Quantum Field Theory, problem sheet 1

Solutions to be discussed on 20/09/2023

Problem 1: Fourier transform

We define the d-dimensional Fourier transformed function $\tilde{f}$ for a suitable function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\tilde{f}(k) = \int \mathrm{d}^d x\, f(x)\, e^{-ik\cdot x}.$$

Show that

$$\begin{aligned} \widetilde{\lambda f + g} &= \lambda \tilde{f} + \tilde{g} \qquad (\lambda \in \mathbb{C})\,, \\ \widetilde{\partial_\mu f} &= ik_\mu \tilde{f}\,, \\ f(x) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \tilde{f}(k)\, e^{ik\cdot x}\,, \\ \int \mathrm{d}^d x\, f(x) g^*(x) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \tilde{f}(k) \tilde{g}^*(k)\,. \end{aligned}$$

Hint: Use the following, extremely useful representation of the Dirac delta function:

$$\int \mathrm{d}^d k\, e^{ik\cdot(x-y)} = (2\pi)^d\, \delta^{(d)}(x-y)\,.$$

Problem 2: The classical free complex scalar field

For a complex-valued scalar field $\phi(x)$, it is convenient to treat ϕ and its complex conjugate ϕ^* as the two independent degrees of freedom (rather than the real and imaginary parts of ϕ). The action for a free field is

$$S[\phi, \phi^*] = \int d^4 x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \left|\phi\right|^2 \right)\,.$$

1. Show that this action describes two canonically normalised free real scalar fields $\varphi_+ = \frac{1}{\sqrt{2}}(\phi + \phi^*)$ and $\varphi_- = \frac{i}{\sqrt{2}}(\phi - \phi^*)$. ("Canonically normalised" means that the factor in front of the kinetic terms is $\frac{1}{2}$).
2. Derive the equations of motion for ϕ and ϕ^*.
3. Show that the action is invariant under a phase rotation $\phi(x) \to e^{i\alpha}\phi(x)$, where α is a real constant. Calculate the associated conserved Noether current.

Problem 3: Lagrangian formalism for classical electrodynamics

The action of classical electrodynamics is (in Heaviside-Lorentz units)

$$S[A] = \int d^4 x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \right)\,.$$

Here $A_\mu = (\Phi, \vec{A})$ is the four-potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $J_\mu = (\rho, \vec{\jmath})$ is a fixed external current density, with J^μ and all its derivatives vanishing at infinity.

1. Show that the homogeneous Maxwell equations $\partial_\mu F_{\kappa\lambda}\epsilon^{\mu\nu\kappa\lambda} = 0$ are identically satisfied, by definition of $F^{\mu\nu}$. Show that the equations of motion for A_μ obtained from this action are the inhomogenous Maxwell equations $\partial_\mu F^{\mu\nu} = J^\nu$.

2. Consider a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ with $\Lambda = \Lambda(x)$ some smooth function. Show that $S[A]$ is invariant under this transformation if J^μ obeys the continuity equation $\partial_\mu J^\mu = 0$. Choosing Λ such that $\partial_\mu A^\mu = 0$ ("Lorenz gauge"), how do the equations of motion simplify?

3. Show that gauge symmetry is broken upon adding a "mass term" $\frac{1}{2}m^2 A_\mu A^\mu$ for the gauge field to the Lagrangian. What are the equations of motion now for a gauge field A^μ satisfying the Lorenz gauge condition?

4. Calculate the electrostatic potential Φ of a point charge at rest, for $m > 0$. To do so, start by writing down its Fourier transform, and then evaluate the inverse transform using the identity

$$\int_0^\infty \mathrm{d}k \, \frac{k \, \sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)$$

which you will prove on Problem Sheet 3. What do you obtain in the physical limit $m \to 0$?