Practical Hastings-Metropolis algorithm

Exercise 1

We want to get simulation from the distribution associated to the following density

$$f(x) = \frac{1}{Z} \exp(-x^2) \left(2 + \sin(5x) + \sin(2x)\right), \quad x \in [-3, 3]$$

whre Z is the normalizing constant of f.

- 1 Draw the graph of the unnormalized density.
- **2** Give an approximation of Z and with this, plot the graph of the normalized density.
- **3** Write a Metropolis-Hastings algorithm to simulate according to f. You can use a Gaussian random walk proposal with variance σ^2 .
- 4 Simulate the first 50,000 of the corresponding Markov chain.
- **5** Using an histogram, check that $(X_n)_{n>n_0}$ (for n_0 large enough) is a sequence of random variables which can be considered as simulation from f.
- **6** What is the influence of the choice of n_0 (burn-in period)?
- 7 Analyse the influence of the choice of σ^2 .

Exercise 2 We consider the set $\underline{x} = (x_1, \dots, x_n)$ of iid observation from a Gaussian distribution with unknown mean θ and variance equal to 2. We are interested in estimating θ using a Bayesian method. We adopt a uniform prior on [0, 5] for θ .

- 1 Give the density of the posterior distribution $\pi(\theta|x)$.
- **2** Propose an Metropolis-Hastings algorithm to get simulations from $\pi(\theta|x)$.
- **3** Use this algorithm to simulate a sample of N=1,000 random variables approximately distributed from $\pi(\theta|\underline{x})$ (a sample of size n=20 will first have been generated with $\theta=3$).
- 4 Give an approximation of $\mathbb{E}^{\pi}[\theta|\underline{x}]$.