two retail competitors adopting a particular form of pricing strategy. Let y_t be the price charged by retailer number 1, and let x_t be the price charged by retailer number 2. They sell the same product and are located across the street from one another. Because they sell the same product, they have the option of cooperating (colluding) to set a common price or not cooperating. The first option is unlawful per se. The second can lead to price wars. Suppose that they were to not cooperate. Further suppose that retailer number 1 adopted the following pricing strategy:

$$y_{t+1} = y_t - \alpha(y_t - x_t)$$

This says that retailer number 1 will set price in period $t + 1$ equal to what it was in the previous period minus a fraction α times the *difference* between its price and the rival's price in the previous period. If the rival's price (x_t) was lower last period than his own price in the previous period (y_t), retailer 1 will lower his own price this period. Let us assume that retailer number 2 adopts a symmetric strategy

$$x_{t+1} = x_t - \beta(x_t - y_t)$$

The only difference is that we allow retailer 2 to have β, which can differ from α. Like retailer 1, retailer 2 will lower her price in period $t + 1$ if her rival's price in the previous period was lower than her own. Conversely, each of them will raise their price in this period if their rival's price in the previous period was higher than their own price.

An example makes this model more concrete. Imagine that the two retailers operate gasoline stations on opposing corners of a busy intersection. The length of a time period might be one hour. Each owner then is assumed to adopt a price-setting strategy that follows the lead of the other at rates of α and β respectively.

The problem at hand is to determine the implications of these pricing strategies. Where do they lead? Do prices tend to converge to stationary values? Are rivals likely to adopt such pricing strategies?

To answer these questions, the first step is to solve the system of homogeneous difference equations. Write them in matrix form

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

The determinant of the coefficient matrix is $(1-\alpha)(1-\beta)-\alpha\beta$, which simplifies to $1-\alpha-\beta$. The eigenvalues (roots) therefore are

$$r_1, r_2 = \frac{2-\alpha-\beta}{2} \pm \frac{1}{2}\sqrt{(2-\alpha-\beta)^2 - 4(1-\alpha-\beta)}$$

The roots are given by

$$r_1 = 1, \quad r_2 = 1-(\alpha+\beta)$$

The solutions are

$$\begin{aligned} y_t &= C_1 + C_2(1-\alpha-\beta)^t \\ x_t &= C_1 - C_2(1-\alpha-\beta)^t \end{aligned}$$

Now suppose that we start the system off at a time that we call $t = 0$ with prices established at y_0 and x_0. We use these as starting values to determine the values of the constants C_1 and C_2. Setting $t = 0$ gives

$$\begin{aligned} y_0 &= C_1 + C_2 \\ x_0 &= C_1 - C_2 \end{aligned}$$

Solving for C_1 and C_2 gives

$$\begin{aligned} C_1 &= \frac{y_0 + x_0}{2} \\ C_2 &= \frac{y_0 - x_0}{2} \end{aligned}$$

The solutions then become

$$y_t = \frac{y_0 + x_0}{2} + \frac{y_0 - x_0}{2}(1 - \alpha - \beta)^t$$

$$x_t = \frac{y_0 + x_0}{2} - \frac{y_0 - x_0}{2}(1 - \alpha - \beta)^t$$

Now we can use the solutions to determine the implications of these pricing strategies. Inspection of the solutions reveals that if $\alpha + \beta < 1$, the solutions do converge to the *average* of the two starting prices as $t \to \infty$. Thus, if neither retailer *reacts* too much to the gap between his or her own price and the rival's price (i.e., $\alpha + \beta < 1$), the pricing strategies result in each retailer charging the same price, the average of y_0 and x_0.

What about the case in which $\alpha + \beta > 1$? In this case, $1 - \alpha - \beta < -1$. As a result prices do not converge. On the contrary, they diverge in ever-increasing oscillations. The dynamic system will evolve into a situation in which one retailer's price is so high one day, that sales are zero, while the rival's price is so low that sales are enormous but profits are negative because price is below cost. On the next day the situation is reversed, and reversed again the next day, and so on. This is clearly a situation that is not beneficial to either retailer. We conclude that if we observe pricing strategies like these, we should also expect to observe $\alpha + \beta < 1$.

EXERCISES

1. Given $y_0 = 6$ and $x_0 = -1$, solve the following system of difference equations:

$$y_{t+1} = y_t + 5x_t - 10$$

$$x_{t+1} = \frac{1}{4}y_t - x_t + 10$$

2. Given $y_0 = 4$ and $x_0 = -2$, solve the following system of difference equations:

$$y_{t+1} = 2y_t + \frac{1}{2}x_t + 3$$

$$x_{t+1} = \frac{7}{2}y_t - x_t + 3$$

3. Given $y_0 = 5$ and $x_0 = 8$, solve the following system of difference equations:

$$y_{t+1} = -y_t + \frac{3}{4}x_t$$
$$x_{t+1} = -3y_t + 2x_t$$

4. Given y_0 and x_0, solve the following system of difference equations:

$$y_{t+1} = 2y_t - 2x_t$$
$$x_{t+1} = 2y_t + 2x_t$$

5. Consider a Cournot duopoly model in which two firms share the market for some product. Each chooses a strategy of producing an amount given by the following “reaction" functions:

$$y_{t+1} = 60 - \frac{1}{4}x_t$$
$$x_{t+1} = 60 - \frac{1}{4}y_t$$

where y is firm 1’s output and x is firm 2’s output. Solve this system of difference equations and determine whether the steady state is stable.

6. Replace the reaction functions in exercise 5 with

$$y_{t+1} = 60 - \frac{1}{2}x_t$$
$$x_{t+1} = 60 - \frac{1}{2}y_t$$

Assuming $x_0 = 60$ and $y_0 = 0$, solve the system of difference equations and plot y_t and x_t against t for $t = 0, 1, 2, 3, 4, 5$.

7. Two rival retailers each adopt a price-setting strategy of setting price today 10% lower than its rival’s price yesterday. Write out the system of difference equations for the two prices, calling retailer 1’s price y and retailer 2’s price x. Solve the system assuming that $y_0 = x_0$. Show that prices converge to zero.

CHAPTER REVIEW

Key Concepts

center
characteristic equation
complete solutions
direct method
eigenvalues
eigenvectors
general form
global stability
homogeneous solutions
improper stable node
improper unstable node
isocline
isosectors
local stability
particular solutions
phase diagram
phase plane
saddle path
saddle point
saddle-point equilibrium
simultaneous system
stable focus
stable node
steady state
substitution method
trajectory
unstable focus
unstable node

Review Questions

1. Explain how the characteristic equation can be derived and how it is used to find the characteristic roots for a differential or difference equation.

2. Under what conditions is the particular solution given by the steady-state solution for (a) a system of two differential equations, and (b) a system of two difference equations?

3. If the steady state of a system of two differential equations is a stable focus, sketch the paths that y_1 and y_2 would follow as a function of time.

4. Why is a saddle-point steady state said to be unstable even though the saddle path converges to the steady state?

5. State the conditions under which a system of differential equations is stable.

6. State the conditions under which a system of difference equations is stable.

Review Exercises

1. Solve each of the following systems of linear differential equations using the substitution method and determine the stability property of the steady state:

(a)

$$\dot{y}_1 = \frac{1}{2}y_1 + \frac{1}{4}y_2 + 3$$

$$\dot{y}_2 = 3y_1 + \frac{1}{2}y_2 + 2$$

(b)

$$\dot{y}_1 = -y_1 + \frac{3}{4}y_2 - 4$$
$$\dot{y}_2 = -3y_1 + 2y_2 - 1$$

2. Solve each of the following systems of linear differential equations using the direct method and determine the stability property of the steady state:

(a)

$$\dot{y}_1 = y_1 + 5y_2 - 10$$
$$\dot{y}_2 = \frac{1}{4}y_1 - y_2 + 10$$

(b)

$$\dot{y}_1 = 2y_1 + \frac{1}{2}y_2 + 1$$
$$\dot{y}_2 = \frac{7}{2}y_1 - y_2 - 8$$

(c)

$$\dot{y}_1 = 2y_1 - 6y_2 + 1$$
$$\dot{y}_2 = -3y_1 + 5y_2 + 2$$

3. Let y be the stock of pollution and x be the flow of pollution from industrial sources. Some of the stock of pollution is assimilated into the environment by natural processes. The differential equation for the stock of pollution is

$$\dot{y} = -\alpha y + x$$

where α determines the rate of natural assimilation. Assume that the flow of industrial emissions of pollution is governed by

$$\dot{x} = -\beta y + a$$

where $a > 0$ is a constant. This equation implies that the change in emissions is negative, the larger is the stock of pollution (e.g., negative feedback owing to increased government regulation).

Solve this system of linear differential equations and interpret your results, including a discussion of the stability property of the steady state.

4. Let x be the stock of fish in a commercial fishery, and let N be the stock of capital (number of fishing boat units). The natural growth rate of the fishery is governed by the logistic equation studied in earlier chapters: $rx(1 - x)$. The actual growth of the fishery is its natural growth minus the commercial harvest. Assuming that each unit of fishing capital catches 1 unit of fish, then N is the total commercial harvest. The differential equation for the fish stock then is

$$\dot{x} = rx(1 - x) - N$$

Fishing capital is attracted to the industry if profits can be earned, and leaves the industry if losses are being made. Assuming that the price of fish is constant and equal to p, and the unit cost of fishing is c/x, where $c > 0$ is a constant, then the differential equation for fishing capital is

$$\dot{N} = \alpha\left(p - \frac{c}{x}\right)$$

where $\alpha > 0$ is a speed-of-adjustment parameter that determines how rapidly the stock of fishing capital adjusts.

Draw the phase diagram for this nonlinear system of differential equations with N on the vertical axis and x on the horizontal axis. Determine the conditions under which the steady state is asymptotically stable.

5. In the following nonlinear differential equation system, $I(t)$ is a firm's investment at time t, $K(t)$ is its capital stock at time t, δ is the rate of depreciation of capital, and α is a parameter of the firm's production function with $0 < \alpha < 1$.

$$\dot{I} = \delta I - \frac{\alpha K^{\alpha-1}}{2}$$
$$\dot{K} = I - \delta K$$

Find the steady-state point, show that it is a saddle point, and construct the phase diagram.

6. The following nonlinear differential equation system is a predator-prey model of two fish species. Let y_1 be the population size (in million of kilograms say) of species 1 and y_2 be the population size of species 2. Find the four steady-state points, determine the local stability properties of each, and draw the phase diagram.

$$\dot{y}_1 = 0.8y_1\left(1 - \frac{y_1}{200}\right) - 0.02y_1y_2$$
$$\dot{y}_2 = 0.5y_2\left(1 - \frac{y_2}{100}\right) - 0.01y_1y_2$$

7. Solve each of the following systems of linear difference equations using the substitution method:

(a)
$$y_{t+1} = y_t + x_t + 1$$
$$x_{t+1} = y_t + x_t + 2$$

(b)

$$y_{t+1} = -2y_t + 2x_t + 1$$
$$x_{t+1} = y_t - 3x_t + 2$$

8. Solve each of the following systems of linear difference equations using the direct method:

(a)

$$y_{t+1} = 5y_t - 4x_t + 2$$
$$x_{t+1} = 2y_t + x_t - 10$$

(b)

$$y_{t+1} = -y_t + 8x_t + 10$$
$$x_{t+1} = 3y_t + 4x_t - 15$$

(c)

$$y_{t+1} = \frac{1}{2}y_t + \frac{1}{2}x_t + 5$$
$$x_{t+1} = \frac{3}{10}y_t + \frac{1}{10}x_t + 15$$

Chapter 25 Optimal Control Theory

The reader is encouraged to consult our website http://mitpress.mit.edu/math_econ3 for the following material useful to the discussion of this chapter:

- A Derivation of the Necessary Conditions in Optimal Control Theory
- Interpretation of λ
- Derivation of the $H(T) = 0$ Condition
- Practice Exercise

In this chapter we take up the problem of optimization over time. Such problems are common in economics. For example, in the theory of investment, firms are assumed to choose the time path of investment expenditures to maximize the (discounted) sum of profits over time. In the theory of savings, individuals are assumed to choose the time path of consumption and saving that maximizes the (discounted) sum of lifetime utility. These are examples of dynamic optimization problems. In this chapter, we study a new technique, optimal control theory, which is used to solve dynamic optimization problems.

It is fundamental in economics to assume optimizing behavior by economic agents such as firms or consumers. Techniques for solving *static* optimization problems have already been covered in chapters 6, 12, and 13. Why do we need to learn a new mathematical theory (optimal control theory) for handling *dynamic* optimization problems? To demonstrate the need, we consider the following economic example.

Static versus Dynamic Optimization: An Investment Example

Suppose that a firm's output depends only on the amount of capital it employs. Let

$$Q = q(K)$$

where Q is the firm's output level, q is the production function and K is the amount of capital employed. Assume that there is a well-functioning rental market for the kind of capital the firm uses and that the firm is able to rent as much capital as it wants at the price R per unit, which it takes as given. To make this example more concrete, imagine that the firm is a fishing company that rents fully equipped units of fishing capital on a daily basis. (A unit of fishing capital would include boat, nets, fuel, crew, etc.). Q is the number of fish caught per day and K is the number of units of fishing capital employed per day. If p is the price of fish, then current profit depends on the amount of fish caught, which in turn depends on the amount of K used and is given by the function $\pi(K)$:

$$\pi(K) = pq(K) - RK$$

If the firm's objective is to choose K to maximize current profit, the optimal amount of K is given implicitly by the usual first-order condition:

$$\pi'(K) = pq'(K) - R = 0$$

But why should the firm care only about current profit? Why would it not take a longer-term view and also care about future profits? A more realistic assumption is that the firm's objective is to maximize the discounted *sum* of profits over an interval of time running from the present time ($t = 0$) to a given time horizon, T. This is given by the functional $J[K(t)]$

$$\max J[K(t)] = \int_0^T e^{-\rho t}\pi[K(t)]\,dt$$

where ρ is the firm's discount rate and $e^{-\rho t}$ is the continuous-time discounting factor. $J[K(t)]$ is called a **functional** to distinguish it from a *function*. A function maps a single value for a variable like K, (or a finite number of values if K is a vector of different types of capital) into a single number, like the amount of current profit. A functional maps a function like $K(t)$—or finite number of functions if there is more than one type of capital—into a single number, like the discounted sum of profits.

It appears we now have a dynamic optimization problem. The difference between this and the static optimization problem is that we now have to choose a *path* of K values, or in other words we have to choose a *function* of time, $K(t)$, to maximize J, rather than having to choose a single *value* for K to maximize $\pi(K)$. This is the main reason that we require a new mathematical theory. Calculus helps us find the value of K that maximizes a function $\pi(K)$ because we can differentiate $\pi(K)$ with respect to K to find the maximum of $\pi(K)$. However, calculus is not, in general, suited to helping us find the function of time $K(t)$ that maximizes the functional $J[K(t)]$ because we cannot differentiate a functional $J[K(t)]$ with respect to a function $K(t)$.

It turns out, however, that we do not have a truly dynamic optimization problem in this example. As a result calculus works well in solving this particular problem. The reason is that the amount of K rented in any period t affects only profits in that period and not in any other period. Thus it is fairly obvious that the maximum of the discounted sum of profits occurs by maximizing profits at each point in time. As a result this dynamic problem is really just a sequence of static optimization problems. The solution therefore is just a sequence of solutions to a sequence of static optimization problems. Indeed, this is the justification for spending as much time as we do in economics on static optimization problems.

An optimization problem becomes truly dynamic only when the economic choices made in the current period affect not only current payoffs (profit) but also payoffs (profits) at a later date. The intuition is straightforward: if current output affects only current profit, then in choosing current output, we need only be concerned with its effect on current profit. Hence we choose current output to maximize current profit. But if current output affects current profit *and* profit at a later date, then in choosing current output, we need to be concerned about its effect on current and future profit. This is a dynamic problem.

To turn our fishing firm example into a truly dynamic optimization problem, let us drop the assumption that a rental market for fishing capital exists. Instead, we suppose that the firm must purchase its own capital. Once purchased, the capital lasts for a long time. Let $I(t)$ be the amount of capital purchased (investment) at time t, and assume that capital depreciates at the rate δ. The amount (stock) of capital owned by the firm at time t is $K(t)$ and changes according to the differential equation

$$\dot{K} = I(t) - \delta K(t)$$

which says that, at each point in time, the firm's capital stock increases by the amount of investment and decreases by the amount of depreciation.

Let $c[I(t)]$ be a function that gives the cost of purchasing (investing) the amount $I(t)$ of capital at time t; then profit at time t is

$$\pi[K(t), I(t)] = pq[K(t)] - c[I(t)]$$

The problem facing the fishing firm at each point in time is to decide how much capital to purchase. This is a truly dynamic problem because current investment affects current profit, since it is a current expense, and also affects future profits, since it affects the amount of capital available for future production. If the firm's objective is to maximize the discounted sum of profits from zero to T, it maximizes

$$J[I(t)] = \int_0^T e^{-\rho t} \pi[K(t), I(t)]\, dt$$

$$\text{subject to} \quad \dot{K} = I(t) - \delta K$$
$$K(0) = K_0$$

Once a path for $I(t)$ is chosen, the path of $K(t)$ is completely determined because the initial condition for the capital stock is given at K_0. Thus, the functional J depends on the particular path chosen for $I(t)$.

There is an infinite number of paths, $I(t)$, from which to choose. A few examples of feasible paths are as follows:

(i) $I(t) = \delta K_0$. This is a constant amount of investment, just enough to cover depreciation so that the capital stock remains intact at its initial level.
(ii) $I(t) = 0$. This is the path of no investment.
(iii) $I(t) = Ae^{\alpha t}$. This is a path of investment that starts with $I(0) = A$ and then increases over time at the rate α, if $\alpha > 0$, or decreases at the rate α, if $\alpha < 0$.

These are just a few arbitrary paths that we mention for illustration. In fact any function of t is a feasible path. The problem is to choose the path that maximizes $J[I(t)]$. Since we know absolutely nothing about what this function of time might look like, choosing the right path would seem to be a formidable task.

It turns out that in the special case in which $T = \infty$ and the function $\pi[K(t), I(t)]$ takes the quadratic form

$$\pi[K(t), I(t)] = K - aK^2 - I^2$$

the solution to the above problem is

$$I^*(t) = \frac{a(K_0 - \bar{K})}{r_1 - \delta - \rho} e^{r_1 t} + \delta \bar{K}$$

where r_1 is the negative root of the characteristic equation of the differential equation system that, as we shall see, results from solving this dynamic optimization problem, and $\bar{K}$ is the steady-state level of the capital stock that the firm desires, and is given by

$$\bar{K} = \frac{1}{2[\delta(\rho + \delta) + a]}$$

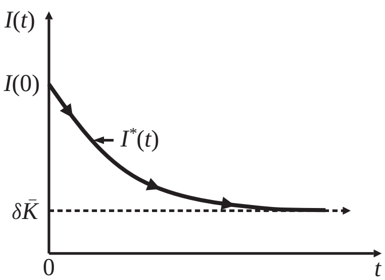


Figure 25.1 Optimal path of investment over time

Figure 25.1 displays the optimal path of investment for the case in which $K_0 < \bar{K}$. Along the optimal path, investment declines. In the limit as $t \to \infty$, investment converges to a constant amount equal to $\delta\bar{K}$ (since $r_1 < 0$) so that in the long run the firm's investment is just replacement of depreciation.

How did we find this path? We found it using optimal control theory, which is the topic we turn to now.

25.1 The Maximum Principle

Optimal control theory relies heavily on the **maximum principle,** which amounts to a set of necessary conditions that hold only on optimal paths. Once you know how to apply these necessary conditions, then a knowledge of basic calculus and differential equations is all that is required to solve dynamic optimization problems like the one outlined above. In this section we provide a statement of the necessary conditions of the maximum principle and then provide a justification. In addition we provide examples to illustrate the use of the maximum principle.

We begin with a definition of the general form of the dynamic optimization problem that we shall study in this section.

Definition 25.1 The general form of the **dynamic optimization problem** with a finite time horizon and a free endpoint in continuous-time models is

$$\max J = \int_0^T f[x(t), y(t), t]\, dt \tag{25.1}$$

$$\text{subject to} \quad \dot{x} = g[x(t), y(t), t]$$
$$x(0) = x_0 > 0 \qquad \text{(given)}$$

The term **free endpoint** means that $x(T)$ is unrestricted, and hence is *free* to be chosen optimally. The significance of this is explored in more detail below.

In this general formulation, J is the value of the functional which is to be maximized, $x(t)$ is referred to as the **state variable** and $y(t)$ is referred to as the **control variable.** As the name suggests, the control variable is the one directly chosen or controlled. Since the control variable and state variables are linked by a differential equation that is given, the state variable is indirectly influenced by the choice of the control variable.

In the fishing firm example posed above, the state variable is the amount of capital held by the firm; the control variable is investment. The example was a free-endpoint problem because there was no constraint placed on the final amount of the capital stock. As well, the integrand function, $f[x(t), y(t), t]$, was equal to $\pi[K(t), I(t)]e^{-\rho t}$, and the differential equation for the state variable, $g[x(t), y(t), t)]$, was simply equal to $I(t) - \delta K(t)$.

We will examine a number of important variations of this general specification in later sections. In section 25.3 we examine the **fixed endpoint** version of this problem. This means that $x(T)$, the final value of the state variable, is specified as an equality constraint to be satisfied. In section 25.4 we consider the case in which T is infinity. Finally in section 25.6 we consider the case in which the time horizon, T, is also a *free* variable to be chosen optimally.

Suppose that a unique solution to the dynamic optimization problem in definition 25.1 exists. The solution is a path for the control variable, $y(t)$. Once this is specified, the path for the state variable is automatically determined through the differential equation for the state variable, combined with its given initial condition. We assume that the control variable is a continuous function of time (we relax this assumption in section 25.5) as is the state variable. The necessary conditions that constitute the maximum principle are stated in terms of a *Hamiltonian function,* which is akin to the Lagrangean function used to solve constrained optimization problems. We begin by defining this function:

Definition 25.2

The **Hamiltonian function,** H, for the dynamic optimization problem in definition 25.1 is

$$H[x(t), y(t), \lambda(t), t] = f[x(t), y(t), t)] + \lambda(t)g[x(t), y(t), t]$$

where $\lambda(t)$, referred to as the **costate** variable, is akin to the Lagrange multiplier in constrained optimization problems.

Forming the Hamiltonian function is straightforward: take the integrand (the function under the integral sign), and add to it the equation for $\dot{x}$ multiplied by an, as yet, unspecified function of time, $\lambda(t)$.

We can now state the necessary conditions.

Theorem 25.1

The optimal solution path for the control variable, $y(t)$, for the dynamic optimization problem in definition 25.1 must satisfy the following necessary conditions:

(i) The control variable is chosen to maximize H at each point in time: $y(t)$ maximizes $H[x(t), y(t), \lambda(t), t]$. That is,

$$\frac{\partial H}{\partial y} = 0$$

(ii) The paths of $x(t)$ and $\lambda(t)$ (state and costate variables), are given by the solution to the following system of differential equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = g[x(t), y(t), t]$$

(iii) The two boundary conditions used to solve the system of differential equations are given by

$$x(0) = x_0, \quad \lambda(T) = 0$$

In writing the first necessary condition, we have assumed that the Hamiltonian function is strictly concave in y. This assumption implies that the maximum of H with respect to y will occur as an interior solution, so it can be found by setting the derivative of H with respect to y equal to zero at each point in time. In section 25.5 we relax this assumption.

The second set of necessary conditions is a system of differential equations. The first one is obtained by taking the derivative of the Hamiltonian function with respect to the state variable, x, and setting $\dot{\lambda}$ equal to the negative of this derivative. The second is just the differential equation for the state variable that is given as part of the optimization problem.

Necessary conditions (i) and (ii) comprise the maximum principle. The necessary conditions in (iii) are typically referred to as **boundary conditions.** In free-endpoint problems, one boundary condition is given, $x(0)$, and the other is provided by a **transversality condition,** $\lambda(T) = 0$. A justification for this transversality condition is provided later in the chapter; for now, we will just say that this is a necessary condition for determining the optimal value of $x(T)$, when $x(T)$ is free to be chosen optimally.

The maximum principle provides the *first-order* conditions. What are the *second-order* conditions in optimal control theory? In other words, when are the necessary conditions also sufficient to ensure the solution path maximizes the objective functional in equation (25.1)? Although it is beyond our scope to prove it, we state the answer as

Theorem 25.2 The necessary conditions stated in theorem 25.1 are also sufficient for the maximization of J in equation (25.1) if the following conditions are satisfied:

(i) $f(x, y, t)$ is differentiable and jointly concave in x and y.
(ii) One of the following is true:

$g(x, y, t)$ is linear in (x, y)
$g(x, y, t)$ is concave in (x, y) and $\lambda(t) \geq 0$ for $t \in (0, T)$
$g(x, y, t)$ is convex in (x, y) and $\lambda(t) \leq 0$ for $t \in (0, T)$

The sufficiency conditions are satisfied for all of the problems examined in this chapter. As a result we need look no further than the necessary conditions to solve the dynamic maximization problems.

Example 25.1 Solve the following problem:

$$\max \int_0^1 (x - y^2)\, dt$$

$$\text{subject to} \quad \dot{x} = y$$
$$x(0) = 2$$

Solution

Step 1 Form the Hamiltonian function.

$$H = x - y^2 + \lambda y$$

Step 2 Apply the maximum principle. Since the Hamiltonian is strictly concave in the control variable y and there are no constraints on the choice of y, we can find the maximum of H with respect to y by applying the first-order condition:

$$\frac{\partial H}{\partial y} = -2y + \lambda = 0$$

This gives

$$y(t) = \frac{\lambda(t)}{2} \tag{25.2}$$

Step 3 The differential equation for $\lambda(t)$ is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -1$$

We now have a system of two differential equations which, after using equation (25.2), is

$$\dot{\lambda} = -1 \tag{25.3}$$

$$\dot{x} = \frac{\lambda}{2} \tag{25.4}$$

Step 4 We obtain the boundary conditions. This is a free-endpoint problem because the value for $x(1)$ is not specified in the problem. Therefore the boundary conditions are

$$x(0) = 2, \quad \lambda(1) = 0$$

Step 5 Solve or analyze the system of differential equations. In this example we have a system of linear differential equations, so we proceed by obtaining explicit solutions. Because the first differential equation, (25.3), does not depend on x, we can solve it directly and then substitute the solution into the second equation. Solving equation (25.3) gives $\lambda(t) = C_1 - t$, where C_1 is an arbitrary constant of integration, the value of which is determined by using the boundary condition, $\lambda(1) = 0$. This gives $0 = C_1 - 1$ for which the solution is $C_1 = 1$. Therefore we have $\lambda(t) = 1 - t$.

Substituting this solution into equation (25.4) gives

$$\dot{x} = \frac{1-t}{2}$$

to which the solution is

$$x(t) = \frac{t}{2} - \frac{t^2}{4} + C_2$$

where C_2 is an arbitrary constant of integration. Its value is determined from the boundary condition $x(0) = 2$. This gives $2 = C_2$.

The solution then becomes

$$x(t) = \frac{t}{2} - \frac{t^2}{4} + 2$$

To complete the solution to this maximization problem we substitute the solutions to the differential equations back into equation (25.2). Doing this gives

$$y(t) = \frac{1-t}{2}$$

as the solution path for the control variable. At $t = 0$, $y(0) = 1/2$. It then declines over time and finishes at $t = 1$ with $y(1) = 0$. ■

An Investment Problem

Suppose that a firm's only factor of production is its capital stock, K, and that its production function is given by the relation

$$Q = K - aK^2, \qquad a > 0$$

where Q is the quantity of output produced. Assuming that capital depreciates at the rate $\delta > 0$, then the change in the capital stock is equal to the firm's investment, I, less depreciation, δK:

$$\dot{K} = I - \delta K$$

If the price of the firm's output is a constant \$1, and the cost of investment is equal to I^2 dollars, then the firm's profits at a point in time are

$$\pi = K - aK^2 - I^2$$

The optimization problem we now consider is to maximize the integral sum of profits over a given interval of time $(0, T)$. A more realistic objective would be to maximize the present-valued integral sum of profits but we postpone treatment of this problem to the next section.

$$\max \int_0^T (K - aK^2 - I^2)\, dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0 \qquad \text{(given)}$$

To solve this, we take the following steps:

Step 1 Form the Hamiltonian

$$H = K - aK^2 - I^2 + \lambda(I - \delta K)$$

Step 2 Apply the maximum principle: since the Hamiltonian is strictly concave in the control variable I, we look for the I that maximizes the Hamiltonian by using the first-order condition

$$\frac{\partial H}{\partial I} = -2I + \lambda = 0 \tag{25.5}$$

Since $\partial^2 H/\partial I^2 = -2$ is negative, this gives a maximum. The solution is

$$I(t) = \frac{\lambda(t)}{2} \tag{25.6}$$

Step 3 Form the system of differential equations. λ must obey the differential equation

$$\dot{\lambda} = -\frac{\partial H}{\partial K} = -(1 - 2aK - \lambda\delta)$$

Using equation (25.6) to substitute for $I(t)$, the system is

$$\dot{\lambda} = \lambda\delta + 2aK - 1 \tag{25.7}$$

$$\dot{K} = \frac{\lambda}{2} - \delta K \tag{25.8}$$

Step 4 Obtain the boundary conditions. The boundary condition for $K(t)$ is given by the initial condition $K(0) = K_0$. The boundary condition for $\lambda(t)$ is $\lambda(T) = 0$.
Step 5 Solve or analyze the system of differential equations. If the system is linear, as it is in this example, use the techniques of chapter 24 to obtain an explicit solution. We do this next. If the system is nonlinear, it is probably not possible to obtain an explicit solution. In that case, use the techniques of chapter 24 to undertake a qualitative analysis, preferably with the aid of a phase diagram. In either case keep in mind that the system of differential equations obtained from employing optimal control theory provides the solution to the optimization problem.

An explicit solution to the system of differential equations (25.7) and (25.8) is obtained using the techniques shown in chapter 24. The homogeneous form of this system, written in matrix form is

$$\begin{bmatrix} \dot{\lambda} \\ \dot{K} \end{bmatrix} = \begin{bmatrix} \delta & 2a \\ 1/2 & -\delta \end{bmatrix} \begin{bmatrix} \lambda \\ K \end{bmatrix} \tag{25.9}$$

The determinant of the coefficient matrix of the homogeneous system is $(-\delta^2 - a)$, which is negative. We therefore know immediately that the steady-state equilibrium is a saddle point.

By theorem 24.2, the solutions to the system of differential equations in (25.7) and (25.8) are

$$\lambda(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{\lambda} \tag{25.10}$$

$$K(t) = \frac{r_1 - \delta}{2a} C_1 e^{r_1 t} + \frac{r_2 - \delta}{2a} C_2 e^{r_2 t} + \bar{K} \tag{25.11}$$

where r_1 and r_2 are the eigenvalues or roots of the coefficient matrix in equation (25.9), C_1 and C_2 are arbitrary constants of integration, and $\bar{\lambda}$ and $\bar{K}$ are the steady-state values of the system, and serve as particular solutions in finding the complete solutions.

If A denotes the coefficient matrix in equation (25.9), its characteristic roots (eigenvalues) are given by the equation

$$r_1, r_2 = \frac{\text{tr}(A)}{2} \pm \frac{1}{2}\sqrt{\text{tr}(A)^2 - 4|A|}$$

where $\text{tr}(A)$ denotes the trace of A (sum of the diagonal elements). The roots of equation (25.9) then are

$$r_1, r_2 = \pm\sqrt{\delta^2 + a}$$

The steady-state values of λ and K are found by setting $\dot{\lambda} = 0$ and $\dot{K} = 0$. Doing this and simplifying yields

$$\lambda = \frac{1 - 2aK}{\delta}$$

$$K = \frac{\lambda}{2\delta}$$

Solving these for λ and K give the steady-state values

$$\bar{\lambda} = \frac{\delta}{\delta^2 + a}, \quad \bar{K} = \frac{1}{2(\delta^2 + a)}$$

Because the steady state is a saddle point, it can be reached only along the saddle path and only if the exogenously specified time horizon, T, is large enough to permit it to be reached.

