Modèle de Ramsey

$$U = \int_0^\infty u\left(c\left(t\right)\right) e^{nt} e^{-\rho t} dt \quad (1)$$

$$\dot{a} = w + ra - c - na \quad (2)$$

$$\lim_{t \to \infty} \left\{ a\left(t\right) \exp\left[-\int_0^t \left[r(\tau) - n \right] d\tau \right] \right\} \geq 0 \quad (3)$$

$$J = u\left(c\right) e^{(n-\rho)t} + v\left[w + \left(r - n\right) a - c \right] \quad (4)$$

$$\frac{\partial J}{\partial c} = 0 \Leftrightarrow u'\left(c\right) e^{(n-\rho)t} - v = 0 \quad (5)$$

$$-\frac{\partial J}{\partial a} = \dot{v} \Leftrightarrow \dot{v} = -\left(r - n\right) v \quad (6)$$

$$\lim_{t \to \infty} \left[v\left(t\right) a\left(t\right) \right] = 0 \quad (7)$$

Dérivée de (5)

$$\frac{du'}{dt} e^{(n-\rho)t} + u'\left(n - \rho\right) e^{(n-\rho)t} = \dot{v}$$

La relation (6) devient :

$$\frac{du'}{dt} e^{(n-\rho)t} + u'\left(n - \rho\right) e^{(n-\rho)t} = -\left(r - n\right) u'\left(c\right) e^{(n-\rho)t}$$

soit :

$$-\frac{du'}{dt}\frac{1}{u'} + \rho = r$$

$$r = \rho - \frac{u''(c)c}{u'(c)} \frac{\dot{c}}{c} \quad (8)$$

$$\sigma = -\frac{\dfrac{d\left[c\left(t_1\right)/\,c\left(t_2\right)\right]}{c\left(t_1\right)/\,c\left(t_2\right)}}{\dfrac{d\left[u'\left(c\left(t_1\right)\right)/\,u'\left(c\left(t_2\right)\right)\right]}{u'\left(c\left(t_1\right)\right)/\,u'\left(c\left(t_2\right)\right)}}$$

$si : t_2 \to t_1$

$$\sigma = -\frac{\frac{dc}{c}}{\frac{du'}{u'}} = -\frac{dc}{du'}\frac{u'}{c} = \frac{-1}{\frac{du'}{dc}}\frac{u'}{c} = -\frac{u'(c)}{c\,u''(c)}$$

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta} \;\; ; \;\; \sigma = \frac{1}{\theta} \tag{9}$$

$$u'(c) = \frac{(1-\theta)\,c^{1-\theta-1}}{1-\theta} = c^{-\theta} \;\; ; \;\; u''(c) = -\theta c^{-\theta-1} \; ; \quad -\theta = \frac{u''(c)\,c}{u'(c)}$$

(8) devient :

$$\frac{\dot{c}}{c} = \frac{1}{\theta}(r-\rho) \tag{10}$$

(6) a pour solution :

$$v(t) = v(0)\exp\left\{-\int_0^t [r(\tau)-n]\,d\tau\right\}$$

(7) devient :

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(\tau)-n]\,d\tau\right]\right\} = 0 \tag{11}$$

$$\overline{r}(t) = \frac{1}{t}\int_0^t r(\tau)d\tau \tag{12}$$

$$\dot{a} = w + ra - na - c$$

a pour solution $\forall T > 0$:

$$a(T)e^{-(\overline{r}(T)-n)T} + \int_0^T c(t)e^{-(\overline{r}(t)-n)t}dt = a(0) + \int_0^T w(t)e^{-(\overline{r}(t)-n)t}dt$$

si $T \to \infty$ $a(T)\,e^{-(\overline{r}(T)-n)T} \to 0$ en raison de (11).
Il reste :

$$\int_0^\infty c(t)e^{-(\overline{r}(t)-n)t}dt = a(0) + \int_0^\infty w(t)e^{-(\overline{r}(t)-n)t}dt = a(0) + \widetilde{w}(0) \tag{13}$$

(10) devient par intégration :

$$c(t) = c(0)\, e^{\frac{1}{\theta}(\overline{r}(t)-\rho)t} \text{avec } \overline{r}(t) \text{ tel défini en } (12)$$

