Quantum Field Theory, problem sheet 1

Solutions to be discussed on 11/09/2024

Problem 1: Fourier transform

We define the d-dimensional Fourier transformed function $\tilde{f}$ for a suitable function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\tilde{f}(k) = \int \mathrm{d}^d x \, f(x) \, e^{-ik \cdot x}.$$

Show that

$$\widetilde{\lambda f + g} = \lambda \tilde{f} + \tilde{g} \qquad (\lambda \in \mathbb{C}),$$
$$\widetilde{\partial_\mu f} = ik_\mu \tilde{f},$$
$$f(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \tilde{f}(k) \, e^{ik \cdot x},$$
$$\int \mathrm{d}^d x \, f(x) g^*(x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \tilde{f}(k) \tilde{g}^*(k).$$

Hint: Use the following, extremely useful representation of the Dirac delta function:

$$\int \mathrm{d}^d k \, e^{ik \cdot (x-y)} = (2\pi)^d \, \delta^{(d)}(x - y).$$

Problem 2: The classical free complex scalar field

For a complex-valued scalar field $\phi(x)$, it is convenient to treat ϕ and its complex conjugate ϕ^* as the two independent degrees of freedom (rather than the real and imaginary parts of ϕ). The action for a free field is

$$S[\phi, \phi^*] = \int d^4 x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \, |\phi|^2 \right).$$

1. Show that this action describes two canonically normalised free real scalar fields $\varphi_+ = \frac{1}{\sqrt{2}}(\phi + \phi^*)$ and $\varphi_- = \frac{i}{\sqrt{2}}(\phi - \phi^*)$. ("Canonically normalised" means that the factor in front of the kinetic terms is $\frac{1}{2}$).

2. Derive the equations of motion for ϕ and ϕ^*.

3. Show that the action is invariant under a phase rotation $\phi(x) \to e^{i\alpha} \phi(x)$, where α is a real constant. Calculate the associated conserved Noether current.

Problem 3: Lagrangian formalism for classical electrodynamics

The action of classical electrodynamics is (in Heaviside-Lorentz units)

$$S[A] = \int d^4 x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \right).$$

Here $A_\mu = (\Phi, \vec{A})$ is the four-potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $J_\mu = (\rho, \vec{j})$ is a fixed external current density, with J^μ and all its derivatives vanishing at infinity.

1. Show that the homogeneous Maxwell equations $\partial_\mu F_{\kappa\lambda} \epsilon^{\mu\nu\kappa\lambda} = 0$ are identically satisfied, by definition of $F^{\mu\nu}$. Show that the equations of motion for A_μ obtained from this action are the inhomogenous Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$.

2. Consider a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ with $\Lambda = \Lambda(x)$ some smooth function. Show that $S[A]$ is invariant under this transformation if J^μ obeys the continuity equation $\partial_\mu J^\mu = 0$. Choosing Λ such that $\partial_\mu A^\mu = 0$ ("Lorenz gauge"), how do the equations of motion simplify?

3. Show that gauge symmetry is broken upon adding a "mass term" $\frac{1}{2}m^2 A_\mu A^\mu$ for the gauge field to the Lagrangian. What are the equations of motion now for a gauge field A^μ satisfying the Lorenz gauge condition?

4. Calculate the electrostatic potential Φ of a point charge at rest, for $m > 0$. To do so, start by writing down its Fourier transform, and then evaluate the inverse transform using the identity

$$\int_0^\infty dk \, \frac{k \, \sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)$$

which you will prove on Problem Sheet 3. What do you obtain in the physical limit $m \to 0$?