This leaves only the values of the arbitrary constants of integration to be determined. As usual, they are determined using the boundary conditions $K(0) = K_0$ and $\lambda(T) = 0$. First, requiring the solution for $K(t)$ to satisfy its initial condition gives

$$K_0 = \frac{r_1 - \delta}{2a}C_1 + \frac{r_2 - \delta}{2a}C_2 + \bar{K}$$

After simplifying, this gives

$$C_1 = \frac{2a(K_0 - \bar{K}) - (r_2 - \delta)C_2}{r_1 - \delta}$$

Next, requiring the solution for $\lambda(t)$ to satisfy its terminal condition gives

$$0 = C_1 e^{r_1 T} + C_2 e^{r_2 T} + \bar{\lambda}$$

from which we get an equation for C_2 in terms of C_1:

$$C_2 = -C_1 e^{(r_1 - r_2)T} - \bar{\lambda} e^{-r_2 T}$$

Substituting this into the expression for C_1 and simplifying gives the solution for C_1:

$$C_1 = \frac{2a(K_0 - \bar{K}) + (r_2 - \delta)\bar{\lambda} e^{-r_2 T}}{(r_1 - \delta) - (r_2 - \delta) e^{(r_1 - r_2)T}}$$

Substituting this solution into the equation for C_2 and simplifying gives the explicit solution for C_2:

$$C_2 = \frac{-2a(K_0 - \bar{K}) e^{(r_1 - r_2)T} - \bar{\lambda}(r_1 - \delta) e^{-r_2 T}}{r_1 - \delta - (r_2 - \delta) e^{(r_1 - r_2)T}}$$

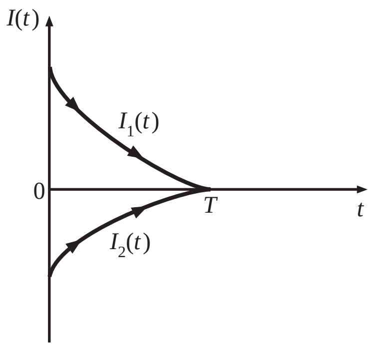


Figure 25.2 Solution path $I_1(t)$ for investment when $K_0 < \bar{K}$; solution path $I_2(t)$ for investment when $K_0 > \bar{K}$

This completes the solution.

The optimal path of investment is obtained using equation (25.6). If we denote the solution for $\lambda(t)$ in equation (25.10) as $\lambda^*(t)$, then the solution for investment, denoted $I^*(t)$ is

$$I^*(t) = \frac{\lambda^*(t)}{2}$$

This solution gives the path of investment that maximizes total profits over the planning horizon. Figure 25.2 shows two possible solution paths. When $K_0 < \bar{K}$, the solution is a path like $I_1(t)$ that starts high and declines monotonically to 0 at time T. When $K_0 > \bar{K}$, the solution is a path of disinvestment like $I_2(t)$ that stays negative from zero to T.

An Economic Interpretation of λ and the Hamiltonian

We introduced $\lambda(t)$ as a sequence or path of Lagrange multipliers. It turns out that there is a natural economic interpretation of this co-state variable. Intuitively $\lambda(t)$ can be interpreted as the marginal (imputed) value or shadow price of the state variable $x(t)$. This interpretation follows informally from the Lagrange multiplier analogy. But it can also be shown formally, as we do at http://mitpress.mit.edu/math_econ3, that $\lambda(0)$ is the amount by which J^* (the maximum value function) would increase if $x(0)$ (the initial value of the state variable) were to increase by a small amount. Therefore $\lambda(0)$ is the value of a marginal increase in the state variable at time $t = 0$ and therefore can be interpreted as the most we would be willing to pay (the shadow price) to acquire a bit more of it at time $t = 0$. By extension, $\lambda(t)$ can be interpreted as the shadow price or imputed value of the state variable at any time t.

In the investment problem just examined, $\lambda(t)$ gives the marginal (imputed) value or shadow price of the firm's capital stock at time t. Armed with this interpretation, the first-order condition (25.5) makes economic sense: it says that at each moment of time, the firm should carry out the amount of investment that satisfies the following equality:

$$2I(t) = \lambda(t)$$

The left-hand side is the marginal cost of investment; the right-hand side is the marginal (imputed) value of capital and, as such, gives the marginal benefit of investment. Thus the first-order condition of the maximum principle leads to a very simple investment rule: invest up to the point that marginal cost equals marginal benefit.

The Hamiltonian function too can be given an economic interpretation. In general, H measures the instantaneous total economic contribution made by the control variable toward the integral objective function. In the context of the investment problem, H is the sum of total profits earned at a point in time and the accrual of capital that occurs at that point in time valued at its shadow price. Therefore H is the instantaneous total contribution made by the control variable to the integral of profits, J. It makes sense then to choose the control variable so as to maximize H at each point in time. This, of course, is what the maximum principle requires.

EXERCISES

1. Solve

$$\max \int_0^T -(ay + by^2)\, dt$$

$$\text{subject to} \quad \dot{x} = x - y$$
$$x(0) = x_0$$

where a, b are positive constants.

2. Solve

$$\max \int_0^T (xy - y^2 - x^2)\,dt$$

$$\text{subject to} \quad \dot{x} = y$$
$$x(0) = x_0$$

3. Solve

$$\max \int_0^T -(ay + by^2 + cx)\,dt$$

$$\text{subject to} \quad \dot{x} = \alpha x + \beta y$$
$$x(0) = x_0$$

4. Solve

$$\max \int_0^T (ay - by^2 + fx - gx^2)\,dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$

5. Solve

$$\max \int_0^T (y - y^2 - 4x - 3x^2)\,dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$

6. In equations (25.7) and (25.8) the differential equation system was written in terms of λ and K. For the same model, transform the differential equation system into a system in I and K. Solve this system of equations for $I(t)$ and $K(t)$.

7. Assume that price is a constant, p, and the cost of investment is bI^2, where b is a positive constant. Then solve the following:

$$\max \int_0^T [p(K - aK^2) - bI^2]\, dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$

25.2 Optimization Problems Involving Discounting

Discounting is a fundamental feature of dynamic optimization problems in economic dynamics. In the remainder of this chapter, we assume that ρ is the going rate of return in the economy, that there is no uncertainty about this rate of return and that it is constant over time. Recall from chapter 3 that $y_0 = y(t)e^{-\rho t}$ is the discounted value (or present value) of $y(t)$. In all of the subsequent models and examples examined in the chapter, firms and consumers will be assumed to maximize the discounted value (present value) of future streams of revenues or benefits net of costs.

The General Form of Autonomous Optimization Problems

Most dynamic optimization problems in economics involve discounting. As a result time enters the objective function explicitly through the term $e^{-\rho t}$. However, if this is the *only way* the variable t explicitly enters the dynamic optimization problem, the system of differential equations can be made *autonomous*. The importance of this fact is that autonomous differential equations (ones in which t is *not* an explicit variable) are much easier to solve than nonautonomous differential equations.

We specified the general form of the integrand function in definition 25.1 as $f(x, y, t)$. If this reduces to some function of just x and y multiplied by the term $e^{-\rho t}$, say $F(x, y)e^{-\rho t}$, and if the differential equation given for the state variable does not depend explicitly on t (is autonomous), so that $g(x, y, t)$ specializes to $G(x, y)$, then we may state

Definition 25.3 The general form of the **autonomous optimization problem** is

$$\max J = \int_0^T F[x(t), y(t)]e^{-\rho t}\,dt \tag{25.12}$$

$$\text{subject to} \quad \dot{x} = G[x(t), y(t)]$$
$$x(0) = x_0$$

The Hamiltonian function for this problem is

$$H = F(x, y)e^{-\rho t} + \lambda G(x, y)$$

The maximum with respect to y occurs when

$$F_y(x, y)e^{-\rho t} + \lambda G_y(x, y) = 0$$

assuming an interior solution. Alternatively, we can write this as

$$F_y(x, y) + (\lambda e^{\rho t})G_y(x, y) = 0$$

Implicitly, this gives y as a function of x and $\lambda e^{\rho t}$, say $\phi(x, \lambda e^{\rho t})$. The differential equation for λ is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -e^{-\rho t}F_x(x, y) - \lambda G_x(x, y)$$

After substituting $y = \phi(x, \lambda e^{\rho t})$, and multiplying both sides by $e^{\rho t}$, this becomes

$$\dot{\lambda}e^{\rho t} = -F_x[x, \phi(x, \lambda e^{\rho t})] - \lambda e^{\rho t}G_x[x, \phi(x, \lambda e^{\rho t})] \tag{25.13}$$

The second differential equation in the system is

$$\dot{x} = G[x, \phi(x, \lambda e^{\rho t})] \tag{25.14}$$

and the two boundary conditions are $x(0) = x_0$ and $\lambda(T) = 0$.

We have derived the general form of the system of differential equations, (25.13) and (25.14), for this kind of problem but it is clearly not autonomous as promised. However, a simple change of variable will transform it into an autonomous system. When we do this, it will become clear why we wrote the differential equation for λ in a slightly different form. Define a new variable $\mu(t)$:

$$\mu(t) = \lambda(t)e^{\rho t}$$

Taking the time derivative of this expression yields

$$\dot{\mu} = \rho\lambda e^{\rho t} + \dot{\lambda}e^{\rho t} = \rho\mu + \dot{\lambda}e^{\rho t}$$

Substituting for $\lambda e^{\rho t}$ and $\dot{\lambda}e^{\rho t}$ in equation (25.13) then gives the new system of two differential equations

$$\dot{\mu} = \rho\mu - F_x[x, \phi(x, \mu)] - \mu G_x[x, \phi(x, \mu)]$$

$$\dot{x} = G(x, \phi(x, \mu))$$

It is now a system of differential equations in two variables, μ and x, both of which are functions of time; however, the variable t does not itself appear explicitly in either of the differential equations. As a result the transformed system is *autonomous!* This makes it easier to solve the system of differential equations. But perhaps more important, this makes it possible to draw phase diagrams to assist in the qualitative analysis of nonlinear systems of differential equations. This is a major advantage. We can rarely draw a phase diagram for a nonautonomous system because the locus of points at which $\dot{\lambda} = 0$ and $\dot{x} = 0$ depends on t and so moves around over time. In contrast, the locus of points at which $\dot{\mu} = 0$ and $\dot{x} = 0$ is stationary.

It is possible to create an autonomous system of differential equations when t enters only through the discounting term using the procedure shown here. However, an even easier (and more common) procedure is to use a *current-valued Hamiltonian* rather than the regular Hamiltonian. We introduce this next.

The Current-Valued Hamiltonian

The purpose for introducing this minor modification to the definition of the Hamiltonian function is that it leads automatically to an autonomous system of differential equations when the variable t enters the optimization problem explicitly only through the discounting term.

In general, the maximization problem with discounting is defined as

$$\max J = \int_0^T e^{-\rho t} F[x(t), y(t)]\, dt \tag{25.15}$$

subject to $\dot{x} = G[x(t), y(t)]$

$x(0) = x_0 > 0$ (given)

The regular Hamiltonian for this problem is

$$H = e^{-\rho t} F[x(t), y(t)] + \lambda G[x(t), y(t)]$$

The current-valued Hamiltonian, $\mathcal{H}$, is defined as

$$\mathcal{H} = He^{\rho t}$$

That is, it is simply the regular Hamiltonian multiplied by $e^{\rho t}$. Inverting this relationship gives

$$H = \mathcal{H}e^{-\rho t}$$

which leads us to interpret the regular Hamiltonian as a *present-valued* Hamiltonian.

Earlier we defined a new costate variable, μ, as

$$\mu = \lambda e^{\rho t}$$

which suggests that we interpret μ as the *current-valued* costate variable and λ as the *present-valued* costate variable. With these relationships, we can now provide a formal definition of the current-valued Hamiltonian. Because it is slightly different from the regular Hamiltonian, the necessary conditions require some minor modifications. We present all these first and then proceed to show the justification for these minor but important changes.

Definition 25.4

The **current-valued Hamiltonian,** $\mathcal{H}$, for autonomous optimization problems conforming to definition 25.3, is

$$\mathcal{H}[x(t), y(t), \mu(t)] = F[x(t), y(t)] + \mu G[x(t), y(t)] \tag{25.16}$$

and the necessary conditions are

(i) The control variable is chosen to maximize $\mathcal{H}$ at each point in time: $y(t)$ maximizes $\mathcal{H}[x(t), y(t), \mu(t)]$. That is,

$$\frac{\partial \mathcal{H}}{\partial y} = 0$$

(ii) The paths of $x(t)$ and $\mu(t)$ (state and costate variables), are given by the solution to the following system of differential equations:

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial x}$$

$$\dot{x} = g[x(t), y(t)]$$

(iii) The two boundary conditions used to solve the system of differential equations are given by

$$x(0) = x_0, \quad \mu(T)e^{-\rho T} = 0$$

Notice that the only necessary conditions that change are the two involving the new costate variable, μ. It is easy to see why these changes follow from the change-of-variable relationships. First, the current-valued Hamiltonian is just

$$\mathcal{H} = He^{\rho t} = F[x(t), y(t)] + \lambda e^{\rho t} G[x(t), y(t)]$$

This becomes

$$\mathcal{H} = F[x(t), y(t)] + \mu G[x(t), y(t)]$$

The changes that must be made to the necessary conditions also follow directly from the change-of-variable definitions. Since $H = \mathcal{H}e^{-\rho t}$, the y that maximizes H will necessarily maximize $\mathcal{H}$, since $e^{-\rho t}$ plays the role of a constant term when we are maximizing with respect to y. This is seen more clearly perhaps for interior solutions where the derivative of H with respect to y equals 0. This implies that

$$\frac{\partial \mathcal{H}}{\partial y} e^{-\rho t} = 0$$

However, since $e^{-\rho t}$ cancels out, this is equivalent to setting

$$\frac{\partial \mathcal{H}}{\partial y} = 0$$

Thus this part of the maximum principle does not change at all. We still wish to maximize the current-valued Hamiltonian at each point in time.

The maximum principle also requires that the costate variable satisfy the differential equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{25.17}$$

This translates into a comparable differential equation in the new variable μ. To derive it, note that

$$-\frac{\partial H}{\partial x} = -\frac{\partial \mathcal{H}}{\partial x} e^{-\rho t} \tag{25.18}$$

and because $\lambda = \mu e^{-\rho t}$,

$$\dot{\lambda} = -\rho \mu e^{-\rho t} + \dot{\mu} e^{-\rho t}$$

Making the substitutions into equation (25.17) gives

$$-\rho \mu e^{-\rho t} + \dot{\mu} e^{-\rho t} = -\frac{\partial \mathcal{H}}{\partial x} e^{-\rho t}$$

Simplifying this gives the differential equation that the new (current-valued) costate variable must satisfy

$$\dot{\mu} - \rho \mu = -\frac{\partial \mathcal{H}}{\partial x} \tag{25.19}$$

Finally the transversality condition changes slightly. Instead of $\lambda(T) = 0$, we now have $\mu(T) e^{-\rho T} = 0$. If T is finite, this reduces to $\mu(T) = 0$.

Example 25.2 Solve the following maximization problem:

$$\max \int_0^T e^{-\rho t} [ax - bx^2 - cy^2] \, dt$$

$$\text{subject to} \quad \dot{x} = y - \alpha x$$

$$x(0) = x_0$$

Solution

Step 1 The current-valued Hamiltonian function is

$$\mathcal{H} = ax - bx^2 - cy^2 + \mu(y - \alpha x)$$

Step 2 The maximum of $\mathcal{H}$ with respect to the control variable y is found by solving

$$\frac{\partial \mathcal{H}}{\partial y} = -2cy + \mu = 0$$

which gives $y = \mu/2c$.
Step 3 The differential equation for μ is

$$\dot{\mu} - \rho\mu = -\frac{\partial H}{\partial x} = -(a - 2bx - \mu\alpha)$$

The system of differential equations then becomes

$$\dot{\mu} = (\rho + \alpha)\mu + 2bx - a$$

$$\dot{x} = \frac{\mu}{2c} - \alpha x$$

Step 4 The first boundary condition is $x(0) = x_0$. Because this is a free-endpoint problem [the value for $x(T)$ is not given], and because T is finite, the second boundary condition is $\mu(T) = 0$.
Step 5 The solution to the system of linear differential equations is given by theorem 24.2:

$$\mu(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{\mu}$$

$$x(t) = \frac{r_1 - \rho - \alpha}{2b} C_1 e^{r_1 t} + \frac{r_2 - \rho - \alpha}{2b} C_2 e^{r_2 t} + \bar{x}$$

where r_i, $\bar{\mu}$, and $\bar{x}$ can be determined directly from the differential equations, but the C_i are constants of integration whose values are determined using the two boundary conditions. We leave it as an exercise for students to complete the solution. ■

An Investment Problem with Discounting: A Qualitative Analysis

We generalize the investment model here by not committing ourselves to a choice of a functional form for the firm's production function. Instead of assuming that $f(K) = K - aK^2$, we now make a less restrictive assumption: we assume that $f(K)$ can be any function that satisfies the following properties:

$f(K)$ is continuous on $K \geq 0$ and possesses continuous first and second derivatives that satisfy

$$f'(K) > 0 \quad \text{for } K \geq 0$$
$$f''(K) < 0 \quad \text{for } K \geq 0$$

In addition we assume that $f'(0) = \infty$ and $f'(\infty) = 0$. The importance of these two assumptions will become clear below.

We continue to assume that the price of the firm's output is $1 and the cost of investing at the rate I is I^2 dollars. The dynamic optimization problem then is

$$\max \int_0^T e^{-\rho t}[f(K) - I^2]\,dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$

In this problem the variable t enters the optimization problem only through the discounting term. We can be sure then that applying the maximum principle to the current-valued Hamiltonian will lead to an autonomous system of differential equations.

Step 1 The current-valued Hamiltonian is

$$\mathcal{H} = f(K) - I^2 + \mu(I - \delta K)$$

Step 2 The necessary condition to maximize $\mathcal{H}$ is

$$-2I + \mu = 0$$

The concavity of $\mathcal{H}$ in I ensures that this equation gives a maximum.

Step 3 The current-valued costate variable must satisfy

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial K} = -f'(K) + \mu\delta$$

The system of differential equations then becomes

$$\dot{\mu} = \mu(\delta + \rho) - f'(K) \tag{25.20}$$

$$\dot{K} = \frac{\mu}{2} - \delta K \tag{25.21}$$

Step 4 The boundary conditions are $K(0) = K_0$ and $\mu(T) = 0$.
Step 5 Because the system of differential equations involves an implicit function, we cannot obtain an explicit solution. Instead, we perform a qualitative analysis of the system using a phase diagram.

To draw the phase diagram for the differential equation system in equations (25.20) and (25.21), we follow the steps outlined in chapter 24.

First, determine the motion of μ. Begin by drawing the isocline for μ. This is the locus of points in the (μ, K) plane at which $\dot{\mu} = 0$. Setting equation (25.20) equal to 0 gives the equation for these points:

$$\mu = \frac{f'(K)}{\delta + \rho}$$

To graph this equation in the μ, K phase plane, note that it has a negative slope

$$\frac{d\mu}{dK} = \frac{f''(K)}{\delta + \rho} < 0$$

because $f''(K) < 0$ and δ and ρ are both positive. Next note that our assumption that $f'(0) = \infty$ means that the graph of the equation does not touch the vertical axis at a finite value of μ; instead, the graph is asymptotic to the vertical axis. Similarly, our assumption that $f'(\infty) = 0$ means the graph is also asymptotic to the horizontal axis. These two assumptions guarantee that the intersection of the μ isocline and the K isocline occurs at strictly positive values of μ and K. In other words, the steady-state solution is guaranteed to be an interior solution. Figure 25.3 shows the μ isocline.

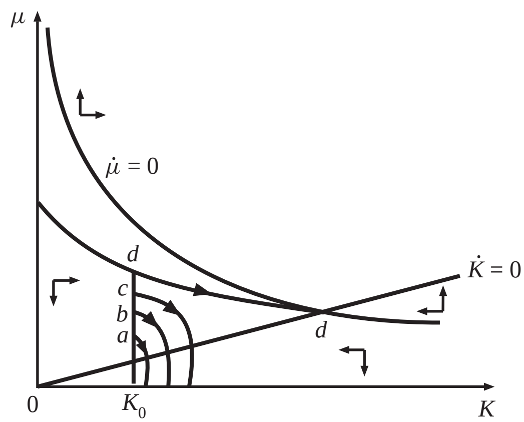


Figure 25.3 Phase diagram for the investment problem with discounting

Determine the sign of $\dot{\mu}$ in each of the two regions separated by the μ isocline. This is done by analyzing the differential equation for μ. Calculate the partial derivative of equation (25.20) with respect to μ. This gives

$$\frac{\partial \dot{\mu}}{\partial \mu} = \delta + \rho > 0$$

which means that, holding K constant, an increase (decrease) in μ leads to an increase (decrease) in $\dot{\mu}$. As a result $\dot{\mu}$ is positive above the $\dot{\mu} = 0$ curve and negative below it.

Second, determine the motion of K. The isocline for K is found by setting $\dot{K} = 0$, which gives

$$\mu = 2\delta K$$

The graph of this equation emanates from the origin with a positive slope equal to 2δ. Determine the motion of K in the two regions separated by this line by partially differentiating equation (25.21) with respect to K. This gives

$$\frac{\partial \dot{K}}{\partial K} = -\delta < 0$$

which means that $\dot{K}$ is negative to the right and positive to the left of the $\dot{K} = 0$ line.

The arrows of motion in the phase diagram give a rough idea of the paths that trajectories must follow. Indeed, they suggest that the steady state is a saddle-point equilibrium. This can be confirmed by checking the signs of the eigenvalues of the system of two differential equations after it has been linearized around the steady-state point (theorem 24.7). The coefficient matrix of the linearized system is

$$\begin{bmatrix} \delta + \rho & -f'' \\ 1/2 & -\delta \end{bmatrix}$$

which has determinant $-\delta(\delta + \rho) + f''/2 < 0$. The negative determinant ensures that the eigenvalues are of opposite sign, which is all the information we need to conclude that we do indeed have a saddle-point equilibrium.

Figure 25.3 depicts a number of representative trajectories in the phase plane. To satisfy the boundary conditions, the optimal trajectory must begin at $K(0) = K_0$ and end at $\mu(T) = 0$. Suppose that the initial capital stock owned by the firm, K_0, is small relative to the steady-state level of the capital stock, $\bar{K}$. With K_0 as shown, the actual optimal trajectory followed by the (μ, K) pair depends on the size of T that is specified in the optimization problem. If T is small, there is not much time available and the trajectory followed cannot cover much distance before finishing its journey at $\mu(T) = 0$. The trajectory beginning at *a* depicts this case.

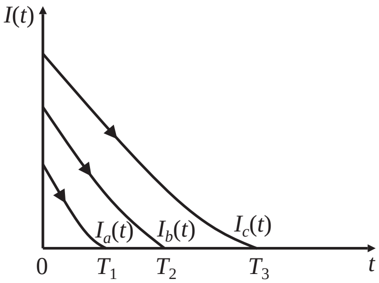


Figure 25.4 Investment paths corresponding to figure 25.3

If T is larger, a longer journey is possible; the trajectory beginning at b depicts this case. If T is larger still, the trajectory beginning at c will be followed. As T approaches infinity, the optimal trajectory will approach the saddle path which is labeled *dd*. Even though the saddle path appears to be of finite length, it takes an infinite amount of time to cover the distance because both μ and K move at a speed approaching zero as they approach the steady state.

Notice in figure 25.3 that no matter what finite value is T, $\mu(T) = 0$, as required by the transversality condition. However, $K(T)$ is quite different for different values of T. Only as the amount of time available to the firm approaches infinity will its capital stock approach the steady-state value.

Figure 25.4 shows the paths of $I(t)$ corresponding to the first three trajectories in Figure 25.3, which have three different finite time horizons, say $T_1 < T_2 < T_3$. On each path, investment declines monotonically and finishes at 0. We know this from the first-order condition for $I(t)$, which implies that

$$I(t) = \frac{\mu(t)}{2}$$

Thus the path of $I(t)$ has the same shape as the path of $\mu(t)$ which, as we know from the phase diagram, declines monotonically to zero.

EXERCISES

1. Solve

$$\max \int_0^T -e^{-\rho t} y^2 \, dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$

2. Solve

$$\max \int_0^T -e^{-\rho t}(ay + by^2) \, dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$

where a, b are positive constants.

3. Solve

$$\max \int_0^T e^{-\rho t}(yx - y^2 - x^2)\,dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$

4. Solve

$$\max \int_0^T e^{-\rho t}(ay - by^2 + fx - gx^2)\,dt$$

$$\text{subject to} \quad \dot{x} = y$$
$$x(0) = x_0$$

5. In this problem a firm's production function depends only on labor, L: output equals $(L - aL^2)$. The firm pays a fixed wage, w, per unit of employed labor and incurs hiring costs equal to qH^2, where H is the amount of new hires, and receives a constant price, p, per unit of output. The firm's labor force has a constant quit rate equal to δ. Solve for the path of hiring that solves

$$\max \int_0^T e^{-\rho t}[p(L - aL^2) - wL - qH^2]\,dt$$

$$\text{subject to} \quad \dot{L} = H - \delta L$$
$$L(0) = L_0$$

6. Conduct a qualitative analysis using a phase diagram [in (μ, K) space] of the following investment problem:

$$\max \int_0^T e^{-\rho t}[K - aK^2 - c(I)]\,dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$

where $c(I)$ is the investment cost function with the following properties: $c'(I) > 0$, $c''(I) > 0$ and $c(0) = 0$.

7. Repeat exercise 6, but this time draw the phase diagram in (I, K) space.

25.3 Alternative Boundary Conditions on $x(T)$

Most dynamic optimization problems place some kind of constraint on the terminal value of the state variable. It may just be a nonnegativity constraint, $x(T) \geq 0$, it may be some other kind of inequality constraint, $x(T) \geq b$, or it may be an equality constraint, $x(T) = b$, where b is some exogenous parameter to the model.

In this section we show that constraints of this type are easily incorporated in optimal control theory by making a minor adjustment to the transversality condition, $\mu(T) = 0$, that applies when $x(T)$ is unconstrained.

Fixed-Endpoint Problems: $x(T) = b$

How do we solve the optimal control problem when the terminal value of the state variable must satisfy a given equality constraint of the form

$$x(T) = b \tag{25.22}$$

where $b > 0$ is a given value? The answer is simple: the necessary conditions of the maximum principle are unchanged, but replace the transversality condition $\mu(T) = 0$ with the boundary condition $x(T) = b$. All together, the necessary conditions are described in the following theorem:

Theorem 25.3 The optimal solution path for the control variable, $y(t)$, for the dynamic optimization problem in definition 25.1, subject to the additional constraint in equation (25.22), must satisfy the following conditions:

(i) The control variable is chosen to maximize H at each point in time: $y(t)$ maximizes $H[x(t), y(t), \lambda(t), t]$. That is,

$$\frac{\partial H}{\partial y} = 0$$

(ii) The paths of $x(t)$ and $\lambda(t)$, the state and costate variables, are given by the solution to the following system of differential equations:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = g[x(t), y(t), t]$$

(iii) The two boundary conditions used to solve the system of differential equations are given by

$$x(0) = x_0, \quad x(T) = b$$

Proof

A rigorous proof is beyond our scope. Instead, we provide a heuristic explanation. Also note that although we have written the necessary conditions in terms of the regular (or present-valued) Hamiltonian, they could just as easily have been presented in terms of the current-valued Hamiltonian.

How did we obtain this minor change to the necessary conditions? An intuitive explanation is possible using the fact that

$$\frac{dJ^*}{dx(T)} = -\lambda(T)$$

as is shown at http://mitpress.mit.edu/math_econ3 for the case of unconstrained $x(T)$. This says that a marginal increase in $x(T)$ leads to a decrease in the value of the objective (e.g., the integral of discounted profits) equal to $\lambda(T)$. If we are free to choose $x(T)$, we obviously choose the amount that maximizes J, which occurs when $dJ/dx(T) = 0$. Setting $\lambda(T) = 0$ then ensures that we choose the optimal $x(T)$. However, if we are not free to choose $x(T)$, but instead must ensure that $x(T)$ equals some exogenously specified number, then $dJ/dx(T)$ will not equal zero at $x(T) = b$. Thus $\lambda(T) \neq 0$ at $x(T) = b$, except by chance. ■

Example 25.3 Solve the following problem:

$$\max \int_0^1 x - y^2 \, dt$$

$$\text{subject to} \quad \dot{x} = y$$
$$x(0) = 2$$
$$x(1) = b$$

Solution

The Hamiltonian for this maximization problem is

$$H = x - y^2 + \lambda y$$

Apart from the fact that $x(1)$ is constrained to equal b, this problem is identical to that solved in example 25.1. There we found that maximizing H with respect to y gave

$$y(t) = \frac{\lambda(t)}{2}$$

After obtaining the differential equation for the costate variable, the system of two differential equations that describe the solution paths for this maximization problem became

$$\dot{\lambda} = -1$$
$$\dot{x} = \frac{\lambda}{2}$$

The boundary conditions for this problem are now different than in example 25.1. When $x(1)$ was unconstrained in example 25.1, we set $\lambda(1) = 0$ and used this as a boundary condition in addition to the given initial condition $x(0) = 2$. Now that $x(1)$ is constrained to equal b, the boundary conditions are

$$x(0) = 2, \quad x(1) = b$$

Solving the differential equation for $\lambda(t)$ gives

$$\lambda(t) = C_1 - t$$

as in example 25.1, where C_1 is a constant of integration. Using this solution allows us to write the differential equation for x as

$$\dot{x} = \frac{C_1 - t}{2}$$

which we can solve directly

$$x(t) = \frac{C_1}{2}t - \frac{t^2}{4} + C_2$$

where C_2 is another constant of integration. The constants of integration are now determined by using the two boundary conditions. First, at $t = 0$, we have $x(0) = 2$. This gives

$$2 = 0 + C_2$$

as the solution for C_2. Next, at $t = 1$ we have $x(1) = b$. This gives

$$b = \frac{C_1}{2} - \frac{1}{4} + C_2$$

which, when rearranged and the solution for C_2 is used, gives

$$C_1 = 2b - \frac{7}{2}$$

Therefore

$$\lambda(t) = 2b - \frac{7}{2} - t$$

and

$$y(t) = b - \frac{7}{4} - \frac{t}{2}$$

is the solution path over the interval $t \in (0, 1)$. ■

Example 25.4 **An Optimal Consumption Model**

An individual has an amount of money x_0 in a bank account at time 0. Thereafter the money earns interest in the account at the rate r. Let $c(t)$ denote the amount of money withdrawn for consumption at time t and assume it is the only source of income for consumption. The differential equation for the bank account is

$$\dot{x} = rx - c$$

and the initial condition is $x(0) = x_0$. We now assume that the individual wishes to choose a consumption path that maximizes lifetime utility that we take to be given simply by

$$u = \int_0^T e^{-\rho t} U(c(t))\, dt$$

where the interval of time, $(0, T)$, is the lifetime, $U(c(t))$ is the instantaneous utility from consuming an amount $c(t)$ at time t, and $\rho \geq 0$ is the individual's personal rate of time preference. We assume that the utility function takes the logarithmic functional form

$$U(c(t)) = \ln c(t)$$

Finally we assume that the individual is constrained to leave a bequest of money in the bank account of a predetermined amount, b. The optimization problem may now be summarized as follows. The individual wishes to

$$\max \int_0^T e^{-\rho t} \ln c \, dt$$

$$\text{subject to} \quad \begin{aligned} \dot{x} &= rx - c \\ x(0) &= x_0 > 0 \\ x(T) &= b > 0 \end{aligned}$$

Solution

Step 1 The current-valued Hamiltonian is

$$\mathcal{H} = \ln c + \mu(rx - c)$$

Step 2 Maximizing the Hamiltonian with respect to the control variable, c, gives

$$\frac{1}{c} - \mu = 0$$

which, when solved for $c(t)$, gives

$$c(t) = \mu(t)^{-1}$$

Step 3 The differential equation for the costate variable is

$$\dot{\mu} - \rho\mu = -\mu r$$

The system of differential equations then is

$$\begin{aligned} \dot{\mu} &= \mu(\rho - r) \\ \dot{x} &= rx - \mu^{-1} \end{aligned}$$

Because these are linear differential equations, we can solve them explicitly. The differential equation for μ depends only on μ, so we solve it first using single-equation techniques. Straightforward application of the techniques explained in chapter 21 gives

$$\mu(t) = C_1 e^{(\rho - r)t}$$

where C_1 is a constant of integration. Using this solution allows us to rewrite the differential equation for x as

$$\dot{x} - rx = -C_1^{-1} e^{-(\rho - r)t}$$

Multiplying both sides by the integrating factor, e^{-rt}, gives

$$\frac{d}{dt}[xe^{-rt}] = -C_1^{-1} e^{-(\rho - r)t} e^{-rt}$$

Simplifying the right-hand side and then integrating both sides gives

$$xe^{-rt} = -C_1^{-1} \int_0^t e^{-\rho s}\, ds + C_2$$

where C_2 is another constant of integration. Carrying out the integration and simplifying leads to

$$x(t) = C_1^{-1} \frac{(e^{-\rho t} - 1)}{\rho} e^{rt} + C_2 e^{rt} \tag{25.23}$$

We complete the solution by using the two boundary conditions to solve for the two constants of integration. First, we have $x(0) = x_0$. This gives

$$x_0 = 0 + C_2$$

as the solution for C_2. Next, we have $x(T) = b$. This gives

$$b = C_1^{-1} \frac{(e^{-\rho T} - 1)}{\rho} e^{rT} + C_2 e^{rT}$$

Simplifying, using $C_2 = x_0$, and solving for C_1 gives

$$C_1 = \frac{(e^{-\rho T} - 1)e^{rT}}{\rho(b - x_0 e^{rT})}$$

Substituting this expression into the solution for $\mu(t)$ and then using it in the solution for $c(t)$ gives the optimal path of consumption chosen by the individual

$$c(t) = \rho \frac{(be^{-rT} - x_0)}{e^{-\rho T} - 1} e^{(r-\rho)t}$$

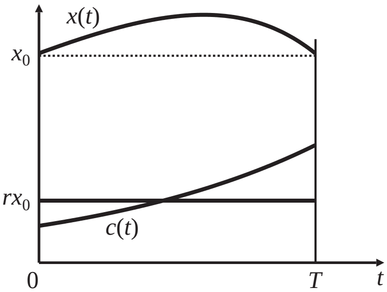


Figure 25.5 Optimal consumption path, $c(t)$, and bank account path, $x(t)$, when $x(T) = x_0$ and $\rho < r$

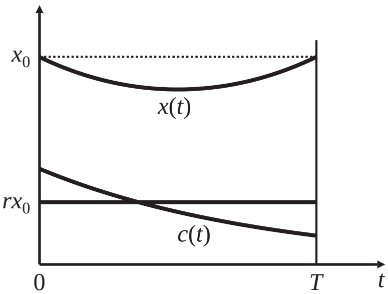


Figure 25.6 Optimal consumption path, $c(t)$, and bank account path, $x(t)$, when $x(T) = x_0$ and $\rho > r$

An interesting case arises when $b = x_0$, which requires the bank account at time T to equal its initial size. An obvious consumption path that satisfies this constraint is to just consume the interest. That is, set $c(t) = rx_0$. Then the bank account never grows or diminishes. However, it is apparent from our solution that this situation will not be the utility-maximizing solution in general. To see this is so, set $b = x_0$. This gives

$$c(t) = \rho x_0 \frac{(e^{-rT} - 1)}{e^{-\rho T} - 1} e^{(r-\rho)t}$$

In the special case in which the private rate of time preference is equal to the market rate of interest, $\rho = r$, this equation reduces to

$$c(t) = rx_0$$

Thus consuming the interest maximizes utility only if $\rho = r$, which occurs if the individual discounts future utility at a rate exactly equal to the market rate of return. This is shown as the line rx_0 in figures 25.5 and 25.6. On the other hand, if the individual discounts future utility at a lower rate than the market rate, $\rho < r$, then the individual is better off foregoing consumption early in life (high savings) to take advantage of the high return on saving and then consuming heavily later in life. In this case, consumption is a rising function of time. This consumption path and the associated path for the bank account, $x(t)$, are shown in figure 25.5. Alternatively, if $\rho > r$, then consumption is a monotonically decreasing function of time. The individual consumes heavily early in life and saves later to bring the bank account back up to its required bequest level. This consumption path and the associated $x(t)$ are shown in figure 25.6. ■

Inequality-Constrained Endpoint Problems: $x(T) \geq b$

When the terminal value of the state variable must satisfy an inequality constraint, we are free to choose $x(T)$ optimally as long as our choice does not violate the constraint. The way we solve the optimal control problem then depends on whether the constraint binds or not. If our optimal (unconstrained) choice of $x(T)$ satisfies the constraint, then we effectively have a free-endpoint problem. The relevant boundary condition for $x(T)$ then is $\lambda(T) = 0$. On the other hand, if our unconstrained choice of $x(T)$ is smaller than b, we have to settle for setting $x(T) = b$, and we effectively have a fixed-endpoint problem. The relevant boundary condition for $x(T)$ then, is $x(T) = b$. As a result, the only thing that makes this a new kind of maximization problem is that we have to decide whether the constraint on $x(T)$ is binding.

