Quantum Field Theory, problem sheet 1

Problem 1: Fourier transform

We define the d-dimensional Fourier transformed function $\tilde{f}$ for a suitable function $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\tilde{f}(k) = \int \mathrm{d}^d x \, f(x) \, e^{-ik\cdot x}.$$

Show that

$$\begin{aligned}
\widetilde{\lambda f + g} &= \lambda \tilde{f} + \tilde{g} \qquad (\lambda \in \mathbb{C}) \,, \\
\widetilde{\partial_\mu f}(k) &= i k_\mu \tilde{f}(k) \,, \\
f(x) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \tilde{f}(k) \, e^{ik\cdot x} \,, \\
\int \mathrm{d}^d x \, f(x) g^*(x) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \, \tilde{f}(k) \tilde{g}^*(k) \,.
\end{aligned}$$

Hint: Use the following, extremely useful representation of the Dirac delta function:

$$\int \mathrm{d}^d k \, e^{ik\cdot(x-y)} = (2\pi)^d \, \delta^{(d)}(x-y) \,.$$

Problem 2: The classical free complex scalar field

For a complex-valued scalar field $\phi(x)$, it is convenient to treat ϕ and its complex conjugate ϕ^* as the two independent degrees of freedom (rather than the real and imaginary parts of ϕ). The action for a free field is

$$S[\phi, \phi^*] = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \, |\phi|^2 \right) .$$

1. Show that this action describes two canonically normalised free real scalar fields $\varphi_+ = \frac{1}{\sqrt{2}}(\phi + \phi^*)$ and $\varphi_- = \frac{i}{\sqrt{2}}(\phi - \phi^*)$. ("Canonically normalised" means that the factor in front of the kinetic terms is $\frac{1}{2}$).

2. Derive the equations of motion for ϕ and ϕ^*.

Problem 3: Lagrangian formalism for classical electrodynamics

The action of classical electrodynamics is (in Heaviside-Lorentz units)

$$S[A] = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \right) .$$

Here $A_\mu = (\Phi, \vec{A})$ is the four-potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $(J^\mu) = (\rho, \vec{j})$ is a fixed external current density, with J^μ and all its derivatives vanishing at infinity.

1. Show that $F^{\mu\nu}$, by construction, satisfies the identity

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0\,, \qquad \text{where } \widetilde{F}^{\mu\nu} = \frac{1}{2} F_{\kappa\lambda} \epsilon^{\mu\nu\kappa\lambda} \qquad (\text{with } \epsilon^{0123} = 1). \tag{1}$$

2. Find the equations of motion for A_μ. You should obtain

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{2}$$

3. We define the 3-vectors $\vec{E}(t, \vec{x})$ and $\vec{B}(t, \vec{x})$ by $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B^k$, such that

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}.$$

   Find the components of $\widetilde{F}$, and express Eqs. (1) and (2) in terms of $\vec{E}$, $\vec{B}$, ρ and $\vec{j}$.

4. Consider a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ with $\Lambda = \Lambda(x)$ some smooth function. Show that $S[A]$ is invariant under this transformation if J^μ obeys the continuity equation $\partial_\mu J^\mu = 0$. Choosing Λ such that $\partial_\mu A^\mu = 0$ ("Lorenz gauge"), how do the equations of motion simplify?

5. Show that gauge symmetry is broken upon adding a "photon mass term" $\frac{1}{2} m^2 A_\mu A^\mu$ to the Lagrangian. What are the equations of motion now for a gauge field A^μ satisfying the Lorenz gauge condition?

6. For $m > 0$, calculate the electrostatic potential Φ of a point charge at rest. To do so, start by writing down its Fourier transform, and then evaluate the inverse transform using the identity

$$\int_0^\infty \mathrm{d}k\, \frac{k \sin(kx)}{k^2 + m^2} = \frac{\pi}{2} e^{-mx} \qquad (x > 0)$$

   which you will prove on Problem Sheet 3. What do you obtain in the physical limit $m \to 0$?