QUANTUM FIELD THEORY, PROBLEM SHEET 1

Problem 1: Fourier transform

We define the d-dimensional Fourier transformed function \tilde{f} for a suitable function $f: \mathbb{R}^d \to \mathbb{C}$ by

$$\tilde{f}(k) = \int \mathrm{d}^d x \, f(x) \, e^{-ik \cdot x}.$$

Show that

$$\widetilde{\lambda f + g} = \lambda \widetilde{f} + \widetilde{g} \qquad (\lambda \in \mathbb{C}),$$

$$\widetilde{\partial_{\mu} f} = i k_{\mu} \widetilde{f},$$

$$f(x) = \int \frac{\mathrm{d}^{d} k}{(2\pi)^{d}} \widetilde{f}(k) e^{ik \cdot x},$$

$$\int \mathrm{d}^{d} x f(x) g^{*}(x) = \int \frac{\mathrm{d}^{d} k}{(2\pi)^{d}} \widetilde{f}(k) \widetilde{g}^{*}(k).$$

Hint: Use the following, extremely useful representation of the Dirac delta function:

$$\int d^d k \, e^{ik \cdot (x-y)} = (2\pi)^d \, \delta^{(d)}(x-y) \,.$$

Problem 2: The classical free complex scalar field

For a complex-valued scalar field $\phi(x)$, it is convenient to treat ϕ and its complex conjugate ϕ^* as the two independent degrees of freedom (rather than the real and imaginary parts of ϕ). The action for a free field is

$$S[\phi, \phi^*] = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 \right) .$$

- 1. Show that this action describes two canonically normalised free real scalar fields $\varphi_{+} = \frac{1}{\sqrt{2}}(\phi + \phi^{*})$ and $\varphi_{-} = \frac{i}{\sqrt{2}}(\phi \phi^{*})$. ("Canonically normalised" means that the factor in front of the kinetic terms is $\frac{1}{2}$).
- 2. Derive the equations of motion for ϕ and ϕ^* .
- 3. Show that the action is invariant under a phase rotation $\phi(x) \to e^{i\alpha}\phi(x)$, where α is a real constant. Calculate the associated conserved Noether current.

Problem 3: Lagrangian formalism for classical electrodynamics

The action of classical electrodynamics is (in Heaviside-Lorentz units)

$$S[A] = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_{\mu} J^{\mu} \right).$$

Here $A_{\mu} = (\Phi, \vec{A})$ is the four-potential, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and $J_{\mu} = (\rho, \vec{\jmath})$ is a fixed external current density, with J^{μ} and all its derivatives vanishing at infinity.

- 1. Show that the homogeneous Maxwell equations $\partial_{\mu}F_{\kappa\lambda}\epsilon^{\mu\nu\kappa\lambda}=0$ are identically satisfied, by definition of $F^{\mu\nu}$. Show that the equations of motion for A_{μ} obtained from this action are the inhomogeneous Maxwell equations $\partial_{\mu}F^{\mu\nu}=J^{\nu}$.
- 2. Consider a gauge transformation $A_{\mu} \to A_{\mu} + \partial_{\mu} \Lambda$ with $\Lambda = \Lambda(x)$ some smooth function. Show that S[A] is invariant under this transformation if J^{μ} obeys the continuity equation $\partial_{\mu}J^{\mu} = 0$. Choosing Λ such that $\partial_{\mu}A^{\mu} = 0$ ("Lorenz gauge"), how do the equations of motion simplify?
- 3. Show that gauge symmetry is broken upon adding a "mass term" $\frac{1}{2}m^2A_{\mu}A^{\mu}$ for the gauge field to the Lagrangian. What are the equations of motion now for a gauge field A^{μ} satisfying the Lorenz gauge condition?
- 4. Calculate the electrostatic potential Φ of a point charge at rest, for m > 0. To do so, start by writing down its Fourier transform, and then evaluate the inverse transform using the identity

$$\int_0^\infty dk \, \frac{k \, \sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)$$

which you will prove on Problem Sheet 3. What do you obtain in the physical limit $m \to 0$?