In practice, the easiest and most reliable way to decide if the constraint on $x(T)$ is binding is to first try to find the unconstrained, optimal value, $x^*(T)$ and compare this value to b. It is then easy to see whether $x^*(T)$ automatically satisfies the constraint.

Example 25.5 Solve the following problem:

$$\max \int_0^1 (x - y^2)\, dt$$

$$\text{subject to} \quad \dot{x} = y$$
$$x(0) = 2$$
$$x(1) \geq b$$

Solution

Except for the constraint on $x(1)$, this is the same as the free-endpoint problem in example 25.1. After forming the Hamiltonian and maximizing with respect to the control variable y, we obtained the following system of differential equations:

$$\dot{\lambda} = -1 \tag{25.24}$$

$$\dot{x} = \frac{\lambda}{2} \tag{25.25}$$

The first boundary condition is $x(0) = 2$. We have to decide now whether the appropriate second boundary condition is $\lambda(1) = 0$ (if the constraint on $x(1)$ is not binding) or $x(1) = b$ (if the constraint on $x(1)$ is binding). To do this, we proceed as if there were no constraint on $x(1)$ (solve the free-endpoint problem). We will find $x^*(1)$, the optimal value, and then compare it to b to see if the constraint is binding or not. In example 25.1 we already solved the free-endpoint problem and found the optimal solution path for the state variable to be

$$x^*(t) = \frac{t}{2} - \frac{t^2}{4} + 2$$

Since $T = 1$, the optimal value for $x(1)$ is

$$x^*(1) = \frac{1}{2} - \frac{1}{4} + 2 = \frac{9}{4}$$

Two cases can arise:

Case 1 $b \leq 9/4$. The constraint is not binding because $x^*(1) \geq b$. The problem is solved.

Case 2 $b > 9/4$. The constraint is binding because $x^*(1) < b$. Thus we cannot have our first choice for $x(1)$. The next best we can do is to set $x(1) = b$. The differential equations now have to be re-solved using the correct boundary conditions: $x(0) = 2$ and $x(1) = b$. This is the fixed-endpoint problem we solved in example 25.5. ■

Example 25.6 Solve the optimal consumption model with an inequality constraint on $x(T)$:

$$\max \int_0^T e^{-\rho t} \ln c \, dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$

$$x(0) = x_0$$

$$x(T) \geq b$$

Solution

We have already solved the model of optimal consumption with a fixed-endpoint constraint. We are now asked to reconsider this problem with an inequality constraint on $x(T)$. The only thing that differs between this problem and its fixed-endpoint version is the boundary condition on $x(T)$. As such, the solution up to the point of finding the boundary conditions is identical. After using the initial condition $x(0) = x_0$, we found the solution for $x(t)$ to be

$$x(t) = -C_1^{-1} \frac{(1 - e^{-\rho t})}{\rho} e^{rt} + x_0 e^{rt} \tag{25.26}$$

We found the solution for $\mu(t)$, the current-valued costate variable, to be

$$\mu(t) = C_1 e^{(\rho - r)t}$$

To find the value of the constant of integration, C_1, we need to use the boundary condition $e^{-\rho T} \mu(T) = 0$ [if the constraint on $x(T)$ is not binding] or $x(T) = b$ [if the constraint on $x(T)$ is binding]. To decide, try to solve for the unconstrained $x^*(T)$. Set $e^{-\rho T} \mu(T) = 0$. Since T is finite, this amounts to setting $\mu(T) = 0$, which gives $C_1 = 0$. However, inspection of equation (25.26) reveals that as $C_1 \to 0, x(t) \to -\infty$ for all values of t, including $t = T$. This clearly violates the constraint that $x(T) \geq b$. Thus, we know that the relevant boundary condition is $x(T) = b$ and that we should proceed to solve this as a fixed-endpoint problem. The solution, therefore, is the same as the fixed-endpoint version already solved.

A related approach to deciding that the constraint is binding in this problem is to note that if $C_1 = 0$, then $\mu(t) = 0$ for all $t \in (0, T)$. However, the first-order condition for $c(t)$ indicates that as $\mu(t) \to 0$, $c(t) \to \infty$. That is, the consumption rate becomes infinitely large as $\mu(t)$ approaches zero. This is clearly impossible, as it would not only drain the bank account but drive it to minus infinity, as we determined above.

We conclude that in the absence of a bequest constraint in this model, the optimal consumption path is one that drives the bank account down to the minimum permitted level, b, by the end of the lifetime, T. The smallest value that b could possibly take in reality is $b = 0$. In that case the optimal consumption path involves consuming all of the capital in the bank account by time T. ■

Optimal Depletion of an Exhaustible Resource

Imagine an individual stranded on a desert island where the only source of food is a fixed stock of an exhaustible resource. The resource is not perishable but also does not accumulate or reproduce. The individual is assumed to live from time 0 (now) to T and is assumed to know this with certainty. The individual must choose a consumption path knowing that the stock of food is exhaustible. We assume the path chosen is the one that maximizes the discounted sum of utility. The maximization problem is

$$\max \int_0^T e^{-\rho t} U[c(t)]\, dt$$

$$\text{subject to} \quad \dot{R} = -c$$
$$R(0) = R_0$$
$$R(T) \geq 0$$

where $U[c(t)]$ is instantaneous utility at t and ρ is the personal rate of time preference.

The differential equation shows that the stock of the resource, $R(t)$, declines by the amount consumed, $c(t)$. The resource stock starts out at size R_0 and cannot decline below a size of zero. We now have an inequality-constrained endpoint problem.

The current-valued Hamiltonian function is

$$\mathcal{H} = U(c) - \mu c$$

Assuming that $U' > 0$ and $U'' < 0$ (positive and diminishing marginal utility) means the following condition holds on the optimal consumption path:

$$U'(c) - \mu = 0 \qquad (25.27)$$

Implicitly this gives a solution for c as a function of μ, which we write as

$$c = \phi(\mu)$$

In addition the costate variable must satisfy

$$\dot{\mu} - \rho\mu = 0$$

since $\partial H/\partial R = 0$ in this model. The system of differential equations then is

$$\dot{\mu} = \rho\mu$$
$$\dot{R} = -\phi(\mu)$$

The relevant boundary conditions are $R(0) = R_0$ and either the transversality condition $\mu(T) = 0$ or the endpoint condition $R(T) = 0$. The solution to the first differential equation is

$$\mu(t) = C_1 e^{\rho t} \tag{25.28}$$

where C_1 is a constant of integration. If we impose the condition $\mu(T) = 0$, we obtain the solution $C_1 = 0$, from which we conclude immediately that $\mu(t) = 0$ for $t \in (0, T)$. Will this lead to the constraint $R(T) \geq 0$ being violated? The answer depends on the form of the utility function. If marginal utility falls to zero at some consumption level, say $\hat{c}$, the solution to equation (25.27) with $\mu(t) = 0$ is $c(t) = \hat{c}$ for all $t \in (0, T)$. Lifetime consumption is $\hat{c}T$. If $\hat{c}T$ is less than R_0, then $R(T)$ is clearly positive, so the resource constraint is nonbinding. In this case the resource is not scarce: more than enough of it is available to sustain the consumer's desired lifetime consumption, so it has no economic value. As a result its shadow price is zero.

On the other hand, if $\hat{c}T$ exceeds R_0, the resource constraint becomes binding, and $\mu(T) = 0$ is the wrong boundary condition. $R(T) = 0$ becomes the correct boundary condition.

If the utility function does not have a satiation point (we typically assume that it doesn't), then marginal utility never becomes zero, no matter how high the consumption rate. This is the case for the utility functions we have examined so far in this chapter. For example, if $U(c) = \ln c$, then $U'(c) = 1/c$, and this tends to zero only as $c \to \infty$. In this case the solution to equation (25.27) with $\mu(t) = 0$ is an infinitely high consumption rate. This would obviously drive the resource stock to negative infinity, thereby violating the constraint. Thus $\mu(T) = 0$ is the wrong boundary condition; $R(T) = 0$ is the correct boundary condition.

To complete the solution, we will assume that the resource is economically scarce, which means that we need to assume either that there is no finite satiation point or, if there is a satiation point, that it is at a high enough consumption rate that the resource constraint becomes binding. The relevant second boundary condition then is $R(T) = 0$.

Because we have not specified a functional form for utility, we cannot obtain an explicit solution for this problem. Instead, we provide the implicit solution as far as it can be taken and draw the phase diagram.

Solving the second differential equation and using equation (25.28) gives

$$R(t) = -\int_0^t \phi(C_1 e^{\rho s})\, ds + C_2$$

Using the initial condition $R(0) = R_0$ in this expression gives $C_2 = R_0$. Evaluating the expression at $t = T$ and imposing the second boundary condition, $R(T) = 0$, gives

$$R(T) = R_0 - \int_0^T \phi(C_1 e^{\rho s})\, ds = 0$$

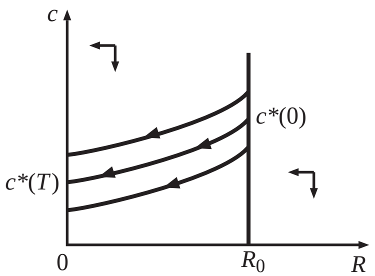


Figure 25.7 Phase diagram for the exhaustible resource problem drawn in R (the resource stock remaining) and c (consumption) space

This is as far as we can go with an implicit solution. However, if a functional form for utility were assumed, then the function ϕ would be known. In principle, the above equation could be solved for C_1. Using this in $\phi[\mu(t)]$ then gives the solution for the path of consumption.

A phase diagram helps to clarify the qualitative properties of the solution. Rather than draw it in (R, μ) space, we shall follow the alternative approach of drawing it in (R, c) space (see figure 25.7).

The differential equation for c is obtained by differentiating equation (25.27) with respect to t. This gives

$$U''(c)\dot{c} = \dot{\mu}$$

Substituting for $\dot{\mu}$ and using equation (25.27) again gives

$$\dot{c} = \frac{U'(c)}{U''(c)}\rho$$

This, combined with

$$\dot{R} = -c$$

makes up the system of differential equations for the phase diagram shown in figure 25.7.

The isocline for c is the locus of points where $\dot{c} = 0$. This occurs where $U'(c) = 0$, for which the solution is $\hat{c}$; this solution was defined above as the satiation point. We have assumed that $\hat{c}$ is large enough (possibly infinitely large) so that it is not possible to consume at that rate and also satisfy the exhaustible resource constraint. For the purpose of our diagram, we simply assume that $\hat{c}$ is sufficiently large that we can safely ignore it and not bother drawing the isocline for c. In other words, all relevant trajectories will be below the isocline. The motion of c below the isocline is negative, since $U' > 0$ and $U'' < 0$.

The isocline for R occurs at $c = 0$. The motion of R is negative for $c > 0$.

The phase diagram shows a number of representative trajectories that are consistent with the system of differential equations. To be also consistent with the boundary conditions, the solution trajectory must start on the vertical line where $R(0) = R_0$ and must finish on the vertical line where $R(T) = 0$. Finally the solution trajectory must also take an amount of time equal to T to travel from $R(0) = R_0$ to $R(T) = 0$. Only one trajectory satisfies all these conditions.

The phase diagram helps us to see the solution. Consumption is at its highest at $t = 0$ and declines monotonically thereafter. Moreover, we note that the slope of the trajectories is largely determined by the personal rate of time preference. How do we know this? The slope of any trajectory is given by

$$\frac{dc}{dR} = \frac{dc/dt}{dR/dt} = \frac{\dot{c}}{\dot{R}} = -\rho \frac{U'(c)}{cU''(c)}$$

Trajectories have a positive slope, since $U'' < 0$ and $\rho, c, U' > 0$; and the smaller is ρ, the smaller is the slope. Thus the optimal consumption path is declining over time, but the steepness of the decline is smaller when ρ is smaller. Intuitively the lower the individual's discount rate, the more equal is the importance of future utility relative to current utility; the higher the discount rate, the less important is future utility relative to current utility. Thus a low discount rate results in a consumption path that is flatter, or more equal, over time.

EXERCISES

1. Solve the following:

$$\max \int_0^T e^{-\rho t}(yx - y^2 - x^2)\, dt$$

$$\text{subject to} \quad \begin{aligned} \dot{x} &= x + y \\ x(0) &= x_0 \\ x(T) &= x_T \end{aligned}$$

2. Solve the following:

$$\max \int_0^T e^{-\rho t}(ay - by^2 + fx - gx^2)\,dt$$

$$\text{subject to} \quad \dot{x} = x + y$$
$$x(0) = x_0$$
$$x(T) = x_T$$

3. Solve for the optimal consumption path, $c^*(t)$, in the following model:

$$\max \int_0^T e^{-\rho t} \frac{c^{1-\alpha}}{1-\alpha}\,dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$
$$x(0) = x_0$$
$$x(T) = x_T$$

where $0 < \alpha < 1$.

4. In this problem, production of a good, y, yields economic benefits but also contributes to the stock of pollution, x, which is an economic bad. Instantaneous net benefits are $y - y^2 - x^2$. If the stock of pollution depreciates (is broken down naturally in the environment) at the rate δ, find the path of consumption that solves the following:

$$\max \int_0^T e^{-\rho t}[y - y^2 - x^2]\,dt$$

$$\text{subject to} \quad \dot{x} = y - \delta x$$
$$x(0) = x_0$$
$$x(T) = x_T$$

5. Draw the phase diagram for exercise 4. Suppose that $x_0 < x_T < \bar{x}$, where $\bar{x}$ is the steady-state value of x. Show that it is possible that if T is not large enough, a solution does not exist. Also show that as T becomes very large, the solution path approaches the saddle path.

6. In this problem you are to analyze a slightly more general version of exercise 4. Let the instantaneous net benefits be given by $B(y) - c(x)$ where $B'(y) > 0$, $B''(y) < 0$, $c'(x) > 0$, and $c''(x) > 0$. Two additional assumptions that will help you draw the phase diagram are: $B'(0) = p > 0$ (net marginal benefits at

$y = 0$ are a positive constant), and $c'(0) = 0$ (the marginal cost $= 0$ at $x = 0$). Draw the phase diagram for the following problem:

$$\max \int_0^T e^{-\rho t}[B(y) - c(x)]\,dt$$

$$\text{subject to} \quad \dot{x} = y - \delta x$$
$$x(0) = x_0$$
$$x(T) = x_T$$

7. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^T e^{-\rho t} \ln y\,dt$$

$$\text{subject to} \quad \dot{x} = -y$$
$$x(0) = x_0$$
$$x(T) \geq 0$$

8. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^T e^{-\rho t} \frac{y^{1-\alpha}}{1-\alpha}\,dt$$

$$\text{subject to} \quad \dot{x} = -y$$
$$x(0) = x_0$$
$$x(T) \geq 0$$

where $0 < \alpha < 1$.

25.4 Infinite–Time Horizon Problems

In some economic models it is unrealistic to assume a finite time horizon. For example, it is difficult to justify the assumption that a firm has a finite planning horizon. That would mean it behaves as if it will exist only over the interval $(0, T)$. Why would it ignore profits earned after T? Similarly, why would we consider a finite planning horizon when trying to solve for an economy's optimal path of consumption and capital accumulation? Why would we ignore the benefits enjoyed by generations alive beyond T? It makes far more sense in cases like these to assume an **infinite time horizon,** $T = \infty$, for the maximization problem.

Definition 25.5 The general form of the **autonomous dynamic optimization problem,** with an *infinite* time horizon, is

$$\max J = \int_0^\infty F(x, y)e^{-\rho t}\,dt$$

$$\text{subject to} \quad \dot{x} = G(x, y)$$
$$x(0) = x_0$$

The first problem that can arise in infinite–time horizon problems is nonconvergence of the integral to be maximized. Because we are integrating over an infinite amount of time, there is a danger that the integral may go to infinity, in which case no maximum exists. However, in autonomous problems with a positive discount rate, the integral does not go to infinity provided $F(x, y)$ is bounded from above. The reason is that the integrand term $F(x, y)e^{-\rho t}$ goes to zero as t goes to infinity. As a result the integral itself is bounded from above.

Fortunately most of the results of optimal control theory that we have already developed for finite–time horizon problems carry over to the infinite time horizon case. We still must maximize the Hamiltonian at each point in time with respect to the control variable

$$\frac{\partial H}{\partial y} = 0$$

or, in terms of the current-valued Hamiltonian,

$$\frac{\partial \mathcal{H}}{\partial y} = 0$$

and the costate variable must still satisfy the differential equation

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

or, in terms of the current-valued Hamiltonian,

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial x}$$

Thus we still obtain a system of two differential equations

$$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial x} \tag{25.29}$$

$$\dot{x} = G(x, y) \tag{25.30}$$

as we would in a finite–time horizon problem. The only difference occurs, as one might expect, with the boundary conditions used to solve the differential equations. For *fixed-endpoint* problems the boundary conditions are specified in the maximization problem. These would appear as

$$x(0) = x_0, \quad \lim_{t\to\infty} x(t) = b$$

Note that the second boundary condition is the infinite-time horizon equivalent of the requirement that $x(T) = b$ when T is finite.

In *free-endpoint* problems, where the value of $x(t)$ as $t \to \infty$ is free to be chosen optimally, the transversality condition used when T is finite [$e^{-\rho T}\mu(T) = 0$] can usually be replaced by the condition $e^{-\rho t}\mu(t) \to 0$ as $t \to \infty$, but this condition turns out to be not terribly helpful and not always correct. Instead, the convention is to assume that the steady-state value of the state variable provides the optimal boundary condition for x as $t \to \infty$. As long as the steady state is a saddle-point equilibrium (which it almost always is), this assumption is correct. The boundary conditions for free-endpoint problems therefore are

$$x(0) = x_0, \quad \lim_{t\to\infty} x(t) = \bar{x}$$

where $\bar{x}$ is the steady-state value of x.

It turns out that in infinite–time horizon problems that are autonomous, for which $F(x, y)$ and $G(x, y)$ are both concave in x and y, and that have only one state variable, the solutions have two interesting properties. First, the dynamic system given by equations (25.29) and (25.30) is either a saddle point (in which case the optimal trajectory is the saddle path), or it is completely unstable. In almost all problems you are likely to encounter, it turns out to be a saddle point. Indeed, this is guaranteed if the discount rate is small. Second, the path of the state variable $x(t)$ along the saddle path is monotonic. That is, the state variable never changes direction in its approach to the steady state; it is either always increasing to a limiting value of $\bar{x}$, or always decreasing to a limiting value of $\bar{x}$, or always constant.

Although we will not attempt to prove the validity of the first property here, we point out that to do so, you would have to determine the signs of the eigenvalues of the system of differential equations in (25.29) and (25.30), after linearizing the system around the steady-state equilibrium. You would find that the concavity of F and G implies that at least one eigenvalue is always positive, so that stable nodes are an impossibility. In addition you would find that the other root will very likely be negative except when the discount rate is very large. As a result saddle points are the most likely kind of equilibrium. The second property can be proven rigorously, but its validity can also be seen by careful inspection of the phase diagrams that have been constructed for saddle points in this chapter and in chapter 24.

Two economic applications of optimal control theory in infinite–time horizon models are presented in this section. We have already analyzed the finite–time horizon versions of these models in this chapter so only the boundary conditions will change. The first is a fixed-endpoint problem; the second is a free-endpoint problem.

Example 25.7 Solve the fixed-endpoint, infinite-time consumption model

$$\max \int_0^{\infty} e^{-\rho t} \ln c \, dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$

$$x(0) = x_0$$

$$\lim_{t \to \infty} x(t) = 0$$

This is the optimal consumption model examined earlier, but now with an infinite time horizon. A sensible endpoint condition has been imposed on the bank account: in the limit as time becomes infinite, the bank account is reduced to zero.

Solution

The solution for an infinite–time horizon model is exactly the same as for a finite–time horizon problem, up to the point of deriving the boundary conditions. Accordingly, applying the maximum principle to this problem leads to the same system of differential equations as found earlier. After maximizing the current-valued Hamiltonian with respect to the control variable, c, and finding the differential equation for μ, we obtained the solution for $x(t)$ written in equation (25.23),

$$x(t) = C_1^{-1} \frac{(e^{-\rho t} - 1)}{\rho} e^{rt} + C_2 e^{rt} \tag{25.31}$$

where C_1 and C_2 are the constants of integration to be determined using the boundary conditions. Imposing the first boundary condition, $x(0) = x_0$, gives $C_2 = x_0$.

Using this value, rewrite the expression for $x(t)$ as

$$x(t) = \frac{e^{(r-\rho)t}}{\rho C_1} - \frac{e^{rt}}{\rho C_1} + x_0 e^{rt} \tag{25.32}$$

Now solve for the constant C_1 that makes this solution satisfy the boundary condition that $x(t) \to 0$ as $t \to \infty$. It is apparent that if $r - \rho < 0$, the first term on the right-hand side goes to zero as $t \to \infty$, which means that if we set

$$C_1 = \frac{1}{\rho x_0}$$

and assume that

$$r - \rho < 0$$

the solution for $x(t)$ satisfies the boundary condition. As a result the solution for $x(t)$ becomes

$$x(t) = x_0 e^{(r-\rho)t}$$

■

Example 25.8 Solve the free-endpoint, infinite-time investment model

$$\max \int_0^{\infty} e^{-\rho t}[K - aK^2 - I^2]\,dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$

This is the investment model examined earlier but now with an infinite time horizon. Notice that the limiting value of $K(t)$ is not specified in this problem.

Solution

The current-valued Hamiltonian is

$$\mathcal{H} = K - aK^2 - I^2 + \mu(I - \delta K)$$

Maximizing with respect to the control variable I and simplifying gives

$$I(t) = \frac{\mu(t)}{2}$$

The differential equation for μ is

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial K} = -(1 - 2aK - \mu\delta)$$

The resulting system of differential equations is

$$\dot{\mu} = \mu(\delta + \rho) + 2aK - 1 \tag{25.33}$$

$$\dot{K} = \frac{\mu}{2} - \delta K \tag{25.34}$$

The solutions are

$$\mu(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \bar{\mu} \tag{25.35}$$

$$K(t) = \frac{r_1 - \delta - \rho}{2a} C_1 e^{r_1 t} + \frac{r_2 - \delta - \rho}{2a} C_2 e^{r_2 t} + \bar{K} \tag{25.36}$$

where $\bar{\mu}$ and $\bar{K}$ are the steady-state values of $\mu(t)$ and $K(t)$ respectively. They are found by setting $\dot{\mu} = 0 = \dot{K}$, which gives

$$\bar{\mu} = \frac{\delta}{\delta(\delta + \rho) + a}$$

$$\bar{K} = \frac{1}{2[\delta(\delta + \rho) + a]}$$

To complete the solution, we require two boundary conditions. The first is

$$K(0) = K_0$$

The second is

$$\lim_{t \to \infty} K(t) = \bar{K}$$

provided that the point $(\bar{\mu}, \bar{K})$ is a saddlepoint equilibrium. To confirm that it is, write the homogeneous form of equations (25.33) and (25.34) as

$$\begin{bmatrix} \dot{\mu} \\ \dot{K} \end{bmatrix} = \begin{bmatrix} \delta + \rho & 2a \\ 1/2 & -\delta \end{bmatrix} \begin{bmatrix} \mu \\ K \end{bmatrix}$$

The determinant of the coefficient matrix is $-\delta(\delta+\rho)-a<0$. Therefore the characteristic roots must be of opposite sign, indicating a saddle-point equilibrium at $(\bar{\mu}, \bar{K})$.

Assuming that r_1 is the negative root and r_2 is the positive root, then it is apparent from inspecting equation (25.36) that the only way the boundary condition for $K(t)$ can be satisfied is if $C_2 = 0$. The solution becomes

$$K(t) = \frac{r_1-\delta-\rho}{2a} C_1 e^{r_1 t} + \bar{K}$$

Using the other boundary condition, $K(0) = K_0$, we find that

$$C_1 = \frac{2a(K_0-\bar{K})}{r_1-\delta-\rho}$$

The final solution then is

$$K(t) = (K_0-\bar{K})e^{r_1 t} + \bar{K}$$

The solution for $\mu(t)$ then follows easily

$$\mu(t) = \frac{2a(K_0-\bar{K})}{r_1-\delta-\rho} e^{r_1 t} + \bar{\mu}$$

Finally the solution for $I(t)$ is then

$$I^*(t) = \frac{a(K_0-\bar{K})}{r_1-\delta-\rho} e^{r_1 t} + \delta\bar{K}$$

where we have used the fact that $\bar{\mu}/2 = \delta\bar{K}$. The optimal investment path starts out high if the initial capital stock is below the steady-state value, $K_0 < \bar{K}$. It then declines monotonically (because $r_1 < 0$) and in the limit converges to $\delta\bar{K}$, which is just enough to cover depreciation and keep the capital stock at its steady-state level. ■

The Neoclassical Growth Model

In chapter 21 we studied the equilibrium characteristics of the Solow model of aggregate economic growth in which the savings rate is assumed to be an exogenous parameter, s. This exogenous savings rate led to a path of capital accumulation and eventually a steady-state equilibrium in which output per capita became constant over time. In the neoclassical growth model we treat the savings rate as endogenous. That is, we look for the path of saving that maximizes the discounted total utility over time.

Recall that the differential equation describing the path of the capital-labor ratio was shown to be

$$\dot{k} = f(k) - c - (\delta + n)k$$

where $f(k)$ is the production function and gives output per person, k is capital per person, c is consumption per person, and δ and n are the depreciation rate of capital and the growth rate of the population (labor force) respectively.

If $k(0)$ is given, then the entire path for $k(t)$ is determined once we choose a path for consumption, $c(t)$. In the Solow growth model, we assumed that $c = (1-s)f(k)$, where $0 \leq s \leq 1$ is the exogenous rate of savings. In that model, we assumed that consumption was always equal to the same fraction of output produced. Now we want to choose the consumption path that maximizes

$$\int_0^\infty e^{-\rho t} U[c(t)] N(t)\, dt$$

where $U[c(t)]$ is the instantaneous utility function of an individual at time t, and $N(t)$ is the total population at time t. Under the simplifying assumption that individuals are identical, $U[c(t)]N(t)$ is the total utility of the population at time t. The population is assumed to grow at the rate n, which gives

$$N(t) = N_0 e^{nt}$$

where N_0 is the initial population level which we set equal to 1, for simplicity. Thus $N(t) = e^{nt}$. The social rate of time preference (the discount rate) is $0 \leq \rho \leq 1$, and we assume an infinite planning horizon.

The problem at hand then is

$$\max \int_0^\infty e^{-(\rho - n)t} U[c(t)]\, dt \tag{25.37}$$

$$\begin{aligned} \text{subject to} \quad & \dot{k} = f(k) - c - (\delta + n)k \\ & k(0) = k_0 > 0 \qquad \text{(given)} \\ & k(t) \geq 0 \\ & c(t) \geq 0 \end{aligned}$$

We assume that $\rho - n > 0$; otherwise, the integral is unbounded. In addition we will not worry about the nonnegativity constraints on $k(t)$ or $c(t)$ at this point. We will make assumptions about the structure of $U(c)$ and $f(k)$ that ensure that the constraints are not binding. In the next section we will explain how to take constraints on the control variable into account in optimal control theory.

The effective discount factor in (25.37) is $(\rho - n)$. Keeping this in mind, the current-valued Hamiltonian function for this problem is

$$\mathcal{H} = U(c) + \mu[f(k) - c - (\delta + n)k]$$

The maximum principle requires that we choose c to maximize $\mathcal{H}$. Assuming that $U'(c) \to \infty$ as $c \to 0$ ensures that the solution is bounded away from zero. Maximizing $\mathcal{H}$ with respect to c gives

$$\frac{\partial \mathcal{H}}{\partial c} = U'(c) - \mu = 0 \tag{25.38}$$

This implicitly makes the choice of c a function of μ. We write this as $c = \phi(\mu)$.

The current-valued costate variable must satisfy

$$\dot{\mu} - (\rho - n)\mu = -\frac{\partial \mathcal{H}}{\partial k} = -\mu[f'(k) - (\delta + n)] \tag{25.39}$$

The system of differential equations then is

$$\dot{\mu} = \mu[\rho + \delta - f'(k)] \tag{25.40}$$

$$\dot{k} = f(k) - \phi(\mu) - (\delta + n)k \tag{25.41}$$

The two boundary conditions are $k(0) = k_0$ (given) and, assuming a saddle-point equilibrium (which we later verify), $\lim k(t) \to \bar{k}$ as $t \to \infty$. We proceed to analyze the system of differential equations using a phase diagram.

Rather than constructing the phase diagram in (k, μ) space for the system in equations (25.40) and (25.41), we instead construct it in (k, c) space. The reason is that our interest is in understanding the properties of the optimal consumption path and this approach allows us to look at it directly. The link between μ and c of course is through equation (25.38), which gives c as a function of μ. Differentiate equation (25.38) to get

$$U''(c)\dot{c} = \dot{\mu}$$

Using equations (25.40) and (25.38), this becomes

$$\dot{c} = \frac{U'(c)}{U''(c)}[\rho + \delta - f'(k)] \tag{25.42}$$

Combined with the differential equation for k

$$\dot{k} = f(k) - c - (\delta + n)k \tag{25.43}$$

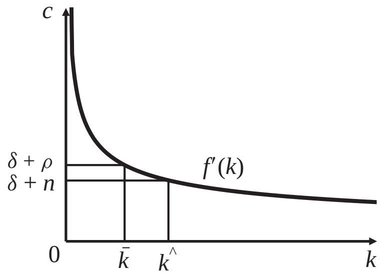


Figure 25.8 Solution values for $\bar{k}$ and $\hat{k}$

equations (25.42) and (25.43) form the system of autonomous, nonlinear differential equations we wish to analyze using a phase diagram.