(13) devient :

$$\begin{aligned} c(0) &= \mu(0)\left[a(0) + \widetilde{w}(0)\right] & (14) \\ \frac{1}{\mu(0)} &= \int_0^\infty \exp\left\{\left[\overline{r}(t)\left(\frac{1-\theta}{\theta}\right) - \frac{\rho}{\theta} + n\right] t\right\} dt & (15) \\ \int_0^\infty c(0)\, e^{-(\overline{r}(t)-n)t} e^{\frac{1}{\theta}(\overline{r}(t)-\rho)t} &= c(0)\left(\int_0^\infty e^{\left(\overline{r}(t)\left(\frac{1-\theta}{\theta}\right)-\frac{\rho}{\theta}+n\right)t} dt\right) \end{aligned}$$

$$\begin{aligned} Y &= F\left(K, \widehat{L}\right) & (16) \\ \widehat{y} &= f\left(\widehat{k}\right) & (17) \\ \frac{\partial Y}{\partial L} &= \left[f\left(\widehat{k}\right) - \widehat{k} f'\left(\widehat{k}\right)\right] e^{xt} & (18) \end{aligned}$$

$$\begin{aligned} \text{Profit} &= F\left(K, \widehat{L}\right) - (r+\delta) K - wL & (19) \\ \text{Profit} &= \widehat{L}\left[f\left(\widehat{k}\right) - (r+\delta)\widehat{k} - we^{-xt}\right] & (20) \end{aligned}$$

$$\begin{aligned} f'\left(\widehat{k}\right) &= r + \delta & (21) \\ \left[f\left(\widehat{k}\right) - \widehat{k} f'\left(\widehat{k}\right)\right] e^{xt} &= w & (22) \end{aligned}$$

$$\begin{aligned} \dot{\widehat{k}} &= f\left(\widehat{k}\right) - \widehat{c} - (x+n+\delta)\widehat{k} & (23) \\ \frac{\dot{\widehat{c}}}{\widehat{c}} &= \frac{\dot{c}}{c} - x = \frac{1}{\theta}\left[f'\left(\widehat{k}\right) - \delta - \rho - \theta x\right] & (24) \\ \widehat{k} &= \frac{K}{\widehat{L}} = \frac{K}{Le^{xt}} \\ \frac{\dot{\widehat{k}}}{\widehat{k}} &= \frac{\dot{K}}{K} - \frac{\dot{L}}{L} - x = \frac{\dot{k}}{k} - x \\ \dot{\widehat{k}} &= \frac{\dot{k}}{k}\widehat{k} - x\widehat{k} = \dot{k}e^{-xt} - x\widehat{k} \end{aligned}$$

de (2) : $\dot{k} = kr - kn + w - c$

$$\dot{k} = k\left(f'\left(\widehat{k}\right) - \delta\right) - kn + \left[f\left(\widehat{k}\right) - \widehat{k}f'\left(\widehat{k}\right)\right]e^{xt} - c$$

d'où

$$\begin{aligned}\dot{\widehat{k}} &= \widehat{k}f'\left(\widehat{k}\right) - \delta\widehat{k} - n\widehat{k} + f\left(\widehat{k}\right) - \widehat{k}f'\left(\widehat{k}\right) - \widehat{c} - x\widehat{k} \\ \dot{\widehat{k}} &= f\left(\widehat{k}\right) - \widehat{c} - (x+n+\delta)\widehat{k}\end{aligned}$$

$$\lim_{t\to\infty}\left\{\widehat{k}\exp\left(-\int_0^t\left[f'\left(\widehat{k}\right) - \delta - x - n\right]d\tau\right)\right\} = 0 \quad (25)$$

$$\widehat{c} = f\left(\widehat{k}\right) - (x+n+\delta)\widehat{k} - \widehat{k}\left(\gamma_{\widehat{k}}\right)^* \quad (26)$$

$$\dot{\widehat{c}} = \dot{\widehat{k}}\left\{f'\left(\widehat{k}\right) - \left[x+n+\delta+\gamma^*_{\widehat{k}}\right]\right\} \quad (27)$$

$$f'\left(\widehat{k}^*\right) = \delta + \rho + \theta x \quad (28)$$

$$\widehat{c}^* = f\left(\widehat{k}^*\right) - (x+n+\delta)\widehat{k}^* \quad (29)$$

$$\rho > n + (1-\theta)x \quad (30)$$