First, analyze the motion of c in the (k, c) phase plane. Begin by sketching the isocline for c. Setting $\dot{c} = 0$ gives the equation for this line

$$f'(k) = \delta + \rho$$

As figure 25.8 indicates, there is a unique solution to this equation. Let this solution be denoted $\bar{k}$. Whenever $k = \bar{k}$, no matter what the value of c, the system has $\dot{c} = 0$. The graph of this in the phase diagram, drawn in figure 25.9, is a vertical line at $\bar{k}$.

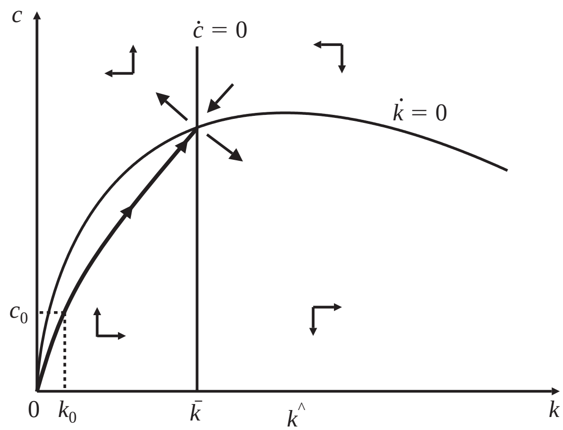


Figure 25.9 Phase diagram for the neoclassical growth model

The $\dot{c} = 0$ isocline divides the positive (k, c) phase plane into two regions. The motion of c in each of these regions is found by taking the partial derivative of the equation for $\dot{c}$ with respect to k. This gives

$$\frac{\partial \dot{c}}{\partial k} = -\frac{U'(c)}{U''(c)} f''(k) < 0$$

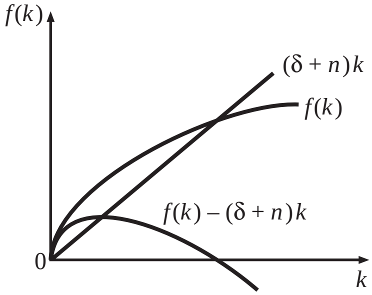


Figure 25.10 Difference between $f(k)$ and $(\delta + n)k$ gives the isocline for k in figure 25.9

The sign of this partial derivative is negative because $U' > 0$, $U'' < 0$, and $f'' < 0$ by assumption. This means that the change in $\dot{c}$ is *opposite* to the change in k. Thus $\dot{c}$ is negative to the right of the isocline and $\dot{c}$ is positive to the right of the isocline.

Second, analyze the motion of k in the (k, c) phase plane. Setting $\dot{k} = 0$ gives the equation for the k isocline:

$$c = f(k) - (\delta + n)k$$

To sketch the graph of the equation, it is useful to examine figure 25.10, which plots $f(k)$, *a* concave function ($f' > 0$, $f'' < 0$ by assumption), and plots the straight line $(\delta + n)k$, and then takes the difference between them. The difference is the plot of the isocline that has been transferred onto figure 25.9.

In figure 25.9, we have placed the peak of the $\dot{k} = 0$ isocline to the right of $\bar{k}$. Why did we do this? The *slope* of the isocline is given by

$$\frac{dc}{dk} = f'(k) - (\delta + n)$$

The peak of this isocline occurs where $f'(k) = \delta + n$. Call the value of k at which this occurs $\hat{k}$. Since we have assumed that $n < \rho$ (i.e., the population growth rate is smaller than the social discount rate), figure 25.8 shows that $\hat{k} > \bar{k}$.

The $\dot{k} = 0$ isocline separates two isosectors of its own. This time, however, we must be careful in using the partial derivative technique to determine the motion of k in each of the two isosectors because the isocline is not monotonic. A move to the right, for example, from the upward-sloping part of the isocline will actually place us to the left of the downward-sloping part. Thus calculating $\partial\dot{k}/\partial k$ could give an ambiguous answer, depending on whether we are considering points on the upward- or downward-sloping part of the isocline. Instead, calculating $\partial\dot{k}/\partial c$ gives the change in $\dot{k}$ unambiguously above and below the $\dot{k} = 0$ isocline. This gives

$$\frac{\partial\dot{k}}{\partial c} = -1$$

Thus *above* the $\dot{k}$ isocline, k is decreasing; *below* the $\dot{k}$ isocline, k is increasing. Appropriate arrows of motion are placed in the phase plane.

The arrows of motion suggest that the steady-state equilibrium point is a saddlepoint. Let us verify this by examining the signs of the roots of the linear version of the coefficient matrix.

The differential equation system in equations (25.42) and (25.43) is of the form

$$\dot{c} = \Phi(c, k)$$
$$\dot{k} = \Omega(c, k)$$

Following the method developed in chapter 24, the coefficient matrix of the linearized version of this system in the neighborhood of the steady-state equilibrium, is the matrix of partial derivatives evaluated at the steady-state equilibrium:

$$A = \begin{bmatrix} \partial\Phi/\partial c & \partial\Phi/\partial k \\ \partial\Omega/\partial c & \partial\Omega/\partial k \end{bmatrix}$$

Calculating this matrix from equations (25.42) and (25.43) gives

$$A = \begin{bmatrix} \dfrac{\partial}{\partial c}\left[\dfrac{U'(c)}{U''(c)}\right][\rho + \delta - f'(\bar{k})] & -\left[\dfrac{U'(c)}{U''(c)}\right]f''(\bar{k}) \\ -1 & f'(\bar{k}) - (\delta + n) \end{bmatrix}$$

Since $f'(k) = \rho + \delta$ at $k = \bar{k}$, the upper-left element of this matrix is zero. As a result the determinant is

$$|A| = -\frac{U'(c)}{U''(c)}f''(\bar{k})$$

The determinant is *negative* because $U' > 0$, $U'' < 0$, and $f'' < 0$ by assumption. As a result we know immediately that the roots are of opposite sign, and the steady-state point $(\bar{k}, \bar{c})$ is a saddle-point equilibrium.

The phase diagram is complete. It depicts the trajectories of the system of differential equations for c and k. We now use the boundary conditions to complete the qualitative analysis of the solution. The solution trajectory must satisfy $k(0) = k_0$, which means that it must start on the vertical line drawn from k_0, and must satisfy $\lim k(t) = \bar{k}$ as $t \to \infty$. Since this is a saddle-point equilibrium, only the saddle path converges to the steady state. Therefore the boundary conditions imply that the optimal trajectory must start *on* the saddle path with $k(0) = k_0$.

Suppose that k_0 is smaller than the steady-state capital stock, as shown in figure 25.9. The solution to the optimal growth problem then is to start consumption low, at $c(0)$, to allow capital to accumulate. As the capital stock grows, the consumption rate grows. Note that both capital and consumption grow monotonically along the saddle path and approach the steady state in the limit.

EXERCISES

1. Solve the following free-endpoint, infinite-time horizon consumption model for the optimal consumption path. Assume that $\rho - r < 0$.

$$\max \int_0^\infty e^{-\rho t}[c - ac^2]\,dt$$

$$\text{subject to} \quad \dot{x} = rx - c$$
$$x(0) = x_0$$
$$x(t) \geq 0$$

2. Analyze the solution to the following optimal growth model using a phase diagram drawn in (k, μ) space:

$$\max \int_0^\infty e^{-\rho t} \ln c\,dt$$

$$\text{subject to} \quad \dot{k} = \frac{k^{1-\alpha}}{1-\alpha} - c - \delta k$$
$$k(0) = k_0$$
$$k(t) \geq 0$$

3. Repeat exercise 2 but draw the phase diagram in (k, c) space.

4. Production of a good y yields economic benefits but also contributes to the stock of pollution x, which is an economic bad. Instantaneous net benefits are $y - y^2 - x^2$. If the stock of pollution depreciates (is broken down naturally in the environment) at the rate δ, find the path of consumption that solves the following:

$$\max \int_0^\infty e^{-\rho t}[y - y^2 - x^2]\,dt$$

$$\text{subject to} \quad \dot{x} = y - \delta x$$
$$x(0) = x_0$$
$$x(t) \geq 0$$

Show that the solution is consistent with most notions of sustainable development. That is, show that in the limit, the level of production of y keeps the stock of pollution constant.

5. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^\infty e^{-\rho t} \ln y \, dt$$

$$\text{subject to} \quad \dot{x} = -y$$
$$x(0) = x_0$$
$$x(t) \geq 0$$

6. A fishery resource has a natural growth function

$$F(x) = rx(1-x), \qquad r > 0$$

Let $U(h)$ be the instantaneous social benefits from harvesting an amount h of the fish. Assume that $U'(h) > 0$ and $U''(h) < 0$. In addition, you may assume $U'(h) \to \infty$ as $h \to 0$, to eliminate the possibility of a corner solution at $h = 0$. Using a phase diagram, analyze the solution to the problem of maximizing the discounted sum of social benefits:

$$\max \int_0^\infty e^{-\rho t} U(h) \, dt$$

$$\text{subject to} \quad \dot{x} = F(x) - h$$
$$x(0) = x_0$$
$$x(t) \geq 0$$

25.5 Constraints on the Control Variable

We often encounter constraints on the control variable in economic problems. The most common is a non-negativity constraint. For example, a firm's output choice is frequently constrained to be greater than or equal to zero. Constraints can take other forms as well, such as upper bounds on the control variable. When the control variable is bounded by a constraint, the maximum principle must be modified to take the constraint into account. Suppose that the control variable y must satisfy at each moment in time the following general form of an inequality constraint:

$$y \leq h(x, y, t)$$

Definition 25.6 The **general optimal control problem** (free-endpoint problem) with an inequality constraint on the control variable is

$$\max J = \int_0^T f[x(t), y(t), t]\,dt$$

$$\text{subject to} \quad \dot{x} = g[x(t), y(t), t]$$
$$x(0) = x_0$$
$$h[x(t), y(t), t)] - y(t) \geq 0$$

The Hamiltonian for this problem is the usual

$$H = f(x, y, t) + \lambda g(x, y, t)$$

We wish to maximize H with respect to the control variable y, but now subject to the inequality constraint on y. To do this, we form the ordinary Lagrangean function

$$\mathcal{L} = H + \theta[h(x, y, t) - y]$$

which incorporates the inequality constraint, where θ is the Lagrange multiplier. The necessary conditions now are

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \tag{25.44}$$
$$\theta \geq 0; \quad \theta[h(x, y, t) - y] = 0$$

$$\dot{\lambda} = -\frac{\partial \mathcal{L}}{\partial x} \tag{25.45}$$

$$\lambda(T) = 0 \tag{25.46}$$

The important differences that arise are in equations (25.44) and (25.45). In equation (25.44) we have the necessary condition for determining the control variable, y. As usual these conditions must hold with complementary slackness. If $\theta = 0$, the constraint is not binding, and we have $y < h(x, y, t)$. If $\theta > 0$, the constraint is binding so we have $y = h(x, y, t)$.

In equation (25.45) the differential equation for the costate variable is equal to the negative derivative of the Lagrangean function, not just the Hamiltonian function. The reason for this is that the time path of the costate variable must also reflect whether or not the constraint on the control variable is binding.

Bang-Bang Controls

Until now we have assumed that the Hamiltonian is strictly concave in the control variable. In some problems studied in economics, however, the Hamiltonian is *linear* in the control variable. This can lead to discontinuities in the solution path for the control variable. While this might seem to be an insurmountable complication since, until now, the control variable has been a continuous function of time, optimal control theory is uniquely suited to solving problems like these. We will demonstrate this with two economic applications.

A Linear Investment Problem

Suppose that a firm's production function is $Q = f(K)$ where K is the stock of capital and Q is the rate of output. Let $p(t)$ be the price of output and $q(t)$ be the price of a unit of capital. If $I(t)$ is the rate of investment (capital purchases) at time t, the firm wishes to

$$\max \int_0^T e^{-\rho t}\{p(t)f[K(t)] - q(t)I(t)\}\,dt$$

$$\text{subject to} \quad \dot{K} = I(t) - \delta K(t)$$

$$K(0) = K_0$$

$$0 \le I(t) \le b$$

The differences between this investment model and the ones examined earlier are, first, that the cost of investment is now linear, $c(I) = qI$, and second, that there are upper and lower bounds placed on the control variable, I. That is, the investment rate is constrained to be greater than or equal to zero and less than or equal to b, an exogenous upper bound.

The current-valued Hamiltonian is

$$\mathcal{H} = pf(K) - qI + \mu(I - \delta K)$$

where we have suppressed t as an argument. The necessary conditions for a maximum require that $\mathcal{H}$ be maximized with respect to I subject to the constraints

$$I \ge 0, \quad b - I \ge 0$$

The Lagrangean function is

$$\mathcal{L} = \mathcal{H} + \theta_1 I + \theta_2(b - I)$$

where θ_1 and θ_2 are the Lagrange multipliers associated with the two constraints. The necessary conditions for a maximum of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial I} = -q + \mu + \theta_1 - \theta_2 = 0 \tag{25.47}$$

and the conditions of complementary slackness are

$$\begin{aligned} \theta_1 \geq 0 \quad & \theta_1 I = 0 \\ \theta_2 \geq 0 \quad & \theta_2(b - I) = 0 \end{aligned}$$

These conditions imply that whenever $I > 0$, we have $\theta_1 = 0$ (and $I = 0$ implies that $\theta_1 \geq 0$) and whenever $I < b$, we have $\theta_2 = 0$ (and $I = b$ implies that $\theta_2 \geq 0$). Although we cannot use these first-order conditions to explicitly solve for I as a function of μ (which is what we normally do), it will become apparent as we work through the solution to this problem that these conditions do, in fact, give us I as a function of μ. We will write this as $I(\mu)$ for convenience.

The costate variable must satisfy

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{L}}{\partial K} = -pf'(K) + \mu\delta$$

The system of differential equations then is

$$\dot{\mu} = \mu(\rho + \delta) - pf'(K) \tag{25.48}$$

$$\dot{K} = I(\mu) - \delta K \tag{25.49}$$

The first boundary condition is $K(0) = K_0$. Because $K(T)$ is free to be chosen, the second boundary condition is provided by the transversality condition

$$e^{-\rho T}\mu(T) = 0$$

We must now analyze the system of differential equations to get some insights into the solution. First, note that if, over an *interval* of time, the optimal solution for I is such that neither constraint is binding $(0 < I < b)$, then $\theta_1 = \theta_2 = 0$ over that interval. Hence, from equation (25.47), we know that $\mu = q$ over the interval. As a result, $\dot{\mu} = \dot{q}$. Making these substitutions in equation (25.48), and rearranging gives an expression that must then hold over this interval of time:

$$pf'(K) = q\left(\rho + \delta - \frac{\dot{q}}{q}\right) \tag{25.50}$$

This expression is the standard rule that defines the optimal capital stock in capital theory. It is interpreted as requiring the marginal value product of capital (left-hand side) to equal the user cost of capital (right-hand side). The user cost of capital is the forgone interest (ρq) plus depreciation (δq) less any capital gains $(\dot{q})$. Implicitly this condition defines a value of K that we shall soon see is the steady-state value, $\bar{K}$.

Now assume that prices are constant ($\dot{q} = 0$ and $\dot{p} = 0$) so that we have a system of autonomous differential equations. Then we can construct the phase diagram. What makes this construction appear to be more difficult than usual is that the first-order condition determining $I(\mu)$ depends on the Lagrange multipliers. The way to circumvent this difficulty is to recognize that three cases can arise: $I = 0$, $I = b$, or $0 < I < b$. We draw the phase diagram for each case separately.

Case 1 With $I = 0$, we have $\theta_1 \geq 0$ and $\theta_2 = 0$. Therefore from equation (25.47) we have

$$\mu = q - \theta_1$$

In particular, we see that $\mu \leq q$, where q is the exogenous price of capital goods.
Case 2 With $I = b$, we have $\theta_1 = 0$ and $\theta_2 \geq 0$. Therefore

$$\mu = q + \theta_2$$

and in particular, $\mu \geq q$.
Case 3 With $0 < I < b$, we have $\theta_1 = \theta_2 = 0$. Therefore $\mu = q$.

The three cases correspond to three regions in the phase plane: $\mu < q$, $\mu > q$, and $\mu = q$. In region 1, $I = 0$, and so $\dot{K} = -\delta K < 0$. The $\dot{K} = 0$ isocline occurs along $K = 0$, and for $K > 0$, K is declining. In region 2, $I = b$, and so $\dot{K} = b - \delta K$. The $\dot{K} = 0$ isocline occurs along $K = b/\delta$. It is easy to see then that $\dot{K} > 0$ for $K < b/\delta$ and that $\dot{K} < 0$ for $K > b/\delta$. We assume that b/δ is quite a large number so that $\dot{K} > 0$. In other words, the maximum investment rate is large enough to overcome depreciation except perhaps at a very large level of the capital stock.

We have now mapped out the motion of K in the (K, μ) phase plane. We have transferred this information onto the phase plane in figure 25.11. Now we must determine the motion of μ. Because the $\dot{\mu}$ equation does not depend explicitly on $I(\mu)$, there is no need to analyze its motion case by case. Instead, we can use the standard procedures. Setting $\dot{\mu} = 0$ gives the equation defining the isocline:

$$\mu = \frac{pf'(K)}{\rho + \delta}$$

To graph this equation, we assume that $f'(0) = \infty$ (i.e., the marginal product of capital becomes infinitely large as the capital stock goes to zero), which means

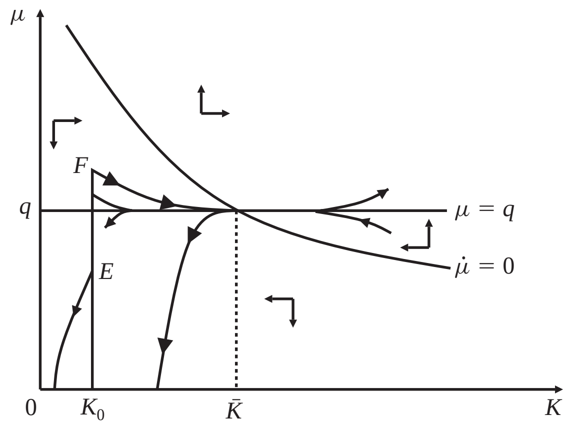


Figure 25.11 Phase diagram for the linear investment problem

that μ becomes infinitely large as K goes to zero. We also note that because $f'(K)$ is a decreasing function of K (i.e., diminishing marginal productivity), the graph has a negative slope. We could verify this by calculating the slope of the graph

$$\frac{d\mu}{dK} = \frac{pf''(K)}{\rho + \delta}$$

Since $f''(K) < 0$, the slope is negative. Furthermore, assuming that the marginal product of capital goes to zero only as the capital stock becomes infinitely large means that μ approaches zero only as K approaches infinity. The μ isocline is graphed in figure 25.11.

The motion of μ is most easily determined by calculating the partial derivative of $\dot{\mu}$ with respect to μ in the neighborhood of $\dot{\mu} = 0$. Taking the partial derivative of equation (25.48) gives

$$\frac{\partial \dot{\mu}}{\partial \mu} = \rho + \delta > 0$$

which indicates that μ is increasing above and decreasing below the μ isocline. These arrows of motion have been transferred onto the phase diagram.

This analysis of the phase plane gives a rough idea of what the trajectories must look like, but a bit more analysis is required to fill in some of the details. In particular, what happens to trajectories as they cross the $\mu = q$ line? First, keep in mind that this is not an isocline. Thus neither μ nor K comes to rest as trajectories cross it. Instead, $\dot{K}$ switches sign discontinuously at this point because I switches discontinuously between 0 and b. As a result the trajectories do not bend as they cross the $\mu = q$ line; rather, they have a kink as they change direction suddenly.

The exceptions are the two trajectories that cross the $\mu = q$ line where it intersects the $\dot{\mu} = 0$ isocline. The system does come to rest at this point. Why? At this point $\mu = q$ and $\dot{\mu} = 0$ so that $\mu = q$ can be sustained for an interval of time. At no other point is this possible. But we know that if $\mu = q$ is sustained over an interval of time, then K is constant at the level $\bar{K}$ defined above in equation (25.50). These two trajectories are like the saddle path in a saddle-point equilibrium. The difference though is that these trajectories reach the steady-state equilibrium in finite time. The reason is that the system does not slow down as it approaches the steady state. Instead, I remains constant at either $I = b$ or $I = 0$ along these trajectories until the instant they hit $K = \bar{K}$, at which point I switches instantly to the singular solution, implicitly defined in equation (25.50).

Let us look at a possible solution. Suppose that $K_0 < \bar{K}$ as shown in figure 25.11. If not much time is available (T is very small), the solution requires starting with $\mu < q$, at a point such as E. From there, $I = 0$ and the capital stock declines because of depreciation until $\mu(T) = 0$. If more time is available, the starting point will be further up the vertical line drawn at K_0. If enough time is available, the solution involves starting as high up as point F. From there, $I = b$, and the system follows the trajectory into the steady state at $\bar{K}$, where it stays for a while and leaves when there is just enough time left to reach $\mu(T) = 0$. If even more time is available, the system still starts at F but spends more time at the steady state.

The name **bang-bang** is given to linear control problems because of the discontinuous switching of the control variable that occurs along the solution path. Along the path starting at point F, for example, the solution is to first set I at its upper bound and keep it there until the steady state is reached. When it is time to leave the steady state, the solution is to set I at its lower bound and keep it there until $\mu(T) = 0$. In a sense, the solution is to approach the steady state with a *bang* (as quickly as possible) and to leave it with a *bang* (as quickly as possible).

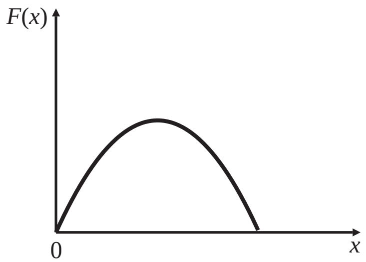


Figure 25.12 Biological growth function for the model of optimal fishing

A Model of Optimal Fishing

Let the growth of a fish stock be given by the differential equation

$$\dot{x} = F(x) - h$$

where x is the stock of fish and h is the harvest rate (both measured in tons, say). $F(x)$ is the biological growth function for the fish. It is a hill-shaped function such as the one depicted in figure 25.12.

The price of harvested fish is a constant p and the cost of harvesting fish is assumed to be

$$\text{total cost } = c(x)h$$

where $c(x)$ is the cost per unit of fish harvested and $c'(x) < 0$ to reflect the assumption that the cost of harvesting fish is lower when the stock of fish is larger (because it takes less effort to find and catch the fish), and higher when the stock of fish is small. We also assume that $c''(x) > 0$.

Let us find the harvesting policy that maximizes the discounted sum of economic returns from the fishery. Formally the problem is to

$$\max \int_0^T e^{-\rho t}[p - c(x)]h\,dt$$

$$\text{subject to} \quad \begin{aligned} &\dot{x} = F(x) - h \\ &x(0) = x_0 > 0 \qquad \text{(given)} \\ &x(T) \geq 0 \\ &0 \leq h \leq h_{\max} \end{aligned}$$

The current-valued Hamiltonian for this problem is

$$\mathcal{H} = [p - c(x)]h + \mu[F(x) - h]$$

The two constraints on the control variable are $h \geq 0$ and $h_{\max} - h \geq 0$. Introducing the Lagrange multipliers, θ_1 and θ_2 gives the Lagrangean expression

$$\mathcal{L} = \mathcal{H} + \theta_1 h + \theta_2(h_{\max} - h)$$

Maximizing $\mathcal{L}$ with respect to the control variable h gives

$$\frac{\partial \mathcal{L}}{\partial h} = p - c(x) - \mu + \theta_1 - \theta_2 = 0$$

From this condition, h is determined as a function depending on the value of μ according to the following three cases:

Case 1 $h = h_{\max}$ implies that $\theta_1 = 0, \theta_2 \geq 0$, and $\mu = p - c(x) - \theta_2$. Therefore $\mu \leq p - c(x)$.
Case 2 $h = 0$ implies that $\theta_1 \geq 0, \theta_2 = 0$, and $\mu = p - c(x) + \theta_1$. Therefore $\mu \geq p - c(x)$.
Case 3 $0 < h < h_{\max}$ implies that $\theta_1 = \theta_2 = 0$, and $\mu = p - c(x)$.

Next the costate variable must satisfy the differential equation

$$\dot{\mu} - \rho\mu = -[-c'(x)h + \mu F'(x)]$$

The differential equation system then becomes

$$\dot{\mu} = \mu[\rho - F'(x)] + c'(x)h(\mu)$$
$$\dot{x} = F(x) - h(\mu)$$

The relevant boundary conditions are the given initial condition, $x(0) = x_0$, and the transversality condition $e^{-\rho t}\mu(T) = 0$ if $x(T) > 0$.

To analyze the solution to the optimal harvesting problem, we first determine the properties of the steady-state solution. To do this, we set $\dot{\mu} = 0$ and $\dot{x} = 0$. First,

$$\dot{x} = F(x) - h = 0$$

gives $h = F(x)$. Next,

$$\dot{\mu} = \mu[\rho - F'(x)] + c'(x)h = 0$$

Simplifying, and rearranging, gives

$$F'(x) = \rho + \frac{c'(x)F(x)}{p - c(x)} \tag{25.51}$$

as the expression that implicitly defines the steady-state value of x. The economic interpretation of this condition is this: the optimal steady-state value of the fish stock is where the change in the growth rate (which can be interpreted as an internal rate of return to the fish stock as a capital asset) equals the discount rate (the external rate of return) plus a term that reflects the cost of fishing. Call this stock size $\bar{x}$. The steady-state harvest rate is $\bar{h} = F(\bar{x})$.

If the dynamic system remains in the steady state over an interval of time, it must be the case that $\mu = p - c(\bar{x})$, for this is the only case (of cases 1 to 3) in which the harvest rate can equal something other than zero or $h_{\max}$. During this interval, both μ and x are constant.

If sufficient time is available (T is large enough), the optimal solution then involves setting $h = 0$ or $h = h_{\max}$ (depending on $p - c(x)$), reaching the steady-state in finite time, spending some time there, and leaving with just enough time to satisfy the transversality condition. The larger is T, the longer is the amount of time spent in the steady state. As T approaches infinity, the amount of time spent in the steady state approaches infinity.

The interesting economic meaning of this solution is that it is optimal to drive the fish stock to its optimal steady-state size as quickly as possible. This is the so-called most-rapid-approach solution. This involves either setting a moratorium on fishing ($h = 0$) if $x(0) < \bar{x}$ or setting $h = h_{\max}$ if $x(0) > \bar{x}$. Figure 25.13 shows the time path of the fish stock for each of these possibilities.

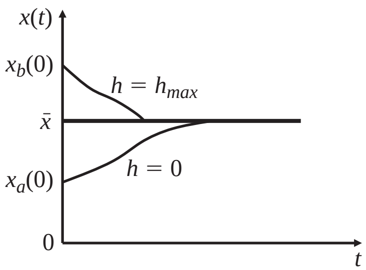


Figure 25.13 Optimal time path of the stock of fish when $x_0 < \bar{x}$ ($h = 0$) and when $x_0 > \bar{x}$ ($h = h_{\max}$)

EXERCISES

1. Assume that $\rho < r$ and solve the following linear optimal consumption model. Use a phase diagram to assist.

$$\max \int_0^T e^{-\rho t} c\, dt$$

$$\text{subject to} \quad \begin{aligned} \dot{x} &= rx - c \\ x(0) &= x_0 \\ x(T) &= x_T \\ 0 \le c &\le c_{\max} \end{aligned}$$

2. Solve the following linear exhaustible resource model for the optimal extraction path:

$$\max \int_0^T e^{-\rho t} (p - c) y\, dt$$

$$\text{subject to} \quad \begin{aligned} \dot{x} &= -y \\ x(0) &= x_0 \\ x(T) &\ge 0 \\ 0 \le y &\le y_{\max} \end{aligned}$$

where $p - c > 0$ is the constant profit per unit extracted.

3. In this investment model we introduce a second choice variable, labor, into the maximization problem. However, we assume that there is no state equation for labor. In essence, we are assuming that there is no stock of labor; it is simply hired as desired at each instant. To solve this problem, proceed by maximizing the Hamiltonian with respect to I and L at each instant. Solve the first-order condition for L in terms of K, and use this expression to eliminate L from any further expressions you derive. Assuming that $K_0 < \bar{K}$, use a phase diagram to solve this problem for the optimal path of investment, capital accumulation, and labor demand,

$$\max \int_0^\infty e^{-\rho t}[pK^\alpha L^\beta - wL - qI]\,dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$
$$0 \le I \le b$$

where L is labor input, K is the stock of capital, I is investment, p, w, and q are the constant prices of output, labor, and investment goods respectively, and α and β are positive constants whose sum is less than one.

25.6 Free-Terminal-Time Problems (T Free)

Until now we have assumed that T is specified exogenously. In some problems studied in economics, however, it is appropriate to allow T to be chosen endogenously. For example, in nonrenewable resource economics problems, T could represent the closure date of a mine, or the date at which an economy optimally switches from an exhaustible source of energy such as oil to a renewable source of energy such as solar.

With one additional endogenous variable to be determined in the optimal control problem, there is one additional necessary condition. We state this condition first and then provide a justification.

Definition 25.7

If T is free to be chosen endogenously, the additional necessary condition required is

$$H[x(T), y(T), \lambda(T), T] = 0$$

or, equivalently, in terms of the current-valued Hamiltonian

$$e^{-\rho T}\mathcal{H}[x(T), y(T), \mu(T), T] = 0$$

To see why this condition must hold at the optimal value of T, consider the general form of the finite-time horizon problem specified in definition 25.1:

$$\max J = \int_0^T f[x(t), y(t), t]\, dt$$

$$\text{subject to} \quad \dot{x} = g[x(t), y(t), t]$$
$$x(0) = x_0$$

The solution to this problem gives a particular value for the objective functional, J that depends on the actual value of T that is specified. We could indicate this by writing the value as $J(T)$. Different values specified for T give a different solution path and, hence, a different value for J. When T is allowed to be chosen optimally, our objective is to find the value of T that yields the largest possible value of J. But since J is a function of T, this occurs when

$$J'(T) = 0$$

$J'(T)$ is just the amount by which the value function J changes when the time horizon is extended slightly. But, as we already argued, the Hamiltonian function gives the total contribution (direct plus the change in the state variable value at its marginal value) to J at any point in time, including time $t = T$. Therefore, if T is extended marginally, the amount by which J changes is given by the value of H at time T. In other words,

$$J'(T) = H(x(T), y(T), \lambda(T), T)$$

and this leads to the condition given in definition 25.7

Further intuition for this additional condition is given by the following concrete example.

Example 25.9 A firm wishes to choose the investment path to maximize the present value of profits:

$$\max \int_0^T e^{-\rho t}[p(t)f(K) - c(I)]\, dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$
$$K(T) \geq 0$$

where $f(K)$ is the production function, $c(I)$ is the investment cost function, $p(t)$ is price, and δ is the depreciation rate.

Solution

At time T one necessary condition is that the present value of the shadow price of capital be zero at $T : e^{-\rho T}\mu(T) = 0$. If T is finite, this implies that $\mu(T) = 0$. The current-valued Hamiltonian at T then is

$$\mathcal{H}(T) = p(T)f[K(T)] - c(I(T))$$

which is just revenue minus cost at time T. If $\mathcal{H}(T) > 0$, it would clearly be desirable to continue production for a bit longer, since that would mean more profits. If $\mathcal{H}(T) < 0$, it means the firm is losing money by continuing production and should have stopped earlier. The optimal shutdown time is when $\mathcal{H}(T) = 0$, that is, when the profitability of continuing just becomes equal to zero.

It is interesting that in this investment problem, if $p(t) = p$, constant, $\mathcal{H}(T)$ is always positive. Profits never go to zero if they are ever positive, which they will be as long as $f(K) > 0$ and $c(0) = 0$. As a result it is always desirable to continue production so the optimal time horizon is actually $T = \infty$. In that case, since T is not finite, the condition in definition 25.7 becomes

$$\lim_{t\to\infty} e^{-\rho t}\mathcal{H}(t) = 0$$

■

An Optimal Mining Problem

A mining firm owns a mineral deposit with known reserves equal to R_0. It receives a constant price p for selling its mined ore and its cost function for extracting ore is

$$C(y) = \frac{y^2}{2}$$

where y is the rate of extraction. There is no uncertainty of any kind. The firm wishes to choose the path of ore extraction $y(t)$ to maximize the present value of profits:

$$\max \int_0^T e^{-\rho t}\left[py(t) - \frac{y(t)^2}{2}\right] dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{R} = -y(t) \\ & R(0) = R_0 > 0 \qquad \text{(given)} \\ & R(T) \geq 0 \\ & T \geq 0 \qquad \text{(free)} \end{aligned}$$

The constraints indicate, respectively, that the remaining stock of reserves declines by the amount extracted, is equal to R_0 initially, must be greater than or equal to 0 at T, and that T is free to be chosen endogenously.

The current-valued Hamiltonian is

$$\mathcal{H} = py - \frac{y^2}{2} - \mu y$$

Maximizing $\mathcal{H}$ with respect to y implies that

$$p - y - \mu = 0 \tag{25.52}$$

Thus

$$y(t) = p - \mu(t) \tag{25.53}$$

The costate variable must satisfy the differential equation

$$\dot{\mu} - \rho\mu = -\frac{\partial \mathcal{H}}{\partial R} = 0$$

The system of differential equations for this optimal control problem then is

$$\dot{\mu} = \rho\mu \tag{25.54}$$

$$\dot{R} = -(p - \mu) \tag{25.55}$$

Given two boundary conditions and a value for T, we can solve these to get the solution path for ore extraction. What are the boundary conditions? First, we are given $R(0) = R_0$. Second, since $R(T)$ is free to be chosen subject to the nonnegativity constraint, we use either $e^{-\rho T}\mu(T) = 0$ if $R(T) > 0$ or $R(T) = 0$ and $e^{-\rho T}\mu(T)$ free. To see which applies, try $\mu(T) = 0$. The solution to equation (25.54) is

$$\mu(t) = C_1 e^{\rho t} \tag{25.56}$$

where C_1 is an arbitrary constant of integration. If $\mu(T) = 0$, then $C_1 = 0$ which implies that $\mu(t) = 0$ for all t. But this in turn implies that $y(t) = p$, a constant, and so clearly leads to exhaustion of the ore deposit in finite time, which means the constraint on $R(T)$ is binding. Hence it was incorrect to assume $\mu(T) = 0$. Thus $R(T) = 0$ is the second boundary condition for this problem.

Using the expression for $\mu(t)$ and solving equation (25.55) gives

$$R(t) = -\int_0^t [p - C_1 e^{\rho s}]\, ds + C_2$$

where C_2 is another arbitrary constant of integration, the value of which is determined from the first boundary condition, $R(0) = R_0$. This gives $C_2 = R_0$. We then have

$$R(t) = R_0 - \int_0^t [p - C_1 e^{\rho s}]\, ds$$

The second boundary condition is $R(T) = 0$. We then have

$$R(T) = R_0 - \int_0^T [p - C_1 e^{\rho s}]\, ds = 0 \tag{25.57}$$

If T were known, this could be solved for C_1 to give the complete solution. However, we have yet to determine the value for T. Using the new necessary condition for determining T implies that

$$e^{-\rho T}\mathcal{H}(T) = py(T) - \frac{y(T)^2}{2} - \mu(T)y(T) = 0$$

We also know from equation (25.52) that $\mu(T) = p - y(T)$. Making this substitution gives

$$py(T) - \frac{y(T)^2}{2} - [p - y(T)]y(T) = 0$$

Simplifying this expression gives

$$\frac{y(T)^2}{2} = 0$$

to which the solution is $y(T) = 0$. This tells us that the optimal rate of ore extraction at $t = T$ is 0. This extra piece of information is all we require to go ahead and solve for the optimal value of T. First, use this information in (25.52) to find $\mu(T)$. This gives

$$\mu(T) = p$$

Next, use this as a boundary condition in equation (25.56) to solve for C_1. This gives

$$\mu(T) = C_1 e^{\rho T} = p$$

which gives

$$C_1 = p e^{-\rho T}$$

With the value for C_1 determined as a function of T, we can now use equation (25.57) to solve for T. Substituting the solution for C_1 into equation (25.57) gives

$$R(T) = R_0 - \int_0^T p\left[1 - e^{-\rho(T-s)}\right] ds = 0$$

Carrying out the integration gives

$$R_0 - p\left[s - \frac{e^{-\rho(T-s)}}{\rho}\right]_0^T = 0$$

Evaluating the integral gives

$$R_0 - p\left[T - \frac{1}{\rho} + \frac{e^{-\rho T}}{\rho}\right] = 0$$

We now have a nonlinear function implicitly defining T. We cannot solve it explicitly. We could solve it numerically given values for R_0, p, and ρ; or we could obtain some qualitative information about T by doing the following. Rewrite the expression for T as

$$\frac{R_0}{p} + \frac{1}{\rho} - T = \frac{e^{-\rho T}}{\rho}$$

Define the left-hand side as $\phi_L(T)$ and the right-hand side as $\phi_R(T)$. Graph these two functions and find where they intersect. In figure 25.14, $\phi_L(T)$ is a negatively sloped function with slope $= -1$, vertical intercept $R_0/p + 1/\rho$, and horizontal intercept at $\hat{T} = R_0/p + 1/\rho$. $\phi_R(T)$ is a negatively sloped function also. Its vertical intercept is $1/\rho$, which is smaller than the vertical intercept of $\phi_L(T)$, and its horizontal intercept is ∞, which is larger than the horizontal intercept of $\phi_L(T)$. The two curves therefore necessarily intersect at a value of T less than $\hat{T}$. We have called the solution T^*. We leave it to the interested reader to show that T^* is smaller the larger is p, the larger is ρ, or the smaller is R_0.

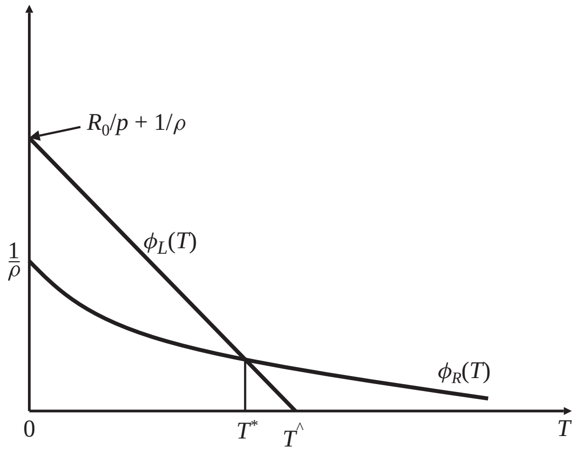


Figure 25.14 Optimal time to stop mining the ore deposit as determined by the intersection of the two curves

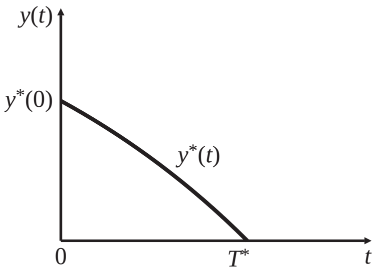


Figure 25.15 Optimal path of extraction in the mining problem

With T^* determined (implicitly), the first constant of integration is determined also (implicitly), and the optimal path of ore extraction is given (implicitly) by substituting the solution for C_1 into equation (25.56) and this into equation (25.53):

$$y^*(t) = p\left[1 - e^{-\rho(T^*-t)}\right]$$

Figure 25.15 graphs the optimal path of extraction. It starts high and then declines monotonically over time, finishing at the optimal mine closing date with a zero rate of extraction.

EXERCISES

1. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^T e^{-\rho t}\left(\frac{y^{1-\alpha}}{1-\alpha} - c\right) dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{x} = -y \\ & x(0) = x_0 \\ & x(T) \geq 0 \\ & T \geq 0 \qquad \text{(free)} \end{aligned}$$

where $c > 0$ is a constant fixed cost of extraction. Assume that $0 < \alpha < 1$.

2. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^T e^{-\rho t} \frac{(y-a)^{1-\alpha}}{1-\alpha}\, dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{x} = -y \\ & x(0) = x_0 \\ & x(T) \geq 0 \\ & T \geq 0 \qquad \text{(free)} \end{aligned}$$

where $a > 0$ is a positive constant which can be interpreted as the minimum consumption level required.

3. Solve the following exhaustible resource problem for the optimal path of extraction:

$$\max \int_0^T e^{-\rho t} \frac{y^{1-\alpha}}{1-\alpha}\, dt$$

$$\begin{aligned} \text{subject to} \quad & \dot{x} = -y \\ & x(0) = x_0 \\ & x(T) \geq 0 \\ & T \geq 0 \qquad \text{(free)} \end{aligned}$$

Show that the optimal T is infinity.

CHAPTER REVIEW

Key Concepts

autonomous optimization problem
bang-bang controls
boundary condition
control variable
costate variable
current-valued Hamiltonian
discounting
dynamic optimization
fixed endpoint
free endpoint
free terminal time
functional
Hamiltonian function
infinite time horizon
maximum principle
perturbing path
shadow price
state variable
transversality condition

Review Questions

1. What is the key difference between static and dynamic optimization problems?
2. Why is a new mathematical theory required to solve dynamic optimization problems?
3. Explain the role of the maximum principle in optimal control theory.
4. Explain the role of boundary conditions in optimal control theory.
5. Under what conditions are the necessary conditions in optimal control theory also sufficient for finding the optimal solution?
6. How does the current-valued Hamiltonian differ from the present-valued Hamiltonian and what is the advantage of using it?
7. What is the difference between free-endpoint problems and fixed-endpoint problems?
8. Explain why bang-bang controls arise when the Hamiltonian is a linear function of the control variable.

Review Exercises

1. Consider the following maximization problem:

$$\max \int_0^T F(x, y)\, dt$$

$$\text{subject to} \quad \dot{x} = G(x, y)$$
$$x(0) = x_0$$

Prove that the *present-valued* Hamiltonian is constant over time along the optimal path. That is, prove that $\dot{H} = 0$ when the necessary conditions hold.

2. Consider the following maximization problem:

$$\max \int_0^T e^{-\rho t} F(x, y)\, dt$$

$$\text{subject to} \quad \dot{x} = G(x, y)$$
$$x(0) = x_0$$

Prove that the *current-valued* Hamiltonian rises over time along the optimal path at the rate $\rho\mu G(x, y)$.

3. A firm's production function is K^α, with $0 < \alpha < 1$. The price of output is \$1 and the cost of investment is I^2. Conduct a qualitative analysis using a phase

diagram [in (K, μ) space] of the solution to

$$\max \int_0^T e^{-\rho t}[K^\alpha - I^2]\,dt$$

$$\text{subject to} \quad \dot{K} = I - \delta K$$
$$K(0) = K_0$$

4. Repeat review exercise 3 but this time draw the phase diagram in (K, I) space.

5. Solve the following consumption model for the optimal path of consumption:

$$\max \int_0^T e^{-\rho t} \frac{y^{1-\alpha}}{1-\alpha}\,dt$$

$$\text{subject to} \quad \dot{x} = rx - y$$
$$x(0) = x_0$$
$$x(T) \geq 0$$
$$T \geq 0 \qquad \text{(given)}$$

where $0 < \alpha < 1$.

6. Solve the following exhaustible resource problem as far as you can for the optimal path of extraction. Draw a phase diagram, and show that the steady state is a saddle point and that the optimal T is infinity.

$$\max \int_0^T e^{-\rho t}\left(py - \frac{y^2}{2R} + 2kR^{1/2}\right)dt$$

$$\text{subject to} \quad \dot{R} = -y$$
$$R(0) = R_0 > 0$$
$$R(T) \geq 0$$
$$T \geq 0 \qquad \text{(free)}$$

where p and k are positive constants.

7. Solve the following exhaustible resource problem as far as you can for the optimal path of extraction. Draw a phase diagram, and show that the optimal T is finite and that the steady state is not reached.

$$\max \int_0^T e^{-\rho t}\left[py - \frac{y^2}{2(R+\alpha)}\right]dt$$

$$\text{subject to}\quad \begin{aligned} &\dot{R} = -y \\ &R(0) = R_0 > 0 \\ &R(T) \geq 0 \\ &T \geq 0 \qquad \text{(free)} \end{aligned}$$

where p and α are positive constants.

8. Draw the phase diagram for the following optimal growth model, and show that the saddle paths reach the steady state within a finite time:

$$\max \int_0^T e^{-\rho t} c\, dt$$

$$\text{subject to}\quad \begin{aligned} &\dot{k} = f(k) - c - \delta k \\ &k(0) = k_0 \\ &k(T) = k_T \\ &0 \leq c \leq f(k) \end{aligned}$$

where $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$. Note that consumption, c, is constrained to be nonnegative and cannot exceed the amount produced. Assume that $r > \rho$.

9. Solve the following optimal fishery model:

$$\max \int_0^\infty e^{-\rho t} \left(p - \frac{c}{x}\right) h\, dt$$

$$\text{subject to}\quad \begin{aligned} &\dot{x} = rx(1 - x) - h \\ &x(0) = x_0 \\ &0 \leq h \leq h_{\max} \end{aligned}$$

ANSWERS

The following are brief answers to the odd-numbered questions of all the exercises. Diagrams are excluded except where they are essential to the answer. Fully worked solutions, including diagrams, are contained in the Student's Solution Manual that accompanies this text.

Chapter 2

2.1 Exercises

1. From $A \subset X$ it follows that $x \in X$. However, the reverse is not true.

3. There are 32 possible subsets.

5. Yes, the order of the elements in a set is not important (definition 2.3).

7. **(a)** B is the set of combinations of goods 1 and 2 that the consumer can afford to buy. C is the set of quantities of goods 1 and 2 that the consumer is physically capable of consuming.

(b) $B \cup C$ is the set of quantities of goods 1 and 2 that the consumer can afford to buy or is physically capable of consuming.

(c) $B \cap C$ is the set of the quantities of goods 1 and 2 that the consumer is physically capable of consuming and can afford to buy.

9. P contains all technologically feasible input-output combinations. $\bar{x}$ is the maximum amount of labor that can be employed.

2.2 Exercises

1. **(a)** Z_+ is bounded below by one. There is no upper bound.

(b) Z is unbounded.

(c) $\mathbb{R}_+$ is bounded below by zero. There is no upper bound.

(d) $\mathbb{R}_+$ is bounded above by zero. There is no lower bound.

(e) S is bounded below by zero and bounded above by $\sqrt{2}$.

3. **(a)** $\dfrac{\text{dollars}}{\text{quantity of output}}$

(b) pure number

(c) $\dfrac{\text{dollars}}{\text{quantity of input}}$

(d) $\dfrac{\text{quantity of goods}}{\text{dollars}}$

(e) $\dfrac{\text{dollars}}{\text{quantity of import good}}$

5. Sets with a maximum are $\mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$ and $T = \{x \in \mathbb{R} : x \leq 5\}$. Sets without a maximum are $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $S = \{x \in \mathbb{R} : x < 5\}$.

2.3 Exercises

1. **(a)**
$$\{1, 2, 3, 4, 5, 6\} \otimes \{7, 8, 9\} = \{\{1, 7\}, \{1, 8\}, \{1, 9\}, \{2, 7\}, \{2, 8\}, \{2, 9\}, \{3, 7\}, \{3, 8\}, \{3, 9\}, \{4, 7\}, \{4, 8\}, \{4, 9\}, \{5, 7\}, \{5, 8\}, \{5, 9\}, \{6, 7\}, \{6, 8\}, \{6, 9\}\}$$

(b) $Z_+ \otimes Z_+ = \{(x, y) : x \in Z_+, y \in Z_+\}$

(c) $\left\{(x, y) : x \in Z_+ \text{ and } \dfrac{x}{2} \in Z_+, y \in Z_+ \text{ and } \dfrac{y+1}{2} \in Z_+\right\}$

3. B is closed, bounded, and convex. Interpreting x' and y' as subsistence consumption, the case $X = \emptyset$ signifies that the consumer cannot afford the consumption bundle necessary for survival.

5. **(a)** $d(4, -5) = \sqrt{[4 - (-5)]^2} = 9$

(b) $d[(-6, 2), (8, -1)]$
$= \sqrt{[(-6) - 8]^2 + [2 - (-1)]^2} \doteq 14.32$

(c) $d[(5, -3, 0, 8), (12, -6, 3, 1)]$
$= \sqrt{(5-12)^2 + [(-3) - (-6)]^2 + (0-3)^2 + (8-1)^2}$
$\doteq 10.77$

7. **(a)** $N_\epsilon(-1) = \{x \in \mathbb{R} : \sqrt{(x+1)^2} < \epsilon\}$
For $\epsilon = 0.1$, $N_\epsilon(-1)$ is the open interval $(-1.1, -0.9)$. For $\epsilon = 10$, $N_\epsilon(-1)$ is the open interval $(-11, -9)$.

(b) $N_\epsilon(-1, 1) =$
$\{(x, y) \in \mathbb{R}^2 : \sqrt{(x+1)^2 + (y-1)^2} < \epsilon\}$
$N_\epsilon(-1, 1)$ is the set of points of $\mathbb{R}^2$ lying inside a circle centered on $(-1, 1)$ with radius ϵ.

(c) $N_\epsilon(-1, 1, -1) = \{(x, y, z) \in \mathbb{R}^3 :$
$\sqrt{(x+1)^2 + (y-1)^2 + (z+1)^2} < \epsilon\}$
$N_\epsilon(-1, 1, -1)$ is the set of points of $\mathbb{R}^3$ lying within a sphere centered at $(-1, 1, -1)$ with radius ϵ.

2.4 Exercises

1. **(a)** $y = 1 - 2x$

(b) $y = -8 - 5x$

(c) $y = -\dfrac{1}{2} - \dfrac{3}{2}x$

3. **(a)** $\bar{x} = 4 - 6\lambda, \lambda \in [0, 1]$

(b) $(\bar{x}, \bar{y}) = (3 - 4\lambda, 4 - 3\lambda), \lambda \in [0, 1]$

(c) $(\bar{x}, \bar{y}, \bar{z}) = (1 - 3\lambda, -2 + 2\lambda, 2 - \lambda)$

5. $x = 10 \pm \sqrt{10}$

7. **(a)** strictly quasiconvex, convex

(b) strictly quasiconvex, convex

(c) strictly quasiconcave, concave

9. The function $y = x_1^2 x_2^2$ is strictly quasiconcave but not concave.

11. $y = x^{1/2} = \sqrt{x}$, $x > 0$ is strictly concave if, for any $\lambda \in (0, 1)$,

$$\sqrt{\lambda x' + (1-\lambda)x''} > \lambda\sqrt{x'} + (1-\lambda)\sqrt{x''}$$

which amounts to $(x' - x'')^2 > 0$.

Review Exercises

1. **(a)** $\lambda(-2) + (1-\lambda)2 = 2 - 4\lambda$

(b) $\lambda(-2, 2) + (1-\lambda)(-3, 3) = (-3 + \lambda, 3 - \lambda)$

(c) $\lambda(0, 0) + (1-\lambda)(x_1, x_2) = (1-\lambda)(x_1, x_2)$

(d) $\lambda(-2, 2, 5) + (1-\lambda)(-3, 3, 8) = (-3 + \lambda, 3 - \lambda, 8 - 3\lambda)$

3. **(a)** $y = 22 + 2x$ (b) $y = \dfrac{5}{2} + \dfrac{3}{4}$

5. **(a)** ab^2

(b) $a^{(1-q)}b^q$

(c) $y = 5^8 x^2$

(d) $3x$

7. To show concavity, we show that

$$10 - [\lambda x' + (1-\lambda)x''] > \lambda(10 - x'^2) + (1-\lambda)(10 - x''^2)$$

which amounts to $(x' - x'')^2 > 0$.

9. To show concavity, we show that

$$[\lambda x_1' + (1-\lambda)x_1'' + \lambda x_2' + (1-\lambda)x_2'']^{1/2} > \lambda(x_1' + x_2')^{1/2} + (1-\lambda)(x_1'' + x_2'')^{1/2}$$

which amounts to

$$[(x_1' + x_2'')^{1/2} - (x_1'' x_2'')^{1/2}]^2 > 0$$

Chapter 3

3.1 Exercises

1. **(a)** 6, 5.5, 5.33, 5.25, 5.20, 5.17, 5.14, 5.125, 5.11, 5.1.

(b) The first 10 terms of the sequence $f(n) = 5n/(2^n)$ are

2.5, 2.5, 1.875, 1.25, 0.78, 0.47, 0.27, 0.16, 0.088, 0.049.

(c) 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

3. The sequence $\hat{f}(n) = 2(n-1),\ n = 1, 2, 3, \ldots$ is identical to the sequence $f(n) = 2n,\ n = 0, 1, 2, \ldots$. (Check the first five or so terms of each.)

5. The sequence $\hat{f}(n) = (1+r)^{n+25},\ n = 1, 2, 3, \ldots$ is identical to the sequence $f(n) = (1+r)^n$, $n = 1, 2, 3, \ldots$, starting with the 26th term; so $f(n) = (1+r)^n, n = 26, 27, 28, \ldots$

3.2 Exercises

1. (a) We need to show that for any $\epsilon > 0$ there must be some value N such that

$$\left|\frac{n}{n+1} - 1\right| < \epsilon$$

for every $n > N$. That is,

$$\frac{1}{n+1} < \epsilon \Rightarrow \frac{1}{\epsilon} < n+1$$

which holds for any $n > (1/\epsilon) - 1$. Thus we can satisfy the condition $|[n/(n+1)] - 1| < \epsilon$ for any $n > N$ by choosing N to be the next integer greater than the number $(1/\epsilon) - 1$.

(b) We need to show that for any $\epsilon > 0$ there must be some value N such that

$$\left|5 + \frac{1}{n} - 5\right| < \epsilon$$

for every $n > N$. That is,

$$\frac{1}{n} < \epsilon \Rightarrow n > \frac{1}{\epsilon}$$

Thus, we can satisfy the condition $|5 + (1/n) - 5| < \epsilon$ for any $n > N$ by choosing N to be the next integer greater than the number $1/\epsilon$.

(c) We need to show that for any $\epsilon > 0$ there must be some value N such that

$$\left|\left(-\frac{1}{2}\right)^n - 0\right| < \epsilon$$

for every $n > N$. That is,

$$\frac{1}{2^n} < \epsilon \Rightarrow 2^n > \frac{1}{\epsilon} \Rightarrow n > \log_2\left(\frac{1}{\epsilon}\right)$$

Thus we can satisfy the condition $|(-1/2)^n - 0| < \epsilon$ for any $n > N$ by choosing N to be the next integer greater than the number $\log_2(1/\epsilon)$.

3. (a)

$$\lim_{n\to\infty} n^2 = \infty$$

Notice that for any value K it is always possible to find an N large enough that

$$n^2 > K$$

for every $n > N$. Choosing N to be the next integer greater than the number $\sqrt{K}$ will satisfy this condition.

(b)

$$\lim_{n\to\infty} (-n)^3 = -\infty$$

Notice that for any value K it is always possible to find an N large enough that

$$(-n)^3 < -K$$

for every $n > N$. Since $(-n)^3 = -n^3$, we can see that, upon multiplying by -1, the inequality above becomes

$$n^3 > K$$

and so choosing N to be the next integer greater than the number $K^{1/3}$ will satisfy this condition.

(c) The sequence $(-c)^n$ is divergent and is not definitely divergent if $|c| > 1$. If $|c| < 1$, then

$$\lim_{n\to\infty} (-c)^n = 0$$

If $c > 0$, then $(-c)^n > 0$ for n even and <0 for n odd. If $c < 0$, then $(-c)^n > 0$ for n even and <0 for

n odd. Therefore, in the case where $|c| > 1$, for any N the sequence $(-c)^n$ will contain both arbitrarily large positive and negative values. (This follows from the fact $|(-c)^n| = |c|^n$ and $\lim_{n\to\infty} |c|^n = +\infty$ for $|c| > 1$.) Therefore the sequence would be divergent and not definitely divergent. If $|c| < 1$, then we can show the sequence converges to the limit 0 in the same way as we did for question 1 (c). That is, we need to show that for any $\epsilon > 0$ there must be some value N such that

$$|(-c)^n - 0| < \epsilon$$

for every $n > N$. That is,

$$|(-c)^n| < \epsilon$$

or

$$|c|^n < \epsilon$$

If we take logs to the base $b = |c| < 1$ of each side we get $n \log_b b > \log_b \epsilon$, namely $n > \log_b \epsilon$ since $\log_b b = 1$. Thus we can satisfy the condition $|(-c)^n - 0| < \epsilon$ for any $n > N$ by choosing N to be the next integer greater than $\log_b \epsilon$.

3.3 Exercises

1. 71.18

3. **(a)** If interest is compounded annually and the present value is the same in each case then

$$\frac{V_2}{(1+r)^{t_2}} = \frac{V_1}{(1+r)^{t_1}} \Rightarrow \frac{V_2}{V_1} = \frac{(1+r)^{t_2}}{(1+r)^{t_1}}$$

Now, since $t_2 > t_1$, we have $(1+r)^{t_2} > (1+r)^{t_1}$, and so

$$\frac{(1+r)^{t_2}}{(1+r)^{t_1}} > 1 \Rightarrow \frac{V_2}{V_1} > 1 \Rightarrow V_2 > V_1$$

(b) We need to show that

$$\frac{V_2}{(1+r)^{t_2+k}} = \frac{V_1}{(1+r)^{t_1+k}}$$

Rewriting gives

$$\frac{V_2}{(1+r)^{t_2}(1+r)^k} = \frac{V_1}{(1+r)^{t_1}(1+r)^k}$$
$$\Rightarrow \frac{V_2}{(1+r)^{t_2}} = \frac{V_1}{(1+r)^{t_1}}$$

5. **(a)** $Z_5 = 100e^{0.02(5)} = 100e^{0.10} = 110.52$ million

(b) $Z_{10} = 100e^{0.02(10)} = 100e^{0.20} = 122.14$ million

(c) $Z_{20} = 100e^{0.02(20)} = 100e^{0.40} = 149.18$ million

3.4 Exercises

1. From theorem 3.3 we know that a monotonic sequence is convergent if and only if it is bounded. Letting $a_t = V/(1+r)^t$ represent a general term in the sequence we first show it is a monotonic sequence.

$$\frac{a_{t+1}}{a_t} = \frac{V/(1+r)^{t+1}}{V/(1+r)^t} = \frac{(1+r)^t}{(1+r)^{t+1}} = \frac{1}{1+r}$$

and so

$$a_{t+1} = \frac{a_t}{1+r}$$

Therefore the sequence is monotonically decreasing if $r > 0$, while it is monotonically increasing if $-1 < r < 0$. In the case with $r > 0$, $a_t = V/(1+r)^t$ is bounded below by 0 and above by V, and so the sequence is convergent. In the case with $-1 < r < 0$, we have $0 < 1 + r < 1$ and $\lim_{t\to\infty} V/(1+r)^t = +\infty$; that is, the sequence is not bounded and so it is not convergent.

3. Steps of the proof are: (i) For any (large) $\bar{K}$, there exists an N_1, such that $b_n > K$ for $n > N_1$. (ii) For any $\epsilon > 0$ there exists an N_2 such that $|a_n - L^a| < \epsilon$ for $n > N_2$. (iii) Show that for any $K > 0$ there is an N large enough that $a_n - b_n < -K$ for $n > N$. (Appropriate choice for N is $N = \max\{N_1, N_2\}$.)

3.5 Exercises

1.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c}{c}\right| = 1$$

and

$$S_n = \sum_{t=1}^{n} a_t = a_1 + a_2 + \cdots + a_n$$
$$= c + c + \cdots + c = nc$$

Since $c > 0$ it is clear that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} nc = +\infty$. That is, the series diverges.

3. $$s_n = a\rho + a\rho^3 + a\rho^5 + \cdots + a\rho^{2n-3} + a\rho^{2n-1}$$

Multiplying this expression by ρ^2 gives

$$\rho^2 s_n = a\rho^3 + a\rho^5 + a\rho^7 + \cdots + a\rho^{2n-1} + a\rho^{2n+1}$$

and so

$$s_n - \rho^2 s_n = a\rho - a\rho^{2n+1}$$

which implies that

$$(1-\rho^2)s_n = a\rho - a\rho^{2n+1}$$

and so

$$s_n = \frac{a\rho - a\rho^{2n+1}}{1-\rho^2}$$

which implies, for $|\rho| < 1$, that

$$\lim_{n\to\infty} s_n = \frac{a\rho}{1-\rho^2}$$

5. **(a)** PV of net operating revenue = \$5,787,037. PV of building costs = \$10 million. Therefore the present value of building costs exceeds the present value of operating revenues and so this project is not profitable.

(b) $r = 0.048$

Review Exercises

1. **(a)** 1, 0.25, 0.111, 0.063, 0.04.

(b) 1, 2.5, 1.67, 2.25, 1.8.

(c) 0.2, 0.25, 0.273, 0.286, 0.294.

(d) −1, −4, −9, −16, −25.

(e) 2, 3, 4, 5, 6.

(f) 6, 5.5, 5.33, 5.25, 5.2.

(g) 4, 4.5, 4.67, 4.75, 4.8.

3. **(a)** **(i)** $PV_1 = \frac{100}{1.08} = \$92.59$

(ii) $PV_5 = \frac{150}{(1.08)^5} = \$102.09$

(b) The individual should rank alternative (ii) as better. Even if the individual wants the money well in advance of 5 years, she should rank alternative (ii) as the better one because at $r = 0.08$ (8%) she could borrow more money now and pay it back after 5 years with the \$150 received at that point in time than she could from receiving the \$100 in one year's time.

5. **(a)** \$1,000

(b) \$907.70

(c) \$92.30

7. **(a)** The NPV of additional income is 23,148, and since this exceeds the costs she should accept the offer.

(b) The internal rate of return is $\hat{r} = 0.092$ or 9.2%. If the interest rate is less than this she should accept the offer.

(c) No.

(d) Forgone income is not an issue for anyone who becomes unemployed.

Chapter 4

4.1 Exercises

1. **(a)** $f(x_n) = 10 - 5/n$, $n = 1, 2, 3, \ldots$. This suggests that

$$\lim_{x\to 2^-} f(x) = 10$$

(b) $f(x_n) = -2 + 3/n$, $n = 1, 2, 3, \ldots$. This suggests that

$$\lim_{x\to 2^-} f(x) = -2$$

(c) $f(x_n) = 2m + b - m/n$, $n = 1, 2, 3, \ldots$. This suggests that

$$\lim_{x \to 2^-} f(x) = 2m + b$$

(d) $f(x_n) = 4 - \frac{4}{n} + \frac{1}{n^2}$, $n = 1, 2, 3, \ldots$. This suggests that

$$\lim_{x \to 2^-} f(x) = 4$$

3. **(a)** This function is not continuous at the point $x = 1$ because the left-hand limit is not equal to the right-hand limit:

$$\lim_{x \to 1^-} f(x) = 5, \quad \lim_{x \to 1^+} f(x) = 6$$

Therefore the second part of condition (i) of definition 4.3 is not satisfied at $x = 1$.

(b) This function is not continuous at the point $x = 0$ because it is not defined there. Moreover the left-hand limit is not equal to the right-hand limit:

$$\lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to 0^+} f(x) = +\infty$$

Therefore, neither part of condition (i) of definition 4.3 is satisfied at $x = 0$.

(c) This function is not continuous at the point $x = 3$ because it is not defined there. However, the left-hand limit is equal to the right-hand limit:

$$\lim_{x \to 3^-} f(x) = +\infty, \quad \lim_{x \to 3^+} f(x) = +\infty$$

Therefore, the first part of condition (i) of definition 4.3 is not satisfied at $x = 3$.

(d) This function is not continuous at the point $x = 2$ because it is not defined there. However, the left-hand limit is equal to the right-hand limit:

$$\lim_{x \to 2^-} f(x) = \frac{1}{3}, \quad \lim_{x \to 2^+} f(x) = \frac{1}{3}$$

Therefore the first part of condition (i) of definition 4.3 is not satisfied at $x = 2$. At $x = -1$, we have

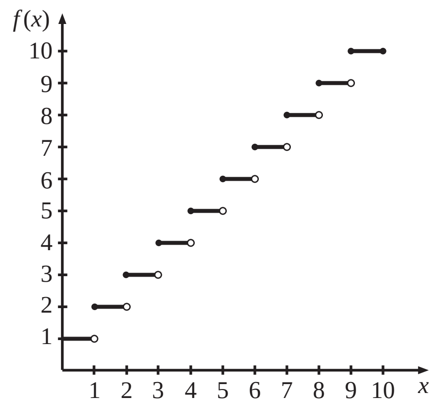
$$\lim_{x \to 1^-} = -\infty, \quad \lim_{x \to 1^+} = +\infty$$

so neither part of (i) of definition 4.3 is satisfied at $x = -1$.

5. **(a)** $f(x) = 4x + 3$ is continuous at every point $x \in \mathbb{R}$. (Choose $\delta = \epsilon/4$ in applying definition 4.4.)

(b) $f(x) = mx + b$ is continuous at every point $x \in \mathbb{R}$. (Choose $\delta = \epsilon/m$ in applying definition 4.4.)

7. This function is defined at every point $x \in [0, 10]$ but is not continuous at the points $x = 1, 2, \ldots, 9$, since at these points the left-hand and right-hand limits are not equal. However, within each subinterval, the function is continuous.



4.2 Exercises

1. $C(Q) = c_0 + (w/b)Q$ so $\pi(Q) = \bar{p}Q - c_0 - (w/b)Q$. The production function is continuous, and so from (i), (ii), and (vi) of theorem 4.1 the cost function is continuous. The revenue function is also continuous, and so from (ii) of theorem 4.1 the difference between two continuous functions (the profit function) is also continuous.

3.

$$P(S) = \begin{cases} \$800 + 0.15S & \text{if } S \le \$10{,}000 \\ \$1{,}800 + 0.15S & \text{if } \$10{,}000 < S \le \$15{,}000 \\ \$4{,}300 + 0.15S & \text{if } S > \$15{,}000 \end{cases}$$

At $S = \$10{,}000$ the left-hand and right-hand limits are:

$$\lim_{S \to 10{,}000^-} P(S) = 2{,}300, \qquad \lim_{S \to 10{,}000^+} P(S) = 3{,}300$$

so the function is not continuous at the point $S = \$10{,}000$. Similarly at $S = \$15{,}000$.

5. Let x be income before tax and $y = f(x)$ be income after tax.

$$f(x) = \begin{cases} x & x \leq \$20{,}000 \\ 5{,}000 + 0.75x & \$20{,}000 < x < \$60{,}000 \\ 4{,}000 + 0.75x & x \geq \$60{,}000 \end{cases}$$

At $x = 60{,}000$, the left-hand limit is 50,000 and the right-hand limit is 49,000, so the function is not continuous at this point.

7. **(a)** The marginal product of labor is:

$$\text{MP}(h) = \begin{cases} 1/20 & 0 \leq h \leq 336{,}000 \\ 0 & h > 336{,}000 \end{cases}$$

The left-hand limit at 336,000 is $1/20$ and the right-hand limit is 0, so the function is discontinuous at this point.

(b)

$$\pi(y) = \begin{cases} 1{,}000 - 600 = 400 & 0 \leq y \leq 12{,}000 \\ 1{,}000 - 1{,}200 = -200 & 12{,}000 < y \leq 16{,}800 \end{cases}$$

There is a discontinuity at $y = 12{,}000$.

9. If firm 2 sets a price $p_2 = 5$, then firm 1 captures the entire market, $y_1 = 20 - p_1$, provided that it charges a price less than 5 (i.e., $p_1 < \$5$). This being the case, its revenue would be

$$R_1(p_1) = p_1 y_1 = p_1(20 - p_1). \qquad p_1 < 5$$

If firm 1 charges the same price as firm 2 (i.e., $p_1 = \$5$), then it will share the market equally with firm 2. Joint sales will be $y = 20 - p = 20 - 5 = 15$, and so each sells 7.5 units at a price of \$5 and so

$$R_1(p_1) = 5 \times 7.5 = 37.5 \qquad p_1 = 5$$

If firm 1 charges a price greater than 5 its sales, and hence revenue, become zero:

$$R_1(p_1) = 0, \qquad p_1 > 5$$

Putting these expressions together gives the firm's revenue function

$$R_1(p_1) = \begin{cases} p_1(20 - p_1), & p_1 < 5 \\ 37.5, & p_1 = 5 \\ 0, & p_1 > 5 \end{cases}$$

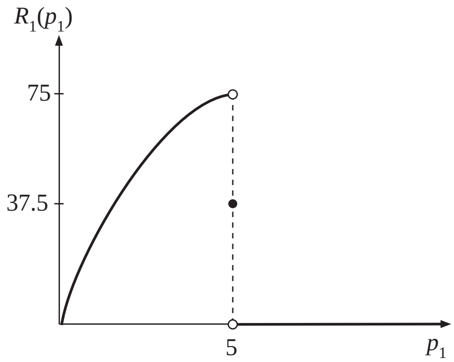


Figure 4.2.10 *(a)i*

To determine the profit function, we simply subtract costs from revenues, which in the case of $p_1 < 5$ is $C(y_1) = 2y_1$, and so since $y_1 = 20 - p_1$ for $p_1 < 5$, we get

$$C_1(p_1) = 2(20 - p_1), \qquad p_1 < 5$$

At $p_1 = 5$, firm 1 produces 7.5 units and so incurs costs of

$$C_1(p_1) = 2 \times 7.5 = 15, \qquad p_1 = 5$$

At $p_1 > 5$, firm 1 produces no output and so incurs no cost. Then

$$C_1(p_1) = 0, \qquad p_1 > 5$$

Putting these cost functions together with the revenue function gives us the following profit function ($\pi_1(p_1) = R_1(p_1) - C_1(p_1)$):

$$\pi_1(p_1) = \begin{cases} p_1(20 - p_1) - 2(20 - p_1), & p_1 < 5 \\ 37.5 - 15 = 22.5, & p_1 = 5 \\ 0, & p_1 > 5 \end{cases}$$

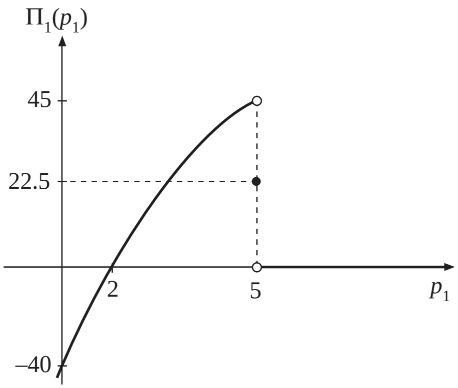


Figure 4.2.10 *(a)ii*

The function $R_1(p_1)$ and $\pi(p_1)$ are discontinuous at the point $p_1 = 5$. For $p_1 < 5$, firm 1 is essentially a monopolist since firm 2 sells zero output when $p_1 < p_2 = 5$. Marginal increases in p_1, when $p_1 < 5$, have marginal effects on revenue and profit since these functions are continuous. However, as soon as p_1 reaches the value 5, firm 2 captures half of the market and so firm 1 loses half of the market. The result is that revenue and profit for firm 1 drop discontinuously at this price. If firm 1 charges a price even marginally exceeding \$5, it loses completely its market share to firm 2, and so revenue and profit drop to zero.

(b) For any price, $\bar{p}_2$, charged by firm 2, where $\bar{p}_2 > 2$, the same pattern applies with respect to $R_1(p_1)$ and $\pi_1(p_1)$ as was the case in part (a). If $p_1 < \bar{p}_2$, 1 captures the entire market, and so its revenue and profit functions are based on its demand being the market demand. However, as soon as firm 1 raises its price to the level $p_1 = \bar{p}_2$, it shares equally the market with firm 2, and so firm 1's revenue and profit drop discontinuously to half their values relative to a price marginally less than $\bar{p}_2$. If firm 1 charges a price $p_1 > \bar{p}_2$, then its revenue and profit drop to zero. Therefore the revenue and profit functions in this case are as follows:

$$R_1(p_1) = \begin{cases} p_1(20 - p_1), & p_1 < \bar{p}_2 \\ \frac{1}{2}p_1(20 - p_1), & p_1 = \bar{p}_2 \\ 0, & p_1 > \bar{p}_2 \end{cases}$$

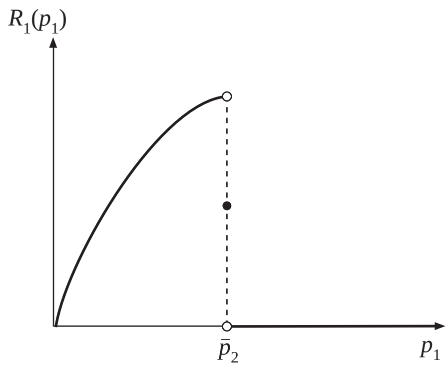


Figure 4.2.10 *(b)i*

$$\pi_1(p_1) = \begin{cases} p_1(20 - p_1) - 2(20 - p_1), & p_1 < \bar{p}_2 \\ \frac{1}{2}[p_1(20 - p_1) - 2(20 - p_1)], & p_1 = \bar{p}_2 \\ 0, & p_1 > \bar{p}_2 \end{cases}$$

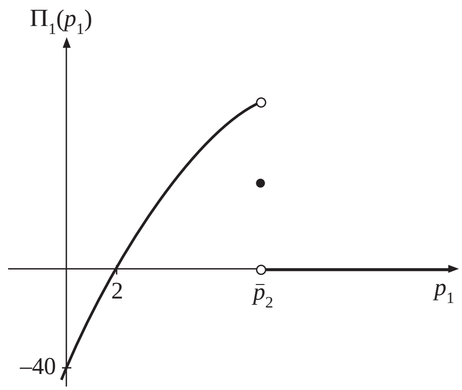


Figure 4.2.10 *(b)ii*

(c) Firm 2 can infer that if it charges a price in excess of \$2, then firm 1 will undercut in order to capture the entire market. (From part (b) we can see that this is indeed the optimal response for firm 1 if $p_2 > 2$.) Of course, the reverse argument also holds. Firm 1 can infer that if it charges a price in excess of \$2, then firm 2's optimal response would be to undercut and capture the entire market. Thus either firm will be shut out of the market if it charges in excess of \$2. The only equilibrium outcome is for each firm to charge \$2 and share the market.

Review Exercises

1. **(a)** At $x = 5$

$$\lim_{x\to 5^-} f(x) = 10, \quad \lim_{x\to 5^+} f(x) = 11$$

so this function is not continuous at $x = 5$

(b) At $x = 1$

$$\lim_{x\to 1^-} f(x) = -\infty, \quad \lim_{x\to 1^+} f(x) = +\infty$$

Moreover, the function is not defined at $x = 1$.

3.

$$y = \begin{cases} x & x \le 25{,}000 \\ 35{,}000 - 0.4x & 25{,}000 < x < 100{,}000 \\ 33{,}000 - 0.4x & x \ge 100{,}000 \end{cases}$$

There is a discontinuity at $x = 100{,}000$ due to the surtax.

5. **(a)** \$1,500

(b) zero

(c) \$800

(d) **(i)** Total cost of taking y passengers is

$$C(y) = \begin{cases} 0 & y = 0 \\ 1{,}500 & 0 < y \le 50 \\ 2{,}300 & 50 < y \le 100 \\ 2{,}900 & 100 < y \le 150 \\ 3{,}100 & 150 < y \le 200 \\ 3{,}900 & 200 < y \le 250 \\ 5{,}100 & 250 < y \le 300 \\ 8{,}500 & 300 < y \le 350 \end{cases}$$

(ii) Average cost is

$$AC(y) = \begin{cases} \text{undefined} & y = 0 \\ 1{,}500/y & 0 < y \le 50 \\ 2{,}300/y & 50 < y \le 100 \\ 2{,}900/y & 100 < y \le 150 \\ 3{,}100/y & 150 < y \le 200 \\ 3{,}900/y & 200 < y \le 250 \\ 5{,}100/y & 250 < y \le 300 \\ 8{,}500/y & 300 < y \le 350 \end{cases}$$

The function has a discontinuity at every integer multiple of 50 passengers.

Chapter 5

5.1 Exercises

1.

Q_i	(25, 625)	(24, 576)	(23, 529)
Δx	5	4	3
Δy	225	176	129
$\Delta y/\Delta x$	45	44	43

Q_i	(22, 484)	(21, 441)
Δx	2	1
Δy	84	41
$\Delta y/\Delta x$	42	41

Yes, the sequence of values looks like it will converge. (Use a graph like figure 5.3 to illustrate.)

3. $y = 6x - 9$. Use a graph such as figure 5.5 to illustrate.

5.2 Exercises

1. **(a)** $f'(x) = \lim_{\Delta x\to 0} \dfrac{[3(x+\Delta x) - 5] - [3x - 5]}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{3\Delta x}{\Delta x} = 3$

(b) $f'(x) = \lim_{\Delta x\to 0} \dfrac{8(x+\Delta x) - 8x}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{8\Delta x}{\Delta x} = 8$

(c) $f'(x) = \lim_{\Delta x\to 0} \dfrac{3(x+\Delta x)^2 - 3x^2}{\Delta x} = \lim_{\Delta x\to 0} \dfrac{6x\Delta x + 3\Delta x^2}{\Delta x} = 6x$

3.

Q_i	(25, 625)	(24, 576)	(23, 529)
Δx	5	4	3
Δy	225	176	129
$dy = f'(x)\,dx$	250	192	138
ϵ	−11.1%	−9.1%	−7.0%

Q_i	(22, 484)	(21, 441)
Δx	2	1
Δy	84	41
$dy = f'(x)\,dx$	88	42
ϵ	−4.8%	−2.4%

This suggests that as Δx or dx gets *smaller* the value of the total differential dy becomes a better approximation to the actual change Δy.

5.3 Exercises

1. **(a)** $\lim_{\Delta x \to 0^-} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^-} \dfrac{[3(x+\Delta x)+2] - [3x+2]}{\Delta x} = 3$

$\lim_{\Delta x \to 0^+} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^+} \dfrac{[(x+\Delta x)+12] - [x+12]}{\Delta x} = 1$

Since the left-hand and right-hand limits are not equal at $x = 5$, the function is not differentiable at this point.

(b) $\lim_{\Delta x \to 0^-} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^-} \dfrac{-(x+\Delta x) - x}{\Delta x} = -1$

$\lim_{\Delta x \to 0^+} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^+} \dfrac{(x+\Delta x) - x}{\Delta x} = 1$

Since the left-hand and right-hand limits are not equal at $x = 0$, the function is not differentiable at this point.

(c) $\lim_{\Delta x \to 0^-} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^-} \dfrac{[4(x+\Delta x)+1] - [4x+1]}{\Delta x} = 4$

$\lim_{\Delta x \to 0^+} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} =$

$\lim_{\Delta x \to 0^+} \dfrac{[11-(x+\Delta x)] - [11-x]}{\Delta x} = -1$

Since the left-hand and right-hand limits are not equal at $x = 2$, the function is not differentiable at this point.

3. **(a)**

$$T(y) = \begin{cases} 0 & 0 \le y \le 6{,}000 \\ 0.20y - 1{,}200 & 6{,}000 < y \le 16{,}000 \\ 0.30y - 2{,}800 & 16{,}000 < y \le 46{,}000 \\ 0.40y - 7{,}400 & y > 46{,}000 \end{cases}$$

(b) The points of nondifferentiability are at $y = 6{,}000$, $y = 16{,}000$, $y = 46{,}000$

(c)

$$AT(y) = \begin{cases} 0 & 0 \le y \le 6{,}000 \\ 0.20 - 1{,}200/y & 6{,}000 < y \le 16{,}000 \\ 0.30 - 2{,}800/y & 16{,}000 < y \le 46{,}000 \\ 0.40 - 7{,}400/y & y > 46{,}000 \end{cases}$$

5. **(a)**

$$P(S) = \begin{cases} 600 & S = 0 \\ 600 + 0.1S & 0 < S \le 10{,}000 \\ -400 + 0.2S & S \ge 10{,}000 \end{cases}$$

(b) There is a single point of nondifferentiability at $S = 10{,}000$.

5.4 Exercises

1. **(a)** $f(L) = 10L$, $f'(L) = 10$, $f''(L) = 0$, so the rate at which output rises with respect to more input being used does not change.

(b) $f(L) = 8L^{1/3}$, $f'(L) = (8/3)L^{-2/3}$, $f''(L) = (-16/9)L^{-5/3} < 0$, so the rate at which output rises with respect to more input being used is falling.

(c) $f(L) = 3L^4$, $f'(L) = 12L^3$, $f''(L) = 36L^2 > 0$, so the rate at which output rises with respect to more input being used is increasing.

3. **(a)** $q_S^A = 0.25p - 2.5$ is firm A's supply curve with $q_S^A \ge 0 \Leftrightarrow p \ge 10$. $q_S^B = 0.5p - 7.5$ is firm B's supply curve with $q_S^B \ge 0 \Leftrightarrow p \ge 15$. These functions are differentiable on every point in their domains.

(b)

$$q = \begin{cases} 0 & p < 10 \\ 0.25p - 2.5 & 10 \le p < 15 \\ 0.75p - 10 & p \ge 15 \end{cases}$$

The total supply function is differentiable on the range of prices $0 \le p < 15$. However, it is nondifferentiable at $p = 15$.

5. $L(q) = (1/4)q^{1/2}$ and so the cost function is $C(q) = c_0 + wL(q) = c_0 + (1/4)wq^{1/2}$. Now $dq/dL = 32L$ which is increasing in L and $dC/dq = (1/8)/q^{1/2}$ which is decreasing in q.

7. **(a)** $\varepsilon = (100 - y)/y$

(b) $\varepsilon = (1200 - 12y)/12y = (100 - y)/y$

5.5 Exercises

1. $f'(x) = 4x^3$ and $f''(x) = 12x^2$ for every $x \in R$ and $f''(x) = 0$ only at $x = 0$, so f is strictly convex.

3. $C(y) = c_0 + ry^3$, so $C''(y) = 6ry > 0$ and so $C(y)$ is strictly convex, and $y = x^{1/3}$ is strictly concave since $d^2y/dx^2 = (-2/9)x^{-5/3} < 0$.

5. $\pi'(y) = -15 - 3y^2 + 18y$ and $\pi''(y) = -6y + 18$, so $\pi''(y) > 0$ ($\pi(y)$ strictly convex) for $y < 3$ and $\pi''(y) < 0$ ($\pi(y)$ strictly concave) for $y > 3$.

y	$\pi(y)$	$\pi'(y)$	$\pi''(y)$
0	−10	−15	18
1	−17	0	12
2	−12	9	6
3	−1	12	0
4	10	9	−6
5	15	0	−12
6	8	−15	−18
7	−17	−36	−24
8	−66	−63	−30
9	−145	−96	−36
10	−260	−135	−42

5.6 Exercises

1.
$$e^{-x} = -1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!} + R_n$$

where $|R_n| = \xi^n/n!$ and ξ between 0 and x. To be correct within 0.001 we need n to be large enough that $|R_n| \le 0.001$, or $n! \ge 1{,}000$, or $n = 7$.

3.
$$\epsilon = -\frac{(x - x_0)^2}{8\xi^{3/2}} \le 0$$

and so $dy \ge \Delta y$ and using the differential leads to an overestimate.

Review Exercises

1.
$$f'(x) = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + 3(x + \Delta x) - 4] - [x^2 + 3x - 4]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + x^2 + 3\Delta x}{\Delta x} = 2x + 3$$

3. **(a)** $f(L) = 64L^{1/4} \Rightarrow f'(L) = 16L^{-3/4} \Rightarrow f''(L) = -12L^{-7/4} < 0$

(b) $f(L) = 10L + 2L^{1/2} \Rightarrow f'(L) = 10 + L^{-1/2} \Rightarrow f''(L) = -(1/2)L^{-3/2} < 0$

(c) $f(L) = 5L^3 \Rightarrow f'(L) = 15L^2 \Rightarrow f''(L) = 30L > 0$

(d) $f(L) = -L^3 + 12L^2 + 3L \Rightarrow f'(L) = -3L^2 + 24L + 3 \Rightarrow f''(L) = -6L + 24$, so

$$f''(L) = 24 - 6L \begin{cases} > 0 & L < 4 \\ = 0 & L = 4 \\ < 0 & L > 4 \end{cases}$$

5. $\epsilon = (-)(-5)p/y = 5p/y = 5p/(200 - 5p)$ so $\epsilon < 1$ for $p < 20$, $\epsilon = 1$ for $p = 20$, and $\epsilon > 1$ for $p > 20$.

7. $L(q) = (q/a)^{1/b}$ and so $C(q) = c_0 + w(q/a)^{1/b}$.

$$\frac{dq}{dL} = \frac{ab}{L^{1-b}} \quad \text{and} \quad \frac{dC}{dq} = \frac{(w/a^{1/b})(1/b)}{q^{1-(1/b)}}$$

If $b < 1$, then dq/dL is decreasing in L and dC/dq is increasing in q. If $b > 1$, then dq/dL is increasing in L and dC/dq is decreasing in q. If $b = 1$ then dq/dL and dC/dq are neither increasing nor decreasing.

9. $C'(y) = 3y^2 - 24y + 50$ and $C''(y) = 6y - 24$, so $C''(y) > 0$ ($C(y)$ is strictly convex) for $y > 4$ and $C''(y) < 0$ ($C(y)$ is strictly concave) for $y < 4$.

y	$C(y)$	$C'(y)$	$C''(y)$
0	20	50	−24
1	59	29	−18
2	80	14	−12
3	89	5	−6
4	92	2	0
5	95	5	6
6	104	14	12
7	125	29	18
8	164	50	24
9	227	77	30
10	320	110	36

Chapter 6

6.1 Exercises

1. **(a)** $x_1 = 0$ (local maximum), $x_2 = 2$ (local minimum)

(b) $x_1 = 0$ (local minimum), $x_2 = 1.5$ (point of inflection)

(c) $x_1 = \sqrt{1/3}$ (local minimum), $x_2 = -\sqrt{1/3}$ (local maximum)

(d) $x_1 = 0$ (local minimum), $x_2 = 0.5$ (local maximum), $x_3 = 2$ (local minimum)

(e) $x_1 = 1$ (local maximum), $x_2 = -1$ (local minimum)

3. $x(p) = 12.5p - 37.5$ for $p \geq \$3$, $x(p) = 0$ for $p < \$3$.

5. If the (inverse) linear demand function is $p(x) = a - bx$, the sales-maximizing output is $x_{\text{sm}} = a/2b$, while the output for zero price is $x_{\text{zp}} = a/b$.

6.2 Exercises

1. **(a)** $f''(x) = 6x - 6$, $f''(0) = -6$, $f''(2) = 6$.

(b) $f''(x) = 12x^2 - 24x + 9$, $f''(0) = 9$, $f''(1.5) = 0$.

(c) $f''(x) = 18x$, $f''(\sqrt{1/3}) = 10.44$, $f''(-\sqrt{1/3}) = -10.44$.

(d) $f''(x) = 36x^2 - 60x + 12$, $f''(0) = 12$, $f''(0.5) = -9$, $f''(2) = 36$.

(e) $f''(x) = \dfrac{4x^3 - 12x}{(x^2+1)^2}$, $f''(1) = -8$, $f''(-1) = 8$

3. $b \leq a^2/80$ and the second derivative of the profit function cannot be positive at the profit maximizing output.

5. **(a)** The inverse of the production function is $L(x)$ and the cost function is $C(x) = wL(x)$, where w is the (constant) wage rate, so setting price equal to marginal cost implies $p = w dL/dx$ or $p(dx/dL) = w$ where the left-hand side is the marginal value product (price multiplied by marginal product).

(b) Similar to (a) except price is replaced by marginal revenue so $p(x) + p'(x)x = w dL/dx$.

(c) The production function must be strictly concave.

6.3 Exercises

1. **(a)** $x = 10$ with $y = 23$

(b) $x = 20$ with $y = 4001$

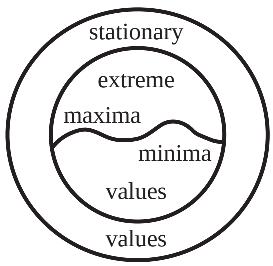
(c) $x = 10$ with $y = -95$
In all cases the first derivatives are not equal to zero at the optimum.

3. **(a)** 0.8

(b) 80

Review Exercises

1.



3. **(a)** $x = 2$ (local minimum)

(b) $x_1 = 0$ (local maximum), $x_2 = 2$ (local minimum)

(c) $x_1 = -1$ (local minimum), $x_2 = 2$ (point of inflection)

(d) $x_1 = -1$ (local maximum), $x_2 = 1$ (local minimum)

(e) $x_1 = -1$ (local maximum), $x_2 = 1$ (local minimum)

(f) $x_1 = 0$ (local minimum), $x_2 = 1/2$ (local maximum), $x_3 = 2$ (local minimum)

(g) $x = 0$ (local maximum)

(h) $x = 0$ (local maximum)

(i) $x_1 = 0$ (local maximum), $x_2 = 4$ (local minimum)

(j) $x_1 = 5$ (local minimum)

5. For $p = \$1$ and $p = \$2$, the firm should produce zero output.

7. The largest bid of the firm is now \$9.5625.

9. $t_1 = 100/9$, $t_2 = 440/9$.

Chapter 7

7.1 Exercises

1. **(a)** $x = 2, y = 4$

(b) Infinitely many solutions. Both equations are equivalent.

(c) No solution. Lines are parallel.

(d) $x = 10/3, y = 20/3$.

3. **(a)** These are parallel if $c = 4$.

(b) They have a solution for any other value of c.

5. $\beta_{21} > 0$ implies that an increase in the price of good 1 increases the supply of good 2.

7. **(a)** $M = 25, R = 10$.

(b) Equations are inconsistent.

7.2 Exercises

1. **(a)** $x = 5, y = -3/5, z = 13/5$.

(b) Linearly dependent: equation 1 = (equation 2 + equation 3) × 2.

(c) $x_1 = 11/2, x_2 = -3/2, x_3 = -7/2$.

(d) $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = -2$.

(e) Linearly dependent: equation 1 = equation 3 + equation 4.

3. **(a)** $x_1 = 4/3, x_2 = 8/3, x_3 = 80/3, x_4 = 106/3$.

(b) $x_1 = -50/3, x_2 = 20, x_3 = -10/3$.

5. $p_1 = 8, p_2 = 5, p_3 = 2, p_4 = 1$.

Review Exercises

1. (d) and (e).

3. (a) Inconsistent. (b) $x = y = z = 0$ is the only solution.

5. $M = 15$.

Chapter 8

8.1 Exercises

1. $x = 2, y = 2$

3. Any x and y for which $x = y$

5. There are no values of y and z that will make these matrices equal.

8.2 Exercises

1.

$$3A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3. $\mathbf{ab} = -1$,

$$\mathbf{ba} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

5. Profit is 94,000

8.3 Exercises

1. **(a)** The transpose of I_3 is I_3 itself.

(b) The transpose of A is A itself.

3. **(a)** $(AB)^T = B^T A^T = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$

(b) $(AB)^T = B^T A^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$

5. B is 5×7.

7. B has 2 rows.

8.4 Exercises

1. $I_3 I_3 = I_3$

3. $AA = A$

5. **(a)** $\text{tr}(A) = \text{tr}(AA) = \text{tr}(AAA) = 1$.
(b) $\text{tr}(A) = \text{tr}(AA) = \text{tr}(AAA) = 2$.

Review Exercises

1. **(a)** AB is 2×2.
(b) $A^T B$ is a scalar.
(c) $A^T BA$ is a scalar.
(d) $AA^T B$ is 5×5.

3. $(A\mathbf{x})^T = [-4 \ \ 2]$, $\mathbf{x}^T A^T = [-4 \ \ 2]$,

$$\mathbf{x}\mathbf{x}^T = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix}$$

and $\mathbf{x}^T\mathbf{x} = 34$. $A^T\mathbf{x}^T$ is not defined.

5. $k = 9$

7. $\mathbf{x}^4 = \begin{bmatrix} 6.3336 \\ 4.9423 \\ 9.6943 \end{bmatrix}$

Chapter 9

9.1 Exercises

1. **(a)** $\begin{bmatrix} 1/5 & 0 \\ 0 & 1/3 \end{bmatrix}$

(b) $\begin{bmatrix} 1/5 & -2/5 \\ 1/5 & 3/5 \end{bmatrix}$

(c) The inverse does not exist.

3. **(a)** $z_1 = 20$, $z_2 = 55$.
(b) $\mathbf{w}^T A\mathbf{y} = 650$—a scalar, representing total cost.

5. $R = 14$, $Y = 440$. Budget deficit is $G - T = 9$. Trade deficit is $-X = 4.4$.

9.2 Exercises

1. **(a)** $|A| = 1$
(b) $|B| = -5$
(c) $|C| = -23$
(d) $|D| = 4$

3. $|B| = 3|A| = 21$

5. $|D| = 3|A| = 9$

7. $|A^3| = -8$

9.3 Exercises

1. The inverses are

(a) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3/2 & 4 & -1/2 \\ 0 & 1 & 0 \\ -5/2 & 6 & -1/2 \end{bmatrix}$

(c) $\begin{bmatrix} -1/3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

3. $z_1 = 55$, $z_2 = 10$, $z_3 = 85$.

5. **(a)** $|A| = 3$
(b) $|B| = 0$
(c) $|C| = 60$

9.4 Exercises

1. $x_1 = 6/4$, $x_2 = 4$, $x_3 = -14/4$.

3. $x_1 = 1$, $x_2 = 0$, $x_3 = 5$.

5.

$$Y = \frac{(a + e + G)h + \bar{M}}{h[1 - b(1 - t)] + lk}$$

$$C = \frac{alk + h[a + (e + G)b(1 - t)] + \bar{M}lb(1 - t)}{h[1 - b(1 - t)] + lk}$$

Review Exercises

1. A has a zero determinant and therefore no inverse.

3. $|A| = -2$.

5. $x_1 = 11/4$, $x_2 = -10/4$, $x_3 = 1/4$.

7. $|A|^2 = 1$, so $|A| = \sqrt{1} = \pm 1$.

9. $|A^3| = 0$ if and only if $|A| = 0$.

Chapter 10

10.1 Exercises

1. $\|\mathbf{y}\| = \sqrt{6}$, $\|\mathbf{w}\| = \sqrt{6}$, $\|\mathbf{z}\| = 1$, $\|\mathbf{v}\| = 1/\sqrt{3}$.

3. **(a)** $\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3 + \lambda_4\mathbf{e}_4 = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

(b) $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 + \lambda_4\mathbf{v}_4 = 0$ implies $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

5. In each case $\mathbf{y}^t\mathbf{w} = 0$.

7. $\mathcal{V}$ describes the positive quadrant, so (a) $\mathbf{u} + \mathbf{v}$ will be nonnegative and therefore in $\mathcal{V}$; (b) if $\lambda < 0$, then $\lambda\mathbf{u}$ will not be in $\mathcal{V}$. $\mathcal{V}$ is not a vector space.

9. $\text{rank}(A) = 3$, $\text{rank}(B) = 3$.

10.2 Exercises

1. **(a)** $(2-\lambda)^2 - 1 = 0$, $\lambda_1 = 3$, $\lambda_2 = 1$.

(b) $\mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{q}_2 = -\mathbf{q}_1$

(c) $Q^T AQ = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

3. **(a)** $PP = X(X^T)^{-1}X^T X(X^T X)^{-1}X^T = X(X^T X)^{-1}X^T = P$

(b) $\text{trace}(P) = \text{rank}(P) = 2$. There are two unit eigenvalues and two zero eigenvalues.

5. Let A and B be two orthogonal matrices, then

$$(AB)^T AB = B^T A^T AB = B^T B = I$$

10.3 Exercises

1. A is positive definite, B is positive definite, C is positive semidefinite.

3. $g(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} =$

$$[x_1 \quad x_2 \quad x_3]\begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5. Leading principal minors are 3, 2, and -10; therefore it is indefinite.

Review Exercises

1. **(a)** Linearly independent, **(b)** linearly dependent, **(c)** linearly independent.

3. **(a)** Largest possible rank is 7.

(b) Largest possible rank is 7.

5. Any 3 linearly independent vectors in $\mathbb{R}^3$ constitute a basis, for example:

(a) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

7. **(a)** $\lambda_1 = 3$, $\lambda_2 = -7$.

(b) $\lambda_1 = 9$, $\lambda_2 = -5$.

9.

$$Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

and $Q^T AQ = \Lambda$.

Chapter 11

11.1 Exercises

1.

$$\begin{aligned} \frac{\partial f(x_1, x_2)}{\partial x_1} &= \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1} \\ &= \lim_{\Delta x_1 \to 0} \frac{[3(x_1 + \Delta x_1) + 5x_2] - [3x_1 + 5x_2]}{\Delta x_1} \\ &= \lim_{\Delta x_1 \to 0} \frac{3\Delta x_1}{\Delta x_1} = 3 \end{aligned}$$

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_2} &= \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2} \\ &= \lim_{\Delta x_2 \to 0} \frac{[3x_1 + 5(x_2 + \Delta x_2)] - [3x_1 + 5x_2]}{\Delta x_2} \\ &= \lim_{\Delta x_2 \to 0} \frac{5\Delta x_2}{\Delta x_2} = 5\end{aligned}$$

3. $$\frac{\partial R(x_1, x_2)}{\partial x_2} =$$

$$\begin{aligned}&\lim_{\Delta x_2 \to 0} \frac{R(x_1, x_2 + \Delta x_2) - R(x_1, x_2)}{\Delta x_2} \\ &= \lim_{\Delta x_2 \to 0} \frac{[p_1 x_1 + p_2(x_2 + \Delta x_2)] - [p_1 x_1 + p_2 x_2]}{\Delta x_2} \\ &= \lim_{\Delta x_2 \to 0} \frac{p_2 \Delta x_2}{\Delta x_2} = p_2\end{aligned}$$

$\partial R/\partial x_2$ is the rate at which revenue increases per unit increase in x_2. This is equal to the price of good 2 for a competitive firm.

5. $$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_2} &= \lim_{\Delta x_2 \to 0} \frac{\left[x_1^2(x_2 + \Delta x_2)\right] - \left[x_1^2 x_2\right]}{\Delta x_2} \\ &= \lim_{\Delta x_2 \to 0} \frac{x_1^2 \Delta x_2}{\Delta x_2} = x_1^2\end{aligned}$$

7. $$\begin{aligned}\frac{\partial y}{\partial x_1} &= \frac{5x_2^{1/3} x_3^{1/4}}{x_1^{1/2}} \\ \frac{\partial y}{\partial x_2} &= \frac{10x_1^{1/2} x_3^{1/4}}{x_2^{2/3}} \\ \frac{\partial y}{\partial x_3} &= \frac{2.5x_1^{1/2} x_2^{1/3}}{x_3^{3/4}}\end{aligned}$$

9. $$\begin{aligned}\frac{\partial y}{\partial x_1} &= (4.8)x_1^{-3/2}\left[0.4x_1^{-1/2} + 0.6x_2^{-1/2}\right]^{-3} \\ \frac{\partial y}{\partial x_2} &= (7.2)x_1^{-3/2}\left[0.4x_1^{-1/2} + 0.6x_2^{-1/2}\right]^{-3}\end{aligned}$$

11. $$\frac{dY}{dt} = 0.1(1+t)^{-1/2} K_0 e^{0.05t} + 0.001(1+t)^{1/2} K_0 e^{0.05t}$$

11.2 Exercises

1. $$\nabla f = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \nabla_2 F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3. $$\nabla f = \begin{bmatrix} 3x_1^2 x_2^4 \\ 4x_1^3 x_2^3 \end{bmatrix} \qquad \nabla_2 F = \begin{bmatrix} 6x_1 x_2^4 & 12x_1^2 x_2^3 \\ 12x_1^2 x_2^3 & 12x_1^3 x_2^2 \end{bmatrix}$$

5. $$\nabla f = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \qquad \nabla_2 F = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

7. $$\begin{aligned}\frac{\partial y}{\partial x_1} &= 25x_1^{-1/2} x_2^{2/3} > 0 \\ \frac{\partial y}{\partial x_2} &= \frac{100}{3} x_1^{1/2} x_2^{-1/3} > 0 \\ \frac{\partial^2 y}{\partial x_1^2} &= \frac{-25}{2} x_1^{-3/2} x_2^{2/3} < 0 \\ \frac{\partial^2 y}{\partial x_2^2} &= -\frac{100}{9} x_1^{1/2} x_2^{-4/3} < 0\end{aligned}$$

$$\frac{\partial^2 y}{\partial x_1 \partial x_2} = \frac{\partial^2 y}{\partial x_2 \partial x_1} = \frac{50}{3} x_1^{-1/2} x_2^{-1/3} > 0$$

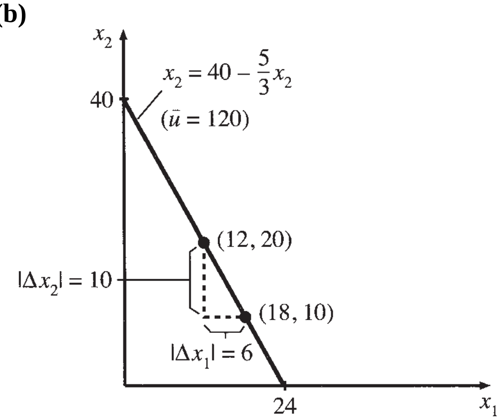
so the marginal products of both inputs are positive, the marginal product of each input falls as more is used and the marginal product of either input is increased by more use of the other input.

9. $$\begin{aligned}f_{12} = f_{21} &= \left(6x_1 + 3x_1^2 x_3\right) e^{3x_2 + x_1 x_3} - 6x_2^2/x_1^2 \\ f_{13} = f_{31} &= \left(3x_1^2 + x_1^3 x_3\right) e^{3x_2 + x_1 x_3}\end{aligned}$$

11.3 Exercises

1. **(a)** $du = 5\,dx_1 + 3\,dx_2$

(b)



(c) Since $u = 5x_1 + 3x_2$ is a linear function, MRS $= 5/3$ (we do not need to take the limit as $\Delta x_1 \to 0$).

3. $dy = 0 \Rightarrow x_2\,dx_1 + x_1\,dx_2 = 0$, or MRTS $= -dx_2/dx_1 = x_2/x_1$. The equation of an isoquant is $x_2 = \bar{y}/x_1$ and so $dx_2/dx_1 = -\bar{y}/x_1^2 < 0$ and $d^2x_2/dx_1^2 = 2\bar{y}/x_1^3 > 0$. So the isoquants are negatively sloped and strictly convex.

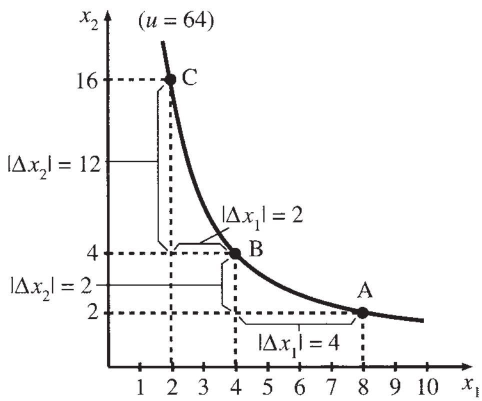
5. **(a)** $dx_2/dx_1 = -x_2/x_1$

(b) $dx_2/dx_1 = -x_2/x_1$

(c) $dx_2/dx_1 = -x_2/x_1$

7. The MRTS $= 3x_2^4/7x_1^4$ gets smaller as one moves along an isoquant from left to right—as x_1 rises and x_2 falls. Thus the isoquants are strictly convex to the origin. To see this more formally, write the equation for an isoquant in the form $x_2 = g(x_1)$ and find dx_2/dx_1 and d^2x_2/dx_1^2.

9. $u(8, 1) = 8^2 \times 1 = 64$, $u(4, 4) = 4^2 \times 4 = 64$ and $u(2, 16) = 2^2 \times 16 = 64$. Thus the points A, B, C lie on the same indifference curve.



Between B and C, $|\Delta x_2|/|\Delta x_1| = 12/2 = 6$, and between A and B, $|\Delta x_2|/|\Delta x_1| = 4/2 = 2$.

11.4 Exercises

1. $d^2y = 2\,dx_1^2 + 4\,dx_1\,dx_2 + 2\,dx_2^2 = 2\,(dx_1 + dx_2)^2 \geq 0$ and so f is convex.

3. Find and show that $|H_1| < 0$ and $|H_2| > 0$. Thus H is negative definite (see theorem 11.9), and so f is strictly concave.

5. $d^2y = -\dfrac{1}{4}(x_1 + x_2)^{-3/2}(dx_1 + dx_2)^2 \leq 0$ and so f is concave.

7. If $f_{ij} = 0$ for all $i \neq j$ then $d^2y = \sum_{i=1}^n f_{ii}\,dx_i^2$. Thus, if $f_{ii} \leq 0$, then $d^2y \leq 0$ (since $dx_i^2 \geq 0$). This proves the necessity part of the claim. Moreover $d^2y \leq 0$ for all dx_i only if $f_{ii} \leq 0$ for all i.

11.5 Exercises

1. **(a)** f is quasiconcave, since

$$|\bar{H}_2| = |\bar{H}| = -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11}$$
$$= \frac{3}{32}x_1^{-1/2}x_2^{-5/4} > 0$$

f is also strictly concave because

$$f_{11} = -\frac{1}{4}x_1^{-3/2}x_2^{1/4} < 0$$

and

$$f_{11}f_{22} > f_{12}^2 \Leftrightarrow \frac{3}{64}x_1^{-1}x_2^{-3/2} > \frac{1}{64}x_1^{-1}x_2^{-3/2}$$

which is the case.

(b) f is quasiconcave, since

$$|\bar{H}| = -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11} = \frac{18}{81}x_1^{-1} > 0$$

f is not strictly concave because $f_{11}f_{22} = f_{12}^2$. However, $|H_1^*| = f_{11}$, f_{22} are both ≤ 0 and $|H_2^*| = |H| = f_{11}f_{22} - f_{12}^2 \leq 0$ and so f is (weakly) concave.

(c) f is quasiconcave, since

$$|\bar{H}| = -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11} = 30x_1^4x_2^7 > 0$$

f is neither strictly concave nor (weakly) concave, since $f_{11}f_{22} < f_{12}^2$.

3. **(a)** For quasiconcavity

$$|\bar{H}_2| = \frac{7}{144}x_1^{-5/4}x_2^{-1}x_3^{3/4} > 0$$

and $|\bar{H}_3| = |\bar{H}| < 0$. We have

$$|\bar{H}| = -f_1 \begin{vmatrix} f_1 & f_{12} & f_{13} \\ f_2 & f_{22} & f_{23} \\ f_3 & f_{32} & f_{33} \end{vmatrix} + f_2 \begin{vmatrix} f_1 & f_{11} & f_{13} \\ f_2 & f_{21} & f_{23} \\ f_3 & f_{31} & f_{33} \end{vmatrix} - f_3 \begin{vmatrix} f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \\ f_3 & f_{31} & f_{32} \end{vmatrix}$$

The first of these terms is

$$-f_1 \frac{12}{576} x_1^{-1/4} x_2^{-1/2} x_3^{-5/4} < 0$$

and expanding the other two terms shows that $|\bar{H}| < 0$ as required for quasiconcavity.

(b) This function is also quasiconcave, following the same steps as in part (a).

5. Computing the partial derivatives f_1 and f_2 gives us

$$f_1 = \alpha A x_1^{\alpha-1} x_2^{\beta} \quad \text{and} \quad f_2 = \beta A x_1^{\alpha} x_2^{\beta-1}$$

and so

$$\begin{aligned} f_1 x_1 + f_2 x_2 &= \left(\alpha A x_1^{\alpha-1} x_2^{\beta}\right) x_1 + \left(\beta A x_1^{\alpha} x_2^{\beta-1}\right) x_2 \\ &= \alpha A x_1^{\alpha} x_2^{\beta} + \beta A x_1^{\alpha} x_2^{\beta} \\ &= (\alpha + \beta) A x_1^{\alpha} x_2^{\beta} \end{aligned}$$

Thus $f_1 x_1 + f_2 x_2 = kf(x_1, x_2)$, where $k = \alpha + \beta$, which is Euler's theorem, and if $\alpha + \beta = 1$, then $f_1 x_1 + f_2 x_2 = f(x_1, x_2)$.

11.6 Exercises

1. $\Delta y = f(3, 4) - f(1, 1) = -15 - 8 = -23$, $dy = 2f_1(1, 1) + 3f_2(1, 1) = -10$, thus $dy > \Delta y$.

Review Exercises

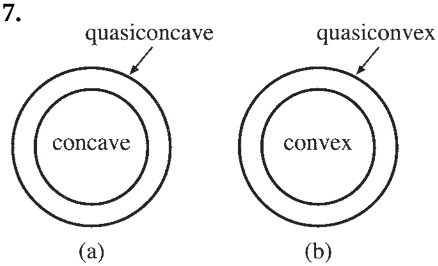
1. $f_1 = \alpha A x_1^{\alpha-1} x_2^{\beta}$, $f_2 = \beta A x_1^{\alpha} x_2^{\beta-1}$

3. $f_1 = a$, $f_{11} = 0$, $f_{12} = 0$, $f_{13} = 0$, $f_2 = \beta x_2^{\beta-1} x_3^{\gamma}$, $f_{21} = 0$, $f_{22} = \beta(\beta - 1) x_2^{\beta-2} x_3^{\gamma}$, $f_{23} = \gamma\beta x_2^{\beta-1} x_3^{\gamma-1}$, $f_3 = \gamma x_2^{\beta} x_3^{\gamma-1}$, $f_{31} = 0$, $f_{32} = \gamma\beta x_2^{\beta-1} x_3^{\gamma-1}$, $f_{33} = \gamma(\gamma - 1) x_2^{\beta} x_3^{\gamma-2}$.

Notice that Young's theorem applies: $f_{12} = f_{21}$, $f_{13} = f_{31}$, $f_{23} = f_{32}$.

5. $|H_1| = f_{11} = -(3/16) x_1^{-7/4} x_2^{1/2} < 0$, $|H_2| = (1/32) x_1^{-6/4} x_2^{-1} > 0$.

7.



9. **(a)** $MRTS = 0.3x_2^3/0.7x_1^3$ falls as one moves along an isoquant left to right (i.e., as x_1 increases and x_2 decreases). Thus isoquants are strictly convex to the origin.

(b) We need to show that $|\bar{H}_2| = -f_1^2 f_{22} + 2f_1 f_2 f_{12} - f_2^2 f_{11} > 0$. We have

$$\begin{aligned} f_1 &= 0.3x_1^{-3}\left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-3/2} > 0 \\ f_2 &= 0.7x_2^{-3}\left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-3/2} > 0 \end{aligned}$$

Also $f_{11} < 0$, $f_{22} < 0$, $f_{12} > 0$, and so $|\bar{H}_2| > 0$. See Student Solutions Manual for details.

(c) To show that f is concave, we refer to theorem 11.9. In part (b) it is established that $|H_1^*| = f_{11}, f_{22} < 0$. We also need to show $|H_2^*| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \geq 0$. That is, $f_{11} f_{22} > f_{12}^2$. Using expressions for f_{11}, f_{22}, f_{12} from part (b) above and making the substitution $z = \lfloor 0.3x_1^{-2} + 0.7x_2^{-2} \rfloor$ one can determine this inequality.

(d)

$$\begin{aligned} f(sx_1, sx_2) &= \left[0.3(sx_1)^{-2} + 0.7(sx_2)^{-2}\right]^{-1/2} \\ &= \left[s^{-2}\left(0.3x_1^{-2} + 0.7x_2^{-2}\right)\right]^{-1/2} \\ &= s\left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-1/2} \\ &= sf(x_1, x_2) \end{aligned}$$

(e) $f_1x_1 + f_2x_2 = \left(0.3x_1^{-3}\left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-3/2}\right)x_1 + \left(0.7x_2^{-3}\left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-3/2}\right)x_2$ simplifying gives

$$f_1x_1 + f_2x_2 = \left[0.3x_1^{-2} + 0.7x_2^{-2}\right]^{-1/2} = f(x_1, x_2)$$

(f) $$\frac{f_1}{f_2} = \frac{0.3}{0.7}\left(\frac{x_2}{x_1}\right)^3 \Rightarrow \left(\frac{x_2}{x_1}\right)^3 = \left(\frac{0.7}{0.3}\right)\left(\frac{f_1}{f_2}\right)$$
$$\Rightarrow \left(\frac{x_2}{x_1}\right) = \left(\frac{0.7}{0.3}\right)^{1/3}\left(\frac{f_1}{f_2}\right)^{1/3}$$

Taking ln, we get

$$\ln\left(\frac{x_2}{x_1}\right) = \frac{1}{3}\ln\left(\frac{0.7}{0.3}\right) + \frac{1}{3}\ln\left(\frac{f_1}{f_2}\right)$$

and so

$$\sigma = \frac{d\ln(x_2/x_1)}{d\ln(f_1/f_2)} = \frac{1}{3}$$

Chapter 12

12.1 Exercises

1. **(a)** (0, 0)
(b) (0, 0)
(c) (17/47, 8/47)
(d) (11/20, 17/20)
(e) (−17/24, 17/32)
(f) (6, 16/9)
(g) (0, 0, 0)
(h) (1/8, 0, −5/2)
(i) (0, 0) and (1, 1)
(j) (0, 0), $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

3.
$$p_1 = 60: \quad q_1^* = 10, \ q_2^* = 20, \ p_2^* = 80$$
$$p_1 = 10: \quad q_1^* = 0, \ q_2^* = 25, \ p_2^* = 75$$

5. Cournot: $q_1^* = 33.33$, $q_2^* = 30.83$, $p^* = 3.58$. Joint-profit maximization: $q_1^* = 48.75$, $q_2^* = 0$, $p^* = 5.13$. Profits increase.

7. $q_1^* = 13.64$, $q_2^* = 9.09$, $p^* = 77.27$.

12.2 Exercises

1. **(a)** minimum
(b) saddle point
(c) maximum
(d) maximum
(e) saddle point
(f) maximum
(g) maximum
(h) maximum
(i) (0, 0) : neither an extremum nor saddle point
(1, 1) : minimum
(j) (0, 0) : neither extremum nor saddle point
$(\sqrt{2}, -\sqrt{2})$: maximum
$(-\sqrt{2}, \sqrt{2})$: maximum

3. The Hessian matrices

$$H_i^* = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \qquad i = 1, 2$$

are negative definite.

5. $K^* = L^* = 4096$. The Hessian matrix

$$H^* = \frac{1}{8192}\begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}$$

is negative definite.

7. (0, 0) maximizes y. The problem is that $|H^*| = 0$.

12.3 Exercises

1. **(a)** (0, 2)
(b) (17/47, 8/47)
(c) (1, 1)
(d) (0.5, 1)
(e) (0, 1)
(f) (0, 1) and (1, 1)

3. Hint: $\partial\pi/\partial q_i$ is independent of q_j, $i \neq j$. The explanation is that any change in output induced by the introduction of a quota in country 2 leaves the shape of the marginal cost curve for q_1 unaffected. Hence no adjustment in price and quantity in country 1 is necessary.

Review Exercises

1. **(a)** (0, 0) (minimum)
   (b) (1, 1) (minimum)
   (c) $(-50/21, -52/21)$ (neither an extremum nor a saddle point)
   (d) (0, 0) (neither an extremum nor a saddle point); $(2/3, 2/3)$ (maximum)
   (e) (0, 0) (neither an extremum nor a saddle point); $(4/3, 4/3)$ (minimum)
   (f) (1, 2) (minimum)
   (g) (0, 0) (saddle point)

3. **(a)** $24.8
   (b) Coke and hot dogs are complements.

5. **(a)** $q_1 = 2\frac{6}{7}$, $q_2 = 1\frac{3}{7}$, $p_1 = p_2 = 7\frac{1}{7}$.
   (b) The solution depends on b as shown in the table below:

	$b \le 2$	$2 \le b \le 5\frac{5}{9}$	$5\frac{5}{9} \le b \le 7\frac{1}{7}$
Home market:			
quantity	0	$(5b - 10)/4$	$10 - b$
price	—	b	b
Foreign market:			
quantity	2	$(10 - b)/4$	$b/5$
price	6	$5 + 0.5b$	$10 - 0.4b$

Chapter 13

13.1 Exercises

1. **(a)** $\left(\frac{5}{3}\sqrt{\frac{18}{19}}, \sqrt{\frac{18}{19}}\right)$
   (b) (7.38, 11.07)
   (c) (3, 4)
   (d) $(5, 0), (-5, 0)$

3. The first-order conditions of the Lagrangean yield

$$\frac{r_1}{r_2} = \frac{a_2}{a_1}\frac{y_1^{(1-\alpha_1)/\alpha_1}}{y_2^{(1-\alpha_2)/\alpha_2}}$$

Combined with $y_1^{1/\alpha_1} + y_2^{1/\alpha_2} = \bar{l}$ this gives implicitly the optimal outputs.

5. $$x_1 = \alpha\frac{m - p_2c_2}{p_1} + (1-\alpha)c_1$$
$$x_2 = (1-\alpha)\frac{m - p_1c_1}{p_2} + \alpha c_2$$

7. Hint: Use the first-order conditions to solve for $\lambda(r, w, \bar{y})$, $K(r, w, \bar{y})$ and $L(r, w, \bar{y})$. Insert these into the Lagrangean and differentiate with respect to $\bar{y}$. Consider that the optimized Lagrangean has the same value as the cost function for all $\bar{y}$.

13.2 Exercises

1. **(a)**
$$H^* = \begin{bmatrix} -1.23 & 0 & -4 \\ 0 & -3.1 & -10 \\ -4 & -10 & 0 \end{bmatrix}, |H^*| > 0 \text{ (check!)}$$

   (b)
$$H^* = \begin{bmatrix} 0.203 & 0.135 & -0.339 \\ 0.135 & 0.090 & -0.678 \\ -0.339 & -0.678 & 0 \end{bmatrix}, |H^*| < 0 \text{ (check!)}$$

   (c)
$$H^* = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix}, |H^*| < 0 \quad \text{(check!)}$$

3. $|H^*| = -r_2a_2(a_2 - 1)l_2^{a_2-2} - r_1a_1(a_1 - 1)l_1^{a_1-2} > 0$ since $(a_2 - 1) < 0$ and $(a_1 - 1) < 0$.

13.3 Exercises

1. We do this for (a): $f(x_1, x_2) = 2x_1 + 3x_2$ and $g(x_1, x_2) = 10 - 2x_1^2 - 5x_2^2$. f is quasiconcave because it has linear level curves, g is quasiconvex because, by theorem 11.2, $|\bar{H}_2| = -160x_1 - 400x_2 < 0$, for $x_1, x_2 > 0$. Applying theorem 13.4 shows that

$$\begin{vmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & g_1 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix} = \lambda\left(120x_1^2 + 400x_2^2\right) > 0$$

for $\lambda > 0$.

3. See Student's Solution Manual for discussion.

5. When there are two variables and one constraint, we have an application such as the utility-maximization problem with two goods and a single budget constraint.

Review Exercises

1. The production functions are $y_1 = f(l_1)$ and $y_2 = g(l_2)$ and so at the optimum $p_1 f'(l_1) = p_2 g'(l_2)$. The value of the increase in profit from an increase in the input available is the value of the Lagrange multiplier at the optimum.

3. If A is the amount invested, M is the amount of money available, r is the annual rate of interest, and Y is profit, then the optimal amount invested is the solution to $Y'(A^*) = 1 + r$ and the individual saves if $A^* < M$ and borrows if $A^* > M$.

Chapter 14

14.1 Exercises

1. The initial equilibrium level of income is $Y_0 = 5{,}000$. The final level of income is $Y_1 = 6{,}000$. The multiplier is 5.

3. At $t = \$1$, profit-maximizing output is $q_1 = 33$. At $t = \$2$ output is $q_2 = 32\frac{2}{3}$.

5. **(a)** $dY^*/dc = 1/(1-c)^2 > 0$

(b)

$$\frac{dp^*}{db} = \frac{dp^*}{d\beta} = -\frac{a - \alpha + cy}{(b+\beta)^2} < 0$$

$$\frac{dp^*}{dc} = \frac{y}{b+\beta} > 0$$

$$\frac{dq^*}{db} = -\frac{\beta p^*}{b+\beta} < 0 \quad \frac{dq^*}{d\beta} = \frac{bp^*}{b+\beta} > 0$$

$$\frac{dq^*}{dc} = \frac{\beta y}{b+\beta} > 0$$

(c)

$$\frac{dq^*}{da} = \frac{1}{2(b+c)} > 0$$

$$\frac{dp^*}{da} = \frac{b+2c}{2(b+c)} > 0$$

$$\frac{dq^*}{dc} = -\frac{q}{b+c} < 0$$

$$\frac{dp^*}{dc} = \frac{bq}{b+c} > 0$$

7.

$$r = \frac{a - \alpha + cY_R - \gamma Y_D}{b+\beta}$$

So

$$\frac{dr}{dY_R} = \frac{c}{b+\beta} > 0, \quad \frac{dr}{dY_D} = -\frac{\gamma}{b+\beta} < 0$$

$$Y_D = \frac{a - \alpha - (b+\beta)r + cY_R}{\gamma}$$

and so

$$\frac{dY_D}{dr} = -\frac{\beta + b}{\gamma} < 0, \quad \frac{dY_D}{dY_R} = \frac{c}{\gamma} > 0$$

If both r and Y_D are treated as endogenous, then there is a continuum of (Y_D, r)—pairs which can produce equilibrium of the trade balance.

14.2 Exercises

1.

$$\frac{\partial Y^*}{\partial M} = \frac{E_R}{(1-E_Y)L_R + L_Y E_R} > 0$$

$$\frac{\partial R^*}{\partial M} = \frac{1 - E_Y}{(1-E_Y)L_R + L_Y E_R} < 0$$

3. **(a)**

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{\lambda^* P_2^2}{|D|} - x_1^* \frac{p_2}{|D|}$$

where $|D| = 2p_1 p_2$. The income effect is negative, therefore $\partial x_1^*/\partial p_1 < 0$. The utility function is symmetric.

(b)

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{\lambda^* p_2^2}{|D|}$$

where $|D| = 0.25p_2^2 x_1^{-1.5}$ and the income effect is zero:

$$\frac{\partial x_2^*}{\partial p_2} = -\frac{\lambda^* p_1^2}{|D|} - x_2^* \frac{0.25x_1^{-1.5} p_2}{|D|}$$

The income effect is negative.

5. The Slutsky equation is

$$\frac{\partial x_1^*}{\partial p_1} = \lambda^* |D|^{-1} \begin{vmatrix} u_{22} & u_{23} & -p_2 \\ u_{32} & u_{33} & -p_3 \\ -p_2 & -p_3 & 0 \end{vmatrix} - x_1^* |D|^{-1} \begin{vmatrix} u_{12} & u_{13} & -p_1 \\ u_{22} & u_{23} & -p_2 \\ u_{32} & u_{33} & -p_3 \end{vmatrix}$$

where

$$|D| = \begin{vmatrix} u_{11} & u_{12} & u_{13} & -p_1 \\ u_{21} & u_{22} & u_{23} & -p_2 \\ u_{31} & u_{32} & u_{33} & -p_3 \\ -p_1 & -p_2 & -p_3 & 0 \end{vmatrix}$$

7. $\partial Y_1^*/\partial I_1^0 = 3.2$, $\partial Y_2^*/\partial I_1^0 = 1.2$, $\partial Y_1^*/\partial I_2^0 = 2$, $\partial Y_2^*/\partial I_2^0 = 2$

14.3 Exercises

1. $L_1 = L_2 = 500$, $x_1 = 2{,}236$, $x_2 = 1{,}118$, $\lambda = 2.236$

3. $V(p, w, r) = Ap^5 w^{-2.5} r^{-1.5}$, where

$$A = 0.2(0.8)^4 \left[\left(\frac{3}{5}\right)^{5/8} + \left(\frac{5}{3}\right)^{3/8} \right]^{-4} \doteq 0.0058$$

5.
$$V(p, m) = \alpha^\alpha (1-\alpha)^{1-\alpha} p_1^{-\alpha} p_2^{\alpha-1} (m - p_1 c_1 - p_2 c_2)$$
$$E(p, u) = \alpha^{-\alpha} (1-\alpha)^{\alpha-1} p_1^\alpha p_2^{1-\alpha} u + p_1 c_1 + p_2 c_2$$

7. Hint: Start from $C(y_a) = c[y_a, k(y_a)]$ and differentiate this equation at y_a with respect to y. Consider the value of $\partial c/\partial k$ at y_a.

Review Exercises

1. The value functions are as follows. The comparative-statics effects of a change in the α-variables can be found by partially differentiating the value function with respect to the α-variables.

(a) $V = 0.88\alpha$

(b) $V = \left(\frac{20}{3\alpha_1} + 2\right)\sqrt{\frac{9\alpha_1\alpha_2}{100 + 45\alpha_1}}$

(c) $V = 0.38\sqrt{\alpha}$

(d) $V = 2^{2.75}\alpha$

(e) $V = \left(\frac{\alpha + 3}{2}\right)^2$

(f) $V = 2(\sqrt{\alpha_1\alpha_2} - \alpha_1) - 1$

3. (a) The condition for the Lagrange multipliers to be equal is

$$\left|\frac{dx_{2i}}{dx_{1i}}\right|_{du=0,\beta\in(0,1)} = \frac{1}{\beta}\left|\frac{dx_{2i}}{dx_{1i}}\right|_{du=0,\beta=0}$$

5. Defining $H_i(r, u) = m_i(r, m_2^*(r, u))$, $i = 1, 2$ where m_2^* is the minimum income in period 2 necessary to reach utility level u given income $\bar{m}_1$ and r, the Slutsky equations are

$$\frac{\partial m_i}{\partial r} = \frac{\partial H_i}{\partial r} - \frac{\partial m_i}{\partial \bar{m}_2}\left(H_1 - \bar{m}_1\right), \qquad i = 1, 2$$

Chapter 15

15.1 Exercises

1. The feasible set contains only the origin and so does not satisfy the K-T conditions. Slater's condition is not satisfied.

5. (a) $L_1 = 7.83$, $L_2 = 4.17$, $\lambda = 8.7$

(b) $L_1 = 10$, $L_2 = 5\frac{1}{3}$, $\lambda = 0$

15.2 Exercises

1. $x_1 = 32$, $x_2 = 12$, $\lambda = 2.3$, $\mu = 7.4$

3.
$$y_1 = 293.6, \quad y_2 = 58.7$$
$$l_1 = 862.1, \quad l_2 = 137.9$$
$$x_1 = 68.1, \quad x_2 = 340.6$$

The shadow wage rate is $\lambda = 0.29$.

Review Exercises

1. Hint: Solve the Lagrangean

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = x_1x_2 + \lambda(m - x_1 + wT - wx_2) + \mu_1(x_2 + H - T) + \mu_2(T - x_2)$$

Then consider the cases: (i) $\mu_1 = \mu_2 = 0$, (ii) $\mu_1 = 0$ and $\mu_2 \geq 0$, and (iii) $\mu \geq 0$ and $\mu_2 = 0$.

3. The investor will borrow approximately $39.

5. For $p_1 = 10$ and $p_2 = 1$: $x_1^* = 4$, $x_2^* = 40$. For $p_1 = 10$ and $p_2 = 1.60$: $x_1^* = 0$, $x_2^* = 50$.

Chapter 16

16.1 Exercises

1. **(a)** $x^5/5 + x^4/2 + 2x^2 + 10x + C$
(b) $3/5x^{5/3} + C$
(c) $10e^x + C$
(d) $3e^{x^2} + C$
(e) $\ln(x^3 + 2x) + C$

3. **(a)** $F(x) = 2x$
(b) $F(x) = 3x^2 + 5$
(c) $F(x) = (5/4)x^4 + x^2 + 6x$
(d) $F(x) = x^2 + 1$

5. $Q(L) = (20/3)L^{3/2}$

7. We have that

$$\frac{d[F(x)]}{dx} = f(x), \quad \frac{d[G(x)]}{dx} = g(x)$$

and so

$$\frac{d[F(x) \pm G(x)]}{dx} = f(x) \pm g(x)$$

which is the integrand of the expression, and so this proves the result.

16.2 Exercises

1. **(a)** area $= 2.5$
(b) $\mathcal{S}_{\min} = 2$, $\mathcal{S}_{\max} = 3$
(c) $\mathcal{S}_{\min} = 2.2$, $\mathcal{S}_{\max} = 2.8$
(d) $\mathcal{S}_{\min} = 2.5 - 1.5/n$, $\mathcal{S}_{\max} = 2.5 + 1.5/n$. It follows that

$$\lim_{n\to\infty} \mathcal{S}_{\min} = \lim_{n\to\infty} \mathcal{S}_{\max} = 2.5$$

3. **(a)** 12.7
(b) 19.2
(c) 6.321
(d) 155.645
(e) 1.386

5. $K(5) \doteq K^0 + 74.53$

7. From the definition of the derivative we have

$$F'(x) = \lim_{\Delta x\to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

where

$$F(x) = \int_a^x f(t)\,dt$$

$$F(x + \Delta x) = \int_a^{x+\Delta x} f(t)\,dt$$

Using property 1 from section 16.3 gives us

$$F'(x) = \lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x)\Delta x}{\Delta x} = f(x)$$

which proves the result.

16.3 Exercises

1. **(a)** $\text{PS}(p_0) = 5\frac{1}{3}$
(b) $\text{PS}(\hat{p}) = 18$
(c) $\Delta\text{PS} = 18 - 5\frac{1}{3} = 12\frac{2}{3}$

3. **(a)** $\text{CS}(p_0 = 1) = 74\frac{2}{3}$

(b) $\text{CS}(\hat{p} = 4) = 54$

(c) $\Delta\text{CS} = -20\frac{2}{3}$

16.4 Exercises

1. 15

3. **(a)** 15

(b) $+\infty$

For the demand function in part (b), the ΔCS cannot be computed (i.e., it is not a finite value).

5. $25,000

16.5 Exercises

1. **(a)** $(1/11)(x^3 + 5x)^{11} + C$

(b) $(1/2)\left(e^{x^2} + 4x\right)^2 + C$

(c) $(e^x + 3x^2)^2 + C$

(d) $-(1/9)(x^2 + 2)^{-9} + C$

(e) $-2(x^3 + 4x)^{-1} + C$

3. **(a)** $x^2 e^x - 2(xe^x - e^x) + C$

(b) $x^2(1 + x^2)^{1/2} - (2/3)(1 + x^2)^{3/2} + C$

(c) $(x^2/2)\ln x - (x^2/4) + C$

Review Exercises

1. **(a)** $x^3/3 + C$

(b) $x^4/2 + (5/3)x^3 + x^2/2 + 5x + C$

(c) $\sum_{i=0}^{n} a_i[x^{i+1}/(i + 1)]$

3. $\text{TP}(L) = (15/4)L^{4/3}$

5. **(a)** $\text{PS}(p_0) = 3/5$

(b) $\text{PS}(\hat{p}) = 614.4$

(c) $\Delta\text{CS} = 613.8$

7. $\text{CS} = 6$

9. $(1/5)(x^3 + 4x^2 + 3)^5 + C$

Chapter 17

1. (a) first order, (b) linear, (c) autonomous, (d) difference equation

3. (a) first order, (b) linear, (c) nonautonomous, (d) difference equation

5. (a) first order, (b) nonlinear, (c) nonautonomous, (d) difference equation

7. (a) second order, (b) linear, (c) nonautonomous, (d) difference equation

9. (a) second order, (b) linear, (c) nonautonomous, (d) difference equation

11. (a) first order, (b) nonlinear, (c) nonautonomous, (d) differential equation

13. (a) second order, (b) linear, (c) nonautonomous, (d) differential equation

15. (a) first order, (b) linear, (c) nonautonomous, (d) differential equation

17. (a) third order, (b) linear, (c) autonomous, (d) differential equation

19. (a) second order, (b) linear, (c) nonautonomous, (d) differential equation

Chapter 18

18.1 Exercises

1. **(i)** **a.** $y_t = C2^t + 10(1 - 2^t)$

b. $y_t = (y_0 - 10)2^t + 10$

c. $\bar{y} = 10$. Does not converge.

(ii) **a.** $y_t = C$

b. $y_t = y_0$

c. $\bar{y}$ does not exist.

(iii) **a.** $y_t = C(0.5)^t + (1 - 0.5^t)/0.5$

b. $y_t = (y_0 - 2)(0.5)^t + 2$

c. $\bar{y} = 2$. Does converge

3. \$1,819.40

5. $K_{t+1} = K_t(1-\delta) + I$;
$K_t = [K_0 - (I/\delta)](1-\delta)^t + (I/\delta)$.

7. $Y_{t+1} = BY_t + A + I$;

$$Y_t = \left(Y_0 - \frac{A+I}{1-B}\right)B^t + \frac{A+I}{1-B}$$

Since $B > 0$ is given in the model set up, we add the restriction $B < 1$ to ensure convergence.
The immediate impact is $\partial Y_t/\partial I = 1$; the long-run impact is $\partial \bar{Y}/\partial I = 1/(1-B)$.

18.2 Exercises

1. $y_t = \alpha^t/t!$

3. $y_t = y_0 t! + \sum_{k=0}^{t-1} b[t!/(k+1)!]$

5. $y_t = y_0\alpha^{t(t-1)/2} + b\sum_{k=0}^{t-1}\left(\prod_{i=k}^{t-1}\alpha^i/\alpha^k\right)$

7.
$$\begin{aligned} y_{t+1} &= (1+r_t)y_t + 100 \\ y_t &= \prod_{i=0}^{t-1}(1+r_i)100 \\ &\quad + 100\sum_{k=0}^{t-1}\prod_{i=k}^{t-1}[(1+r_i)/(1+r_k)] \end{aligned}$$

Review Exercises

1. **(i)** **a.** $y_t = C(0.8)^t + 5(1-0.8^t)$
b. $y_t = (y_0 - 5)(0.8)^t + 5$
c. $\bar{y} = 5$. Does converge.

(ii) **a.** $y_t = C + 10t$
b. $y_t = y_0 + 10t$
c. $\bar{y}$ does not exist.

(iii) **a.** $y_t = C(-0.1)^t + 9(1-(-0.1)^t)$
b. $y_t = (y_0 - 9)(-0.1)^t + 9$
c. $\bar{y} = 9$. Does converge.

3. **(i)** **a.** $y_t = 2(-1)^t + (1-(-1)^t)$
b. $\{y_t\} = \{0, 2, 0, 2, 0\}$
c. $y_5 = 0$

(ii) **a.** $y_t = 3^t + 0.5(1-3^t)$
b. $\{y_t\} = \{2, 5, 14, 41, 122\}$, $\bar{y} = 100$.
c. $y_5 = 122$

(iii) **a.** $y_t = 50(0.5)^t + 100(1-0.5^t)$
b. $\{y_t\} = \{75, 87.5, 93.75, 96.88, 98.44\}$, $\bar{y} = 100$.
c. $y_5 = 98.44$

(iv) **a.** $y_t = 2(-2/3)^t + 0.2[1-(-2/3)^t]$
b. $\{y_t\} = \{-1, 1, -0.33, 0.56, -0.04\}$, $\bar{y} = 0.2$.
c. $y_5 = -0.04$

(v) **a.** $y_t = 2 + \sum_{k=0}^{t-1}(-1)^k$
b. $\{y_t\} = \{3, 2, 3, 2, 3\}$
c. $y_5 = 3$

(vi) **a.** $y_t = 2(-1)^t + \sum_{k=0}^{t-1}(-1)^{t-1}$
b. $\{y_t\} = \{-1, 0, 1, -2, 3\}$
c. $y_5 = 3$

(vii) **a.** $y_t = 2(-1)^{t(t-1)/2} + \sum_{k=0}^{t-1}\left[\prod_{i=k}^{t-1}(-1)^i/(-1)^k\right]$
b. $\{y_t\} = \{3, -2, -1, 2, 3\}$
c. $y_5 = 3$

5.
$$\begin{aligned} Q_{t+1} &= \frac{\theta G}{G-B}Q_t + \frac{AG-BF}{G-B} \\ Q_t &= Q_0\left(\frac{\theta G}{G-B}\right)^t \\ &\quad + \frac{AG-BF}{G(1-\theta)-B}\left[1-\left(\frac{\theta G}{G-B}\right)^t\right] \\ \bar{Q} &= \frac{AG-BF}{G(1-\theta)-B} \end{aligned}$$

The solution converges monotonically to $\bar{Q}$.

7.
$$U_t = \left(U_0 - \frac{\alpha}{1-\beta}\right)\beta^t + \frac{\alpha}{1-\beta}$$

and $\bar{U} = \alpha/(1-\beta)$. We add the restriction $\beta < 1$

(a) $U_t = U_0\beta^t + \sum_{k=0}^{t-1}(\alpha + e_k)\beta^{t-k-1}$

(b) $U_t = U_0\beta^t + \alpha(\beta^{t-1} + \beta^{t-2} + \cdots + \beta + 1) + e_0\beta^{t-1} + e_1\beta^{t-2} + \cdots + e_{t-1}$

(c) $\bar{U} = 6$, $U_0 = 6$, $U_1 = 9$, $U_2 = 7.5$, $U_3 = 6.75$, $U_4 = 6.38$, $U_5 = 5.19$, $U_6 = 5.59$.

(d) $\bar{U} = 6$, $U_0 = 6$, $U_1 = 9$, $U_2 = 8.4$, $U_3 = 7.92$, $U_4 = 7.54$, $U_5 = 6.23$, $U_6 = 6.18$.

Chapter 19

19.1 Exercises

1. $\bar{y} = 3/4$ is unstable; $\bar{y} = 1/4$ is locally stable. The phase diagram shows that if $0 < y_0 < 3/4$, y_t converges to $1/4$. But if $y_0 > 3/4$, y_t diverges to infinity.

3. The positive steady state is stable: $\bar{y} = 9/2$. The phase diagram shows that y_t converges in oscillations to $9/2$ from any positive starting value.

19.2 Exercises

1. $\bar{y} = 1/3$ is locally stable. $\bar{y} = 0$ is not stable. Starting at $y_0 = 2/3$, y_1 overshoots by going below $1/3$. However, the approach thereafter is monotonic.

3. $E_{t+1} = baE_t(1 - E_t)$; $\bar{E} = 0$ and $\bar{E} = 1 - 1/(ba)$. Require $1 < ba < 3$ for stability at $\bar{E} > 0$.

5. $\bar{y} = 0.1$ is unstable. $\bar{y} = 0.4$ is locally stable. Starting from $y_0 = 0.75$, $y_1 = 0.295$, $y_2 = 0.3360$, $y_3 = 0.3662$, $y_4 = 0.3842$.

Review Exercises

1. $\bar{y} = 1/10$ is locally stable. $\bar{y} = 0$ is unstable. The approach to $1/10$ is monotonic if starting from $y_0 < 1/10$. If $y_0 > 1/10$, y_1 overshoots by going below $1/10$ but the approach is monotonic thereafter.

3. $\bar{y} = 0$ and $\bar{y} = (a-1)/b$. $dy_{t+1}/dy_t = 0$ and $2 - a$, respectively, so stability is unaffected by b.

5. $P_{t+1} = 2 - P_t^{0.5}$; $\bar{P} = 1$ is locally stable. The phase diagram confirms that P_t converges to 1 from any starting value.

7. $\bar{y} = 7/12$ is locally stable. $\bar{y} = 1/12$ is not stable.

Chapter 20

20.1 Exercises

1. **(a)** $y_t = C_1 + C_2(-1)^t$

(b) $y_t = C_1 2^t + C_2(.5)^t - 4$

(c) $y_t = (C_1 + C_2 t)(-1)^t + 4$

(d) $y_t = \sqrt{18}^t\left[C_1 \cos\left(\frac{\pi}{4}t\right) + C_2 \sin\left(\frac{\pi}{4}t\right)\right] + 2$

3. $x_{t+2} - \frac{1}{4}x_t = \frac{A}{4B}$

$x_t = C_1(0.5)^t + C_2(-0.5)^t + \frac{A}{3B}$

5.
$$Y_{t+2} - \frac{\alpha}{1-m}Y_{t+1} + \frac{\alpha}{1-m}Y_t = \frac{\bar{G}}{1-m}$$

The roots are

$$r_1, r_2 = \frac{-\alpha}{2(1-m)} \pm \frac{1}{2}\sqrt{\frac{\alpha^2}{(1-m)^2} - \frac{4\alpha}{(1-m)}}$$

For convergence, we require $\frac{\alpha}{1-m} < 1$. The solution is

$$Y_t = \sqrt{\alpha/(1-m)}t(C_1 \cos\theta t + C_2 \sin\theta t) + \bar{G}/(1-m)$$

where $\cos\theta = \sqrt{\alpha/(1-m)}/2$.

20.2 Exercises

1. **(a)** $y_t = C_1 2^t + C_2 - 12t$

(b) $y_t = C_1 2^t + C_2(0.5)^t + \frac{2}{5}3^t$

(c) $y_t = (C_1 + C_2 t)(-1)^t - \frac{1}{4} + \frac{t}{4}$

Review Exercises

1. $y_t = (C_1 + C_2 t)\left(\frac{1}{3}\right)^t + 4$

3. $y_t = C_1(2)^t + C_2\left(\frac{1}{2}\right)^t - 10$

5. $y_t = C_1 + C_2 t + t^2$

7. $y_t = \sqrt{4/3}^t\left[C_1 \cos\left(\frac{\pi}{6}t\right) + C_2 \sin\left(\frac{\pi}{6}t\right)\right] + 14$

9. **(1)** $C_1 = -3$ $C_2 = -12$

(3) $C_1 = 7/3$ $C_2 = 26/3$

(5) $C_1 = 1$ $C_2 = -3$

(7) $C_1 = -13$ $C_2 = -2\sqrt{3}$

11. $x_{t+2} - \beta\alpha x_t = b(1-\beta)$; $x_t = C_1\sqrt{\beta\alpha}^t + C_2(-\sqrt{\beta\alpha})^t + b(1-\beta)/(1-\beta\alpha)$

13. $y_{t+2} - (\rho + \beta)y_{t+1} + \rho\beta y_t = u_{t+2}$

15. $y_t = C_1 r_1^t + C_2 r_2^t$, where

$$r_1, r_2 = \frac{\rho + \beta}{2} \pm \frac{1}{2}\sqrt{(\rho + \beta)^2 - 4\alpha\beta}$$

Convergence is ensured if both ρ and β are between -1 and 1.

Chapter 21

21.1 Exercises

1. **(a)** $y(t) = e^t$

(b) $y(t) = 6e^{-3t} + 4$

(c) $y(t) = -14e^{-t/4} + 24$

(d) $y(t) = 5t + 1$

(e) $y(t) = 2e^{6t} + 1$

3. $p(t) = 100e^{0.05t}$

5. $y(t) = 500e^{-\alpha t} \times 10^6$

7. $q^e = \dfrac{a-g}{h-b}$; $q(t) = [q_0 - \bar{q}]e^{\alpha(b-h)^t} + \bar{q}, \bar{q} = q^e$, $b - h < 0$ required for stability.

21.2 Exercises

1. $y(t) = e^{2t}$

3. $y(t) = -\dfrac{1}{3} + Ce^{-3t^{-1}}$

5. $m(4) = 18\ m(6) = 28$.

7.

$$k(t) = \frac{\beta}{\alpha} + \left(k_0 - \frac{\beta}{\alpha}\right) e^{\alpha t^2/2}$$

$\bar{k} = \beta/\alpha$ is a steady state but $k(t)$ does not converge to it.

Review Exercises

1. $E(t) = 2e^{0.02(t-t_0)}$

3. $K(t) = -1500e^{-0.05t} + 2000$. $\bar{K} = 2000$. $K(t)$ converges to $\bar{K}$ because $e^{-0.05t}$ goes to 0 in the limit.

5.

$$\dot{K}(t) = \alpha(K(t) - K^*)$$
$$K(t) = (K_0 - K^*)e^{\alpha t} + K^*$$

As $t \to \infty$, $K(t)$ converges to the steady state, K^*,

7.

$$\frac{\dot{k}(t)}{k(t)} = s\alpha t 1/2$$
$$\therefore k(t) = k_0 e^{2/3 s\alpha t^{3/2}}$$

Chapter 22

22.1 Exercises

1. The point $\bar{y} = 1/4$ is a stable steady state; $\bar{y} = 3/4$ is an unstable steady state.

3. The point $\bar{y} = 0$ is a stable steady state and $\bar{y} = 1$ is unstable.

5. The equilibrium price, $\bar{p} = 1/2$, is a stable steady state.

22.2 Exercises

1. $y(t) = (Ce^{-4t} + 3/2)^{1/2}$

3. $y(t) = (2t^3/3 + 2C)^{1/2}$

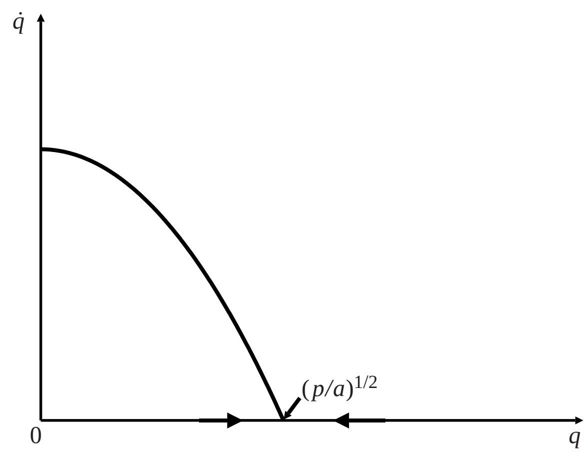
5. $y(t) = (-3t^2/2 + 3C)^{1/3}$

Review Exercises

1. $y(t) = 1/(Ce^{-2t} + 3)$

3. $y(t) = (t^2 + 2C)^{1/2}$

5. The point $\bar{q} = (p/a)^{1/2}$ is a stable steady state.



Chapter 23

23.1 Exercises

1. **(a)** $y(t) = 8e^t + 2$
(b) $y(t) = 2e^{-t/2} + 9te^{-t/2} + 8$
(c) $y(t) = 25.6e^{t/2} + 14.4e^{-t/3} - 30$
(d) $y(t) = 7e^{-t} + 15te^{-t} + 3$

3. $B < 10$

23.2 Exercises

1. $y(t) = C_1e^t + C_2e^{-3t} - 18e^{2t}/15$

3. $y(t) = C_1e^{3t} + C_2e^{2t} + 4e^{-t/2}/35$

5. $y(t) = C_1 + C_2e^{-t/2} + 8t$

Review Exercises

1. **(a)** $y(t) = C_1e^{2t} + C_2e^{-t} - 5$
(b) $y(t) = C_1e^{-3t} + C_2te^{-3t} + 3$
(c) $y(t) = e^{-2t}\sin t + 2$

3. $\ddot{y} - a_{11}\dot{y} - a_{12}a_{21}y = 0$. The solution is $y(t) = C_1e^{r_1t} + C_2e^{r_2t}$. We require that $a_{11} < 0$ and $a_{12}a_{21} < 0$ to ensure convergence.

5. **(a)** $y(t) = C_1e^{-3t} + C_2e^{-6t} + (A - F)/45$
(b) $y(t) = e^{-3t}[A_1\cos(3t) + A_2\sin(3t)] + (A - F)/45$

Chapter 24

24.1 Exercises

1. **(a)** $y(t) = C_1e^{r_1t} + C_2e^{r_2t} + 20$
$x(t) = r_1C_1e^{r_1t} + r_2C_2e^{r_2t} + 0$
where $r_1, r_2 = 1.25 \pm 0.25\sqrt{21}$.
(b) $y(t) = C_1e^{r_1t} + C_2e^{r_2t} - 1/2$
$x(t) = r_1C_1e^{r_1t} + r_2C_2e^{r_2t} + 0$
where $r_1, r_2 = \pm\sqrt{2}$.
(c) $y(t) = C_1e^{r_1t} + C_2e^{r_2t} + 1$
$x(t) = r_1C_1e^{r_1t} + r_2C_2e^{r_2t} + 0$
where $r_1, r_2 = -5 \pm 2\sqrt{6}$.

3. **(a)** $y_1(t) = 2e^{-t} - 7e^{-4t} + 3$
$y_2(t) = e^{-t} + 7e^{-4t} - 3$
(b) $y_1(t) = 4.5e^{7t/2} + 15.3e^{-5t/2} + 0.2$
$y_2(t) = -9e^{7t/2} + 10.2e^{-5t/2} + 0.8$
(c) $y_1(t) = 4\cos(2t)e^{2t} + 2\sin(2t)e^{2t} - 1.5$
$y_2(t) = 4\sin(2t)e^{2t} - 2\cos(2t)e^{2t} + 1$

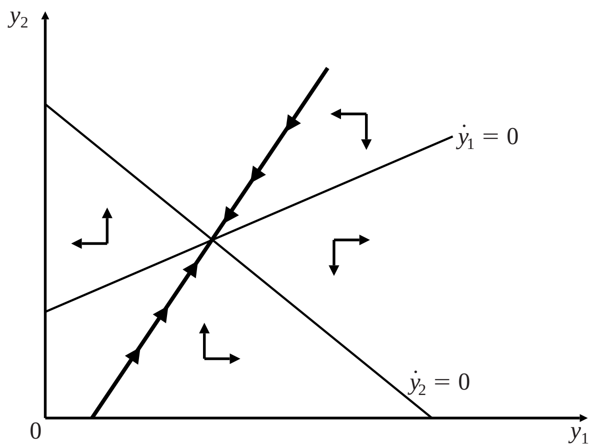
24.2 Exercises

1. **(a)** unstable node
(b) unstable focus
(c) saddle point
(d) stable node.

3. $y_1(t) = C_1e^{5t} + C_2e^{-7t} + 14$

$$y_2(t) = -\frac{C_1}{3}e^{5t} + C_2e^{-7t} + 7$$

The saddlepath equation is $y_2 = y_1 - 7$.



24.3 Exercises

1. $y_t = 20(1.5)^t + 10(-1.5)^t - 24$
$x_t = 2(1.5)^t - 5(-1.5)^t + 2$

3. $y_t = (5 - 3t)(0.5)^t$
$x_t = (8 - 6t)(0.5)^t$

5. $y_t = C_1(0.25)^t + C_2(-0.25)^t + 48$
$x_t = -C_1(0.25)^t + C_2(-0.25)^t + 48$
The steady state is stable.

7. $y_t = y_0(0.9)^t$
$x_t = y_0(0.9)^t$

Review Exercises

1. **(a)** $y_1(t) = C_1e^{r_1t} + C_2e^{r_2t} + 2$
$y_2(t) = 2\sqrt{3}C_1e^{r_1t} - 2\sqrt{3}C_2e^{r_2t} - 16$
where $r_1, r_2 = 1/2 \pm \sqrt{3}/2$
The steady state is a saddle point.

(b) $y_1(t) = (C_1 + C_2t)e^{t/2} + 29$
$y_2(t) = 2(C_1 + C_2t)e^{t/2} + \frac{4}{3}C_2e^{t/2} + 44$
The steady state is an unstable improper node.

3. If the roots are real-valued, the solutions are

$$y(t) = C_1e^{r_1t} + C_2e^{r_2t} + \frac{a}{\beta}$$

$$x(t) = (r_1 + \alpha)C_1e^{r_1t} + (r_2 + \alpha)C_2e^{r_2t} + \frac{\alpha a}{\beta}$$

where $r_1, r_2 = -\alpha/2 \pm \sqrt{\alpha^2 - 4\beta}/2$. The stock of pollution converges to the steady-state size of a/β.

5. The steady-state values are

$$\bar{K} = \left(\frac{2\delta^2}{\alpha}\right)^{1/(\alpha-2)}, \qquad \bar{I} = \delta\bar{K}$$

The determinant of the coefficient matrix of the linearized system is

$$-\delta^2 + \frac{(\alpha - 1)\alpha\bar{K}^{\alpha-2}}{2} < 0$$

It is negative because $\delta > 0$, $0 < \alpha < 1$, and $\bar{K} > 0$. Therefore, the steady state is a saddle point.

7. **(a)** $y_t = C_2(2)^t - 2$
$x_t = C_2(2)^t - 1$

(b) $y_t = C_1(-4)^t + C_2(-1)^t + 0.8$
$x_t = -C_1(-4)^t + \frac{C_2}{2}(-1)^t + 0.7$

Chapter 25

25.1 Exercises

1. $x(t) = (x_0 + a/2b)e^t - a/2b$

3. $x(t) = x_0e^{\alpha t} + \dfrac{\beta^2ce^{-\alpha t}}{4b\alpha^2} + \dfrac{\alpha\beta + \beta^2c}{4b\alpha^2}(\alpha^{\alpha T} - 2)(1 - e^{\alpha t})$

5.
$$\lambda(t) = C_1e^{2t} + C_2e^{-2t} + 1/4$$
$$x(t) = \frac{C_1}{2}e^{2t} - \frac{C_2}{6}e^{-2t} - 5/8$$

where

$$C_1 = \frac{6x_0e^{-4T} + 3.5e^{-2T}}{1 + 3e^{-4T}}$$
$$C_2 = \frac{-(0.75e^{-2T} + 6x_0 + 15/4)}{1 + 3e^{-4T}}$$

7.
$$\lambda(t) = C_1e^{r_1t} + C_2e^{r_2t} + \frac{b\delta p}{b\delta^2 + pa}$$
$$K(t) = \frac{r_1 - \delta}{2pa}C_1e^{r_1t} + \frac{r_2 - \delta}{2pa}C_2e^{r_2t} + \frac{p}{2(b\delta^2 + pa)}$$

where $r_1, r_2 = \pm\sqrt{\delta^2 + pa/b}$ and

$$C_1 = \frac{-2pa(K_0 - \bar{K})e^{(r_2-r_1)T} - (r_2 - \delta)\bar{\lambda}e^{-r_1T}}{r_2 - \delta - (r_1 - \delta)e^{(r_2-r_1)T}}$$
$$C_2 = \frac{2pa(K_0 - \bar{K}) + (r_1 - \delta)\bar{\lambda}e^{-r_1T}}{r_2 - \delta - (r_1 - \delta)e^{(r_2-r_1)T}}$$

25.2 Exercises

1. $x(t) = x_0e^t$, $\mu(t) = 0$

3.
$$\mu(t) = C_1e^{r_1t} + C_2e^{r_2t}$$
$$x(t) = \frac{r_1 - \rho + 3/2}{3/2}C_1e^{r_1t} + \frac{r_2 - \rho + 3/2}{3/2}C_2e^{r_2t}$$

where

$$C_1 = \frac{-x_0e^{(r_2-r_1)T}}{k_2 - k_1e^{(r_2-r_1)T}}$$

$$C_2 = \frac{x_0}{k_2 - k_1 e^{(r_2-r_1)T}}$$

$$k_i = \frac{r_i - \rho + 3/2}{3/2}, \qquad i = 1, 2$$

$$r_1, r_2 = \frac{\rho}{2} \pm \frac{\sqrt{\rho^2 - 6\rho + 12}}{2}$$

5. $$H(t) = \frac{C_1}{2}e^{r_1 t} + \frac{C_2}{2}e^{r_2 t} + \frac{q\delta(p-w)}{2q\delta(p+\delta)+2pa}$$

where

$$C_1 = \frac{-(L_0 - \bar{L})e^{(r_2-r_1)T} - 2k_2\bar{H}e^{-r_1 T}}{k_2 - k_1 e^{(r_2-r_1)T}}$$

$$k_i = \frac{r_i - \rho - \delta}{2pa}, \qquad i = 1, 2$$

$$r_1, r_2 = \frac{\rho}{2} \pm \frac{1}{2}\sqrt{\rho^2 + 4\delta(\rho+\delta) + \frac{4pa}{q}}$$

7. The steady state is a saddle point. The optimal trajectory begins with $K(0) = K_0$ and ends with $\mu(T) = 0$ which means $I(T) = 0$ if $c'(0) = 0$, and takes an amount of time exactly equal to T.

25.3 Exercises

1. $$\mu(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$
$$x(t) = k_1 C_1 e^{r_1 t} + k_2 C_2 e^{r_2 t}$$

where

$$k_i = \frac{r_i - \rho + 3/2}{3/2}, \qquad i = 1, 2$$

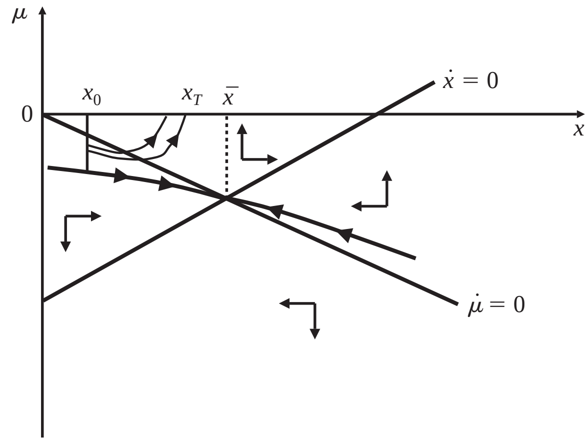
$$r_1, r_2 = \rho/2 \pm \sqrt{\rho^2 - 6\rho + 12}/2$$

$$C_1 = \frac{-x_0 e^{(r_2-r_1)T} + x_T e^{-r_1 T}}{k_2[1 - e^{(r_2-r_1)T}]}$$

$$C_2 = \frac{x_0 - x_T e^{-r_1 T}}{k_2[1 - e^{(r_2-r_1)T}]}$$

3. $$c^*(t) = \frac{(r - \rho - r\alpha)\left(x_0 e^{rT} - x_T\right)}{\alpha[e^{(r-\rho)T/\alpha} - e^{rT}]}$$

5.



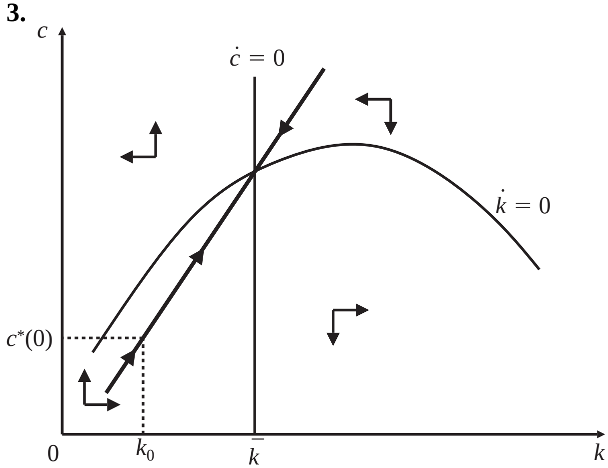
Not all trajectories beginning with $x(0) = x_0$ reach x_T. If T is very small, there may not be any trajectory that goes as far as x_T in that amount of time. As T gets large enough, we can choose a trajectory that reaches x_T and $\mu(t) = 0$. The larger T gets, the closer we get to the saddle path.

7. $$y^*(t) = \frac{\rho x_0 e^{-\rho t}}{1 - e^{-\rho T}}$$

25.4 Exercises

1. $c^*(t) = r\bar{x} - (x_0 - \bar{x})(\rho - 2r)e^{(\rho - r)t}$

3.



If $k_0 < \bar{k}$, the solution requires choosing a low value for consumption, $c^*(0)$, initially, which implies a high level of saving. As a result the capital stock grows as we follow the saddle path, with $c(t)$ and $k(t)$ rising as they approach the steady state.

5. $y^*(t) = \rho x_0 e^{-\rho t}$

25.5 Exercises

1. Suppose that $x_0 < x_T$. Then the solution is to choose a $\mu(0)$ that puts us on a trajectory that moves to the right ($c = 0$ and $\dot{x} > 0$) until it reaches $\mu = 1$, then switches to the left ($c = c_{\max}$ and $\dot{x} < 0$) until it reaches x_T. Many trajectories do this, but we choose the one that takes an amount of time equal to T.

3. The phase diagram has the same properties as figure 25.13. The solution is to follow the saddle path to reach the steady state in finite time and then stay there forever.

25.6 Exercises

1.

$$y^*(t) = \frac{\rho x_0/\alpha}{1 - e^{-\rho T/\alpha}} e^{-\rho t/\alpha}$$

where

$$T = \frac{\alpha}{\rho} \ln\left[1 + \frac{\rho x_0}{\alpha}\left(\frac{c(1-\alpha)}{\alpha}\right)^{-1/(1-\alpha)}\right]$$

3.

$$y^*(t) = \frac{\rho x_0}{\alpha} e^{-\rho t/\alpha}$$

Review Exercises

1. Differentiate H with respect to t to get

$$\dot{H} = F_x \dot{x} + F_y \dot{y} + \dot{\lambda} G(x, y) + \lambda(G_x \dot{x} + G_y \dot{y})$$

Collect terms in $\dot{x}$ and $\dot{y}$. Substitute the first-order condition for y and the equation for $\dot{\mu}$ that must hold along an optimal path. Finally, substitute $\dot{x}$ for $G(x, y)$.

3. This is a free-endpoint problem. If T is fixed, the boundary conditions are $\mu(T) = 0$ and $K(0) = K_0$. The optimal trajectory starts with $K(0) = K_0$ and ends with $\mu(T) = 0$ and takes an amount of time exactly equal to T.

5.

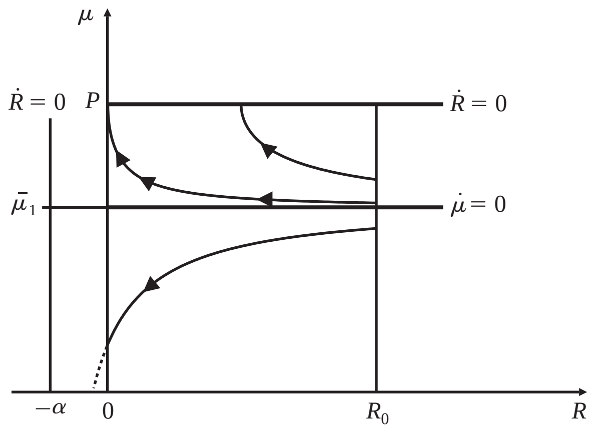
$$y^*(t) = \frac{(\rho - r + r\alpha)x_0}{\alpha\left[1 - e^{-(\rho-r)T/\alpha} e^{-rT}\right]} e^{-(\rho-r)t/\alpha}$$

7.

$$\dot{\mu} = \rho\mu - \frac{(P-\mu)^2}{2}$$

$$\dot{R} = -(P-\mu)(R+\alpha)$$

These differential equations yield the phase diagram shown here:



The phase diagram shows that the saddle path is the horizontal line which reaches the steady-state point at $\bar{R} = -\alpha$ and $\mu = \bar{\mu}_1$. It is clearly not possible to reach a negative resource stock.

The optimal trajectory begins at $R(0) = R_0$, finishes at $R(T) = 0$ and also satisfies $\mu(T) = P$. This occurs in finite time, since the point $(R = 0, \mu = P)$ is not a steady state.

9. The steady-state value of x is the positive solution to

$$x^2 + x\frac{[P(\rho - r) - cr]}{2rP} - \frac{\rho c}{2rP} = 0$$

If $x_0 < \bar{x}$, we set $h = 0$. Then $\mu(t)$ falls and $x(t)$ rises until $\bar{x}$ is reached. At that point we switch to $h = r\bar{x}(1 - \bar{x})$ and remain there forever.

If $x_0 > \bar{x}$, we set $h = h_{\max}$. Then $\mu(t)$ rises and $x(t)$ falls until $\bar{x}$ is reached. Then we switch to $h = r\bar{x}(1 - \bar{x})$ and remain there forever